Hence $Z(f)$ has singularities of the form

$$-\frac{1}{3}\frac{1}{f-nc\lambda}, \qquad n=0, \pm 1, \pm 2, \ldots, \tag{33.13}$$

along the real axis, and only these, so that it must have the form

$$Z(f) = -\frac{\pi}{3c\lambda}\cot\frac{\pi f}{c\lambda} + b, \qquad b = \text{const.} \tag{33.14}$$

From (33.14) $Z \to b - i\pi/3c\lambda$ as $\psi \to -\infty$. Then from (33.11)

$$b = -i\frac{\pi}{c\lambda}\left(a_0 - \frac{1}{3}\right).$$

Since Z must be real for real f, it follows that

$$a_0 = \tfrac{1}{3} \tag{33.15}$$

and

$$Z(f) = -\frac{\pi}{3c\lambda}\cot\frac{\pi f}{c\lambda}. \tag{33.16}$$

It now follows from the definition of $Z(f)$ that

$$\frac{d}{df}\log z'(f) = -\frac{\pi}{3c\lambda}\cot\frac{\pi f}{c\lambda} + i\frac{\pi}{3c\lambda} + i\frac{\pi}{c\lambda}\sum_{n=1}^{\infty} a_n\, e^{-i2n\pi f/c\lambda}, \tag{33.17}$$

which yields, after integration, inversion of the logarithm and use of (33.6) to evaluate a multiplicative constant,

$$\left.\begin{aligned} z'(f) &= \frac{1}{c\sqrt[3]{2}}\, e^{\frac{1}{3}i\pi f/c\lambda}\left(i\sin\frac{\pi f}{c\lambda}\right)^{-\frac{1}{3}} \prod_{n=1}^{\infty}\exp\left(\frac{-c\lambda}{2n\pi}\,a_n\, e^{-i2n\pi f/c\lambda}\right) \\ &= \frac{1}{c\sqrt[3]{2}}\, e^{\frac{1}{3}i\pi f/c\lambda}\left(i\sin\frac{\pi f}{c\lambda}\right)^{-\frac{1}{3}} \sum_{n=0}^{\infty} b_n\, e^{-i2n\pi f/c\lambda}, \end{aligned}\right\} \tag{33.18}$$

where $b_0 = 1$ and the b_n are real. The branch of the root must be chosen so that its argument lies between $\pm\frac{1}{2}\pi$ for $\psi = 0$. From (33.18) one finds immediately also

$$f'(z) = c\sqrt[3]{2}\, e^{-\frac{1}{3}i\pi f/c\lambda}\left(i\sin\frac{\pi f}{c\lambda}\right)^{\frac{1}{3}} \sum_{n=0}^{\infty} c_n\, e^{-i2n\pi f/c\lambda}, \qquad c_0 = 1,\ c_n \text{ real}. \tag{33.19}$$

Aside from the first one, the coefficients in (33.18) or (33.19) are still to be determined. The constant-pressure condition for the surface profile is still available for this purpose, for we have made use of the Eq. (33.4) or (33.5) only through the value of the exponent. The value of the gravitation constant has not entered into (33.18) or (33.19). In fact, a comparison of (33.5) after differentiation and (33.18) in the neighborhood of $f=0$ yields immediately

$$\frac{c^2}{g\lambda} = \frac{3}{4\pi}\left[1 + b_1 + b_2 + \cdots\right]^3 = \frac{3}{4\pi}\left[1 + c_1 + c_2 + \cdots\right]^{-3}, \tag{33.20}$$

so that, once the b_n or c_n are determined, the relation between wavelength and velocity may be found. This method could presumably be pursued to obtain a sequence of further equations for determination of the b_n. However, MICHELL proceeds somewhat differently. If we differentiate (33.1) with respect to φ, we may write the free surface condition as follows [cf. (32.54) and following]:

$$\left.\begin{aligned} &\frac{\partial}{\partial\varphi} q^2 = -\frac{g}{q^2}\frac{\partial\varphi}{\partial y} \qquad \text{for } \psi = 0 \\ \text{or} \qquad & \\ &\frac{\partial}{\partial\varphi} |f'|^4 = 4g \operatorname{Im} f' \qquad \text{for } \psi = 0. \end{aligned}\right\} \tag{33.21}$$

Substitution of (33.19) in (33.21) yields an equation of the following form

$$\left.\begin{aligned} &\frac{4}{3}\pi\, 2^{\frac{4}{3}} \frac{c^3}{\lambda} \sin^{\frac{1}{3}} \frac{\pi\varphi}{c\lambda} \left\{A_1 \cos\frac{\pi\varphi}{c\lambda} + A_3 \cos\frac{3\pi\varphi}{c\lambda} + \cdots\right\} \\ &\qquad = \frac{4}{3} g c \sin^{\frac{1}{3}} \frac{\pi\varphi}{c\lambda} \left\{B_1 \cos\frac{\pi\varphi}{c\lambda} + B_3 \cos\frac{3\pi\varphi}{c\lambda} + \cdots\right\}, \end{aligned}\right\} \tag{33.22}$$

where the B_n's depend linearly upon the c_n's, and the A_n's depend upon them in a more complicated manner. The derivation of (33.22), especially of the right-hand part, and of the particular dependence of the A_n's and B_n's upon the c_n's is rather tedious and we refer to either MICHELL's original paper or preferably to HAVELOCK's more general and systematic treatment. Equating coefficients of the individual cosine terms leads to a set of equations relating $c^2/g\lambda$, c_1, c_2, ... The values as computed by HAVELOCK, which we assume to be somewhat more accurate than MICHELL's own, are as follows:

$$\frac{g\lambda}{c^2} = 0.833 \cdot 2\pi, \quad c_1 = 0.0414, \quad c_2 = 0.0114, \quad c_3 = 0.0042, \quad c_4 = 0.0014. \tag{33.23}$$

The value for $g\lambda/c^2$ should be compared with that for infinitesimal waves, namely 2π. Substitution of $\frac{1}{2}c\lambda$ for f in (33.19) yields the velocity at a trough:

see errata

$$u = c\sqrt[3]{2}\,[1 - c_1 + c_2 - c_3 + \cdots] \approx 1.219c. \tag{33.24}$$

From $q^2 + 2g\eta = 0$ one may now find η for the trough and hence the amplitude-wavelength ratio:

$$\left|\frac{\eta}{\lambda}\right| = \frac{1}{\sqrt[3]{2}} \frac{c^2}{g\lambda} [1 - c_1 + c_2 - \cdots]^2 \approx 0.1418. \tag{33.25}$$

H. JEFFREYS (1951) has recently reexamined the basis of the Michell-Havelock method of approximation and concludes that an apparent discrepancy between the values in (33.23) and Eq. (33.20) does not really indicate a numerical error in the computations.

We note in passing that MICHELL also gave the form of $f'(z)$ analogous to (33.19) which must hold if a highest wave with corner exists in a fluid of finite depth.

Method of NEKRASOV and YAMADA. This method makes use of the ζ-plane introduced in (32.57) and related to the f-plane by (32.58). We may again make use of Fig. 50 but must keep in mind that in the z-plane there is now a corner at 0 with an included angle of 120°. Hence (32.59), the equation relating the z- and ζ-planes, must be replaced by

$$\frac{dz}{d\zeta} = -\frac{\lambda}{2\pi i} \frac{h(\zeta)}{\zeta(1-\zeta)^{\frac{1}{3}}}, \qquad h(\zeta) = 1 + a_1\zeta + a_2\zeta^2 + \cdots \tag{33.26}$$

and (32.60) by

$$w = \frac{df}{dz} = c\,\frac{(1-\zeta)^{\frac{1}{3}}}{h(\zeta)}. \tag{33.27}$$

The coefficients a_n are now to be determined by the constant-pressure condition on the free surface taken in the form (32.94). From

$$q^2|_{\varrho=1} = c^2 \frac{[(1-e^{i\gamma})(1-e^{-i\gamma})]^{\frac{1}{3}}}{h(e^{i\gamma})\,h(e^{-i\gamma})} = c^2 \frac{(2\sin\frac{1}{2}\gamma)^{\frac{2}{3}}}{h(e^{i\gamma})\,h(e^{-i\gamma})} \tag{33.28}$$

and

$$\frac{dz}{d\gamma}\bigg|_{\varrho=1} = -\frac{\lambda}{2\pi}\,\frac{h(e^{i\gamma})}{(1-e^{i\gamma})^{\frac{1}{3}}} = -\frac{\lambda}{2\pi}\left(2\sin\frac{1}{2}\gamma\right)^{-\frac{1}{3}} e^{-i(\gamma-\pi)/6}\,h(e^{i\gamma}) \tag{33.29}$$

one obtains as the equation analogous to (32.96)

$$\frac{d}{d\gamma}\,\frac{(2\sin\frac{1}{2}\gamma)^{\frac{2}{3}}}{h(e^{i\gamma})\,h(e^{-i\gamma})} = \frac{g\lambda}{\pi c^2}\left(2\sin\frac{1}{2}\gamma\right)^{-\frac{1}{3}}\mathrm{Im}\{e^{-i(\gamma-\pi)/6}\,h(e^{i\gamma})\}. \tag{33.30}$$

This yields a set of equations for determination of $g\lambda/c^2$, a_1, a_2, ... The actual computation appears to be as tedious as that of MICHELL's method and, in fact, NEKRASOV's (1920) computations do not seem to have yielded as accurate results as MICHELL's. However, as mentioned earlier, YAMADA (1957) has set up a systematic computation procedure and has obtained results in substantial agreement with those of MICHELL and HAVELOCK. Once $g\lambda/c^2$ and the a_n have been determined, the surface profile can be found in parametric form by integrating (33.29) with respect to γ from 0 to γ. Fig. 51, reproduced from YAMADA's cited paper, shows the form of the profile.

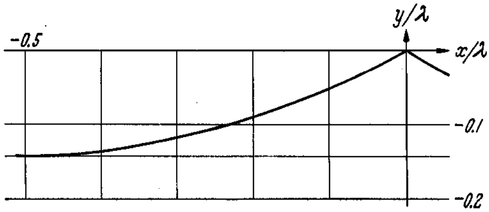


Fig. 51.

β) Havelock's approximation for gravity waves. In a paper already cited several times above HAVELOCK (1919) extended MICHELL's method of construction of periodic waves of maximum amplitude, outlined in the preceding section, to one for construction of periodic waves of any allowable amplitude-length ratio. Up to the present, no one has proved the series involved to converge. However, as HAVELOCK points out, the method has attractive theoretical features: the parameter describing the family of waves occurs in the form $e^{-\beta}$ where β varies from 0, corresponding to the highest wave, to ∞, corresponding to infinitesimal waves.

The method starts out exactly like MICHELL's up to Eq. (33.19) except that it is not assumed that $\psi=0$ corresponds to the free surface. We recall that in MICHELL's analysis the constant-pressure condition did not enter completely until after (33.19), in particular, in (33.21). HAVELOCK assumes instead that this condition is to be satisfied on some other streamline, $\psi=-\alpha$, which will then be taken to correspond to the free surface. The condition may still be written in the form (33.21) provided that one replaces $\psi=0$ by $\psi=-\alpha$. For $\psi=-\alpha$ one may write

$$f' = c\sqrt[3]{2}\,e^{-\frac{1}{3}i\pi(\varphi-i\alpha)/c\lambda}\left(i\sin\pi(\varphi-i\alpha)/c\lambda\right)^{\frac{1}{3}}\sum_{n=0}^{\infty}\gamma_n\,e^{-i2n\pi\varphi/c\lambda}, \tag{33.31}$$

where $\gamma_n = c_n\,e^{-2n\pi\alpha/c\lambda}$, the c_n being the same as those in (33.19). HAVELOCK shows that one may express $\partial|f'|^4/\partial\varphi$ in the following form

$$\left.\begin{aligned}\frac{\partial}{\partial\varphi}|f'|^4 = \frac{4}{3}\pi\,2^{\frac{4}{3}}\frac{c^3}{\lambda}\sin\frac{\pi\varphi}{c\lambda}\left[\sinh^2\frac{\pi\alpha}{c\lambda}+\sin^2\frac{\pi\varphi}{c\lambda}\right]^{-\frac{1}{3}}\times\\ \times\left[A_1\cos\frac{\pi\varphi}{c\lambda}+A_3\cos\frac{3\pi\varphi}{c\lambda}+\cdots\right]\end{aligned}\right\} \quad (33.32$$

and $\operatorname{Im} f'$ in the form

$$\left.\begin{aligned}\operatorname{Im} f' = \frac{1}{3} c\, e^{-4\pi\alpha/3c\lambda} \sin\frac{\pi\varphi}{c\lambda}\left[\sinh^2\frac{\pi\alpha}{c\lambda} + \sin^2\frac{\pi\varphi}{c\lambda}\right]^{-\frac{1}{3}} \times \\ \times\left[B_1\cos\frac{\pi\varphi}{c\lambda} + B_3\cos\frac{3\pi\varphi}{c\lambda} + \cdots\right].\end{aligned}\right\} \tag{33.33}$$

Here the A_n's are rather complicated expressions in the γ_n's but also involve $\cosh \pi\alpha/c\lambda$ linearly; the B_n's are linear expressions in the γ_n with coefficients which are functions of $e^{-2\pi\alpha/c\lambda}$. Havelock finds complete expressions for the B_n's; for A_1, A_3, A_5, A_7 he finds the dependence upon the first few γ_n's. One must refer to the original for details, especially for the scheme for approximate solution for the γ_n's.

When $\alpha=0$ the above analysis is precisely that for the highest wave. The numerical results of Havelock's computations for this case were given in the last section. He also computes $g\lambda/c^2$, γ_1, γ_2 (also γ_3 for the first) for two further cases: $e^{-2\pi\alpha/c\lambda}=0.75$ and 0.3. The agreement with results computed by other methods, either those of subsection 27α or similar ones, is very close. However, to establish the validity of the method, one must prove convergence of the series $\Sigma|\gamma_n|$.

The relation of this method of approximation to Stokes' "second method" (see subsection 32γ) is also clarified by Havelock. For this we refer to the original paper.

34. Explicit solutions. Although it is not in general possible to give an explicit exact solution to a particular problem of interest, it is possible to give various classes of exact solutions and then to determine the associated solid boundaries. This is sometimes referred to as an "inverse method". Several such methods for constructing exact solutions will be discussed in subsection 34α. In addition, there is one periodic wave in infinitely deep fluid which satisfies the boundary conditions exactly, the Gerstner wave. This will be discussed in subsection 34β. In subsection 34γ we shall discuss briefly what may be called pseudo-exact solutions due to Davies and Packham. In these the exact boundary condition is replaced by a closely related one which allows exact solution. They derive their interest from the fact that they contain in one family waves ranging from the smallest amplitude-length ratio up to a counterpart of the Michell wave. Furthermore, the procedure also can be used for pseudo solitary and cnoidal waves. Subsection 34δ will be devoted to an exact solution for pure capillary waves recently discovered by Crapper (1957).

α) Inverse methods. Sautreaux's method. Possibly the earliest method capable of generating a wide class of steady irrotational solutions is due to C. Sautreaux (1893, 1894, 1901). It has been rediscovered several times subsequently, e.g., by Blasius (1910), Wilton (1913), Richardson (1920) and Lewy (1952). F. Aimond (1929) has given a very comprehensive treatment of the method and of various related ones. The method may be easily generalized to include an arbitrary impressed pressure distribution on the free surface (see the papers of Richardson or Aimond).

Let $z=x+iy$, $f=\varphi+i\psi$, and take f as the independent variable. The free surface will be represented by $\psi=0$. We further assume $q^2>\varepsilon>0$. In the constant-pressure condition on the surface, $\frac{1}{2}q^2+g\eta=\text{const}$, it will be convenient to take the position of the x-axis so that the constant is zero, and hence $\eta\leqq 0$. This condition may then be expressed in terms of $z(f)$ as follows [cf. (32.56)]:

$$z'(\varphi)\,\overline{z'(\varphi)}\left[z(\varphi)-\overline{z(\varphi)}\right]=-ig. \tag{34.1}$$

Define

$$\mu(f) = \tfrac{1}{2} i \left[z(f) - \overline{z(\bar{f})}\right]. \tag{34.2}$$

Then $-\mu(\varphi) = y(\varphi)$, the y-coordinate of the free surface. Hence, from (34.1)

$$\mu(\varphi) = \frac{1}{2g} \frac{1}{z'(\varphi)\,\overline{z'(\varphi)}}, \tag{34.3}$$

From (34.2) and (34.3) one may now derive

$$2\,[g\,\mu(\varphi)]^{-1} - 4\mu'^2(\varphi) = \left[z'(\varphi) + \overline{z'(\varphi)}\right]^2. \tag{34.4}$$

Elimination of $\overline{z'}$ between (34.2) and (34.4) yields

$$z'(\varphi) = -i\,\mu'(\varphi) + \sqrt{(2g\,\mu)^{-1} - \mu'^2}, \tag{34.5}$$

where

$$\mu(\varphi) > 0, \qquad 2g\,\mu(\varphi)\,\mu'^2(\varphi) \leqq 1. \tag{34.6}$$

But then, since z' is an analytic function of f, at least near $\psi = 0$,

$$z'(f) = -i\,\mu'(f) + \sqrt{(2g\,\mu)^{-1} - \mu'^2} \tag{34.7}$$

and

$$z(f) = -i\,\mu(f) + \int \sqrt{(2g\,\mu)^{-1} - \mu'^2}\,df. \tag{34.8}$$

One may now reverse the procedure, select an arbitrary analytic function $\mu(f)$ satisfying (34.6) and construct the function $z(f)$ by means of (34.8). The resulting function will describe a flow for which $z(\varphi)$ is the free surface. If (34.6) is satisfied only for some range of φ, then for the remaining range one must treat the streamline $\psi = 0$ as a solid boundary.

One can use the preceding result to construct a flow if the form of the free surface is given. Let the surface be given in the form $x = \xi(y)$ in a neighborhood of some point of the surface. Since $y(\varphi) = -\mu(\varphi)$ on the surface, we may define $\sigma(\mu) = \xi'(y) = x'(\varphi)/y'(\varphi)$; σ is an analytic function of μ for real μ as follows from the theorem of LEWY and GERBER cited near the beginning of subsection 32γ. Hence, from (34.7),

$$\sigma(\mu) = -\left[(2g\,\mu)^{-1} - \mu'^2\right]^{\frac{1}{2}}/\mu'(\varphi). \tag{34.9}$$

Solving for $1/\mu'$, one finds

$$\frac{d\varphi}{d\mu} = \sqrt{2g\,\mu\,(1+\sigma^2)}. \tag{34.10}$$

Since μ is also an analytic function of φ, the same relation holds for $df/d\mu$ when μ is complex, and consequently

$$f = \int \sqrt{2g\,\mu\,(1+\sigma^2(\mu))}\,d\mu. \tag{34.11}$$

It follows similarly from (34.8) and (34.9), first for real μ, then for complex μ that

$$z = -i\,\mu - \int \sigma(\mu)\,d\mu. \tag{34.12}$$

Eqs. (34.11) and (34.12) thus provide a relation between f and z determined by the form of $\sigma(\mu)$ for real μ.

RUDZKI's method. RUDZKI (1898) has given a different formula for deriving exact solutions. The derivation and statement of the formula below differ somewhat from RUDZKI's, but the result is equivalent.

Let $z = z(f)$ and write

$$z' = \frac{1}{q} e^{i\vartheta}, \quad q = q(\varphi, \psi), \quad \vartheta = \vartheta(\varphi, \psi). \tag{34.13}$$

The free-surface condition may be expressed as follows, from (32.61) and the equation preceding it,

$$q^2 \frac{\partial q}{\partial \varphi} = -g \sin\vartheta \quad \text{for} \quad \psi = 0. \tag{34.14}$$

Hence

$$q = [-3g \int \sin\vartheta(\varphi, 0)\, d\varphi]^{\frac{1}{3}} \quad \text{for} \quad \psi = 0, \tag{34.15}$$

where the branch of the cube root is taken which is real for real numbers. Combining (34.15) with (34.13) gives

$$z'(\varphi) = e^{i\vartheta(\varphi,0)} [-3g \int \sin\vartheta(\varphi, 0)\, d\varphi]^{-\frac{1}{3}}. \tag{34.16}$$

This relation must then hold also for $\psi \neq 0$, i.e.

$$z'(f) = e^{i\vartheta(f,0)} [-3g \int \sin\vartheta(f, 0)\, df]^{-\frac{1}{3}}. \tag{34.17}$$

As in SAUTREAUX's method, we may now reverse the above procedure, take $\vartheta(f)$ as an arbitrary analytic function of f such that ϑ is real for f real, and construct $z'(f)$ from (34.17).

RICHARDSON's method. From (34.17) one can derive immediately a formula due to RICHARDSON (1920) for constructing exact solutions. Let $G'(f) = -\sin\vartheta(f)$. Then $e^{i\vartheta} = \sqrt{1 - G'^2} - iG'$ and (34.17) becomes

$$z'(f) = [3g\, G(f)]^{-\frac{1}{3}} \left[-i\, G'(f) + \sqrt{1 - G'^2}\right]. \tag{34.18}$$

Again, inversely, if $G(f)$ is any analytic function such that, for real f, G', $\sqrt{1 - G'^2}$ and G are real, (34.18) gives a corresponding exact free-surface flow.

Examples. The largest collections of specific flows constructed by one of the preceding methods are in the paper of RICHARDSON (1920) and a report of VITOUSEK (1954). Several examples are given below.

1. In (34.17) let $\vartheta(f) = \text{const} = \alpha < 0$. Further, take the constant of integration as zero even though this results in a singularity in z' on the surface. One finds easily

$$f = \tfrac{2}{3} \sqrt{-2g \sin\alpha}\, (z\, e^{-i\alpha})^{\frac{3}{2}}. \tag{34.19}$$

The free surface will consist of only the ray $z = r\, e^{i\alpha}$ unless $\alpha = \pi/6$. However, the ray $z = r\, e^{i(\alpha - \frac{2}{3}\pi)}$ is also a streamline, but not one along which the pressure is constant unless $\alpha = -\pi/6$. Hence it must be taken as a solid boundary in general. The special case $\alpha = -\pi/6$ is just the flow (33.4) considered earlier and has a corner. One may, of course, take any other streamline $\psi = \psi_0 < 0$ as another solid boundary representing a bottom. The pressure remains positive everywhere only if $-\pi/6 < \alpha < 0$. This special family of flows was discussed by WEINGARTEN (1904).

2. If in (34.8) one takes $\mu(f) = f/c$ or in (34.18) takes $G(f) = \frac{2}{3} \sqrt{2g/c^3}\, f^{\frac{3}{2}}$, where c is some fixed velocity, one finds

$$c\, z'(f) = -i + \sqrt{(2g\, f/c^3)^{-1} - 1}. \tag{34.20}$$

This yields a flow of the sort shown in Fig. 52c, taken from RICHARDSON. The internal solid boundary represents some streamline $\psi = \psi_0 < 0$. The free surface corresponds to the segment $\psi = 0$, $0 < \varphi < c^3/2g$ in the f-plane.

3. Let c be some fixed velocity and let

$$G(f) = \frac{3c^3}{g}\left[B + \tanh\left(\alpha \frac{g}{3c^3} f\right)\right], \qquad B > 1, \ \alpha < 1,$$

in (34.18). Then

$$c\,z'(f) = \left[B + \tanh\left(\alpha \frac{g}{3c^3} f\right)\right]^{-\frac{1}{3}} \left\{-i\,\alpha\,\mathrm{sech}^2\left(\alpha \frac{g}{3c^3} f\right) + \sqrt{1 - \alpha^2\,\mathrm{sech}^4\left(\alpha \frac{g}{3c^3} f\right)}\right\}. \quad (34.21)$$

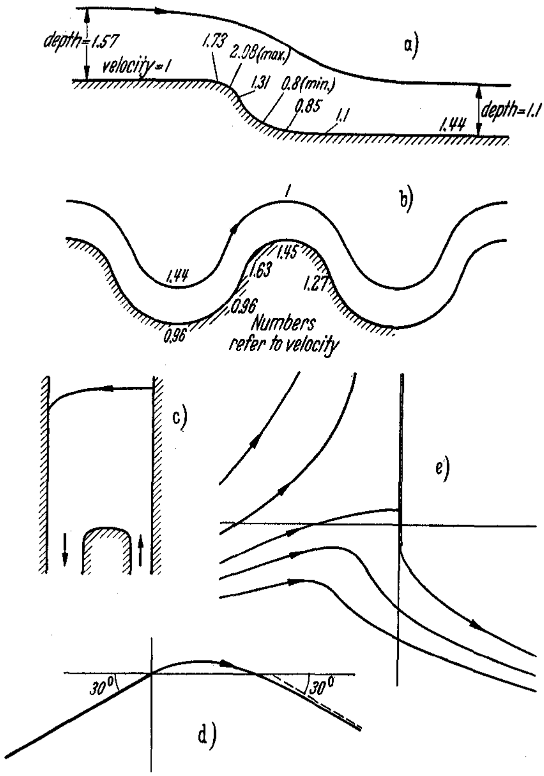


Fig. 52 a—d.

Here $\psi = 0$ corresponds to the free surface. The choice of the bottom streamline is restricted by the necessity of avoiding having the singularity at $B = \tanh \frac{1}{3}\alpha g c^{-3} f$ within the fluid. Fig. 52a, also from RICHARDSON, shows a flow computed from (34.21) for $B = 2$, $\alpha = \frac{1}{2}$ and $c = 1$.

4. Flow over a corrugated bottom has been investigated by both RICHARDSON and RUDZKI by essentially the same method. Following RICHARDSON, we let

$$G(f) = \frac{3c^3}{g}\left[B - \cos\alpha \frac{g}{3c^3} f\right], \qquad B > 1, \ \alpha < 1.$$

Then

$$c\,z'(f) = \left[B - \cos\alpha \frac{g}{3c^3} f\right]^{-\frac{1}{3}} \left\{-i\,\alpha \sin\alpha \frac{g}{3c^3} f + \sqrt{1 - \alpha^2 \cos^2\alpha \frac{g}{3c^3} f}\right\}. \quad (34.22)$$

Fig. 52b shows a flow computed from this formula for $B = 2$, $\alpha = 0.9$.

5. Flows similar to flows over a weir, under a sluice gate, through an opening, etc. have been considered by a number of the cited authors. SAUTREAUX (1901)

applied his formula (34.8) with $\mu = (c^2/2g)\, e^{-2gf/c^2}$ to obtain a number of different flows of this nature. Fig. 52d shows one of them. LAUCK (1925) has also constructed such flows. RICHARDSON obtained a flow through an opening by selecting

$$G(f) = \frac{3c^3}{g}\left[B - e^{gf/3c^3}\right].$$

Possibly the simplest such flow, studied by both BLASIUS and VITOUSEK, is obtained by taking $\mu = \sqrt{cf/g}$ in (34.8); this yields

$$\frac{g}{c^2} z = -i\sqrt{\frac{gf}{c^3}} + \frac{1}{3}\left[2\sqrt{\frac{gf}{c^3}} - 1\right]^{\frac{3}{2}}. \tag{34.23}$$

The flow is shown in Fig. 52e.

FRITZ JOHN'S method. FRITZ JOHN (1953) has devised a method for constructing exact irrotational two-dimensional free-surface flows which may be time-dependent. Let $F(z, t) = \Phi + i\Psi$ denote the complex velocity potential. The flow of particles on the free surface, $y = \eta(x, t)$, will also be described in a Lagrangian system:

$$z = e(\alpha, t), \tag{34.24}$$

where α is a real number associated with a particular particle. Then

$$\frac{dz}{dt} = \frac{\partial e}{\partial t} = F'\big(x + i\eta(x, t), t\big), \tag{34.25}$$

where F' denotes the partial derivative with respect to z. The equations of motion (2.7), reduced to two dimensions and to motion along the free surface, give

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial \alpha} + \left(g + \frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial \alpha} = -\frac{1}{\varrho}\frac{\partial p}{\partial \alpha}. \tag{34.26}$$

Since $p = \text{const}$ on the surface, $\partial p/\partial \alpha = 0$ and (34.26) states that $\partial^2 z/\partial t^2 + ig$ is perpendicular to $\partial z/\partial \alpha$, or that

$$e_{tt} + ig = i r(\alpha, t)\, e_\alpha, \tag{34.27}$$

where $r(\alpha, t)$ is a real function. Thus $e(\alpha, t)$ must satisfy the parabolic partial differential equation (34.27).

If $e(\alpha, t)$ is a solution of (34.27) for some $r(\alpha, t)$ which is real for real α and if e and e_t are analytic functions of α and real for real α, then one may construct the velocity potential $F(z, t)$ for a free-boundary flow as follows. Actually, we shall construct F as a function of α and t, i.e. we shall construct a function G related to F by $G(\alpha, t) = F\big(e(\alpha, t), t\big)$. For real α it follows from (34.25) that

$$G_\alpha = F'\frac{\partial z}{\partial \alpha} = \overline{e_t(\alpha, t)}\, e_\alpha(\alpha, t) = \overline{e_t(\overline{\alpha}, t)}\, e_\alpha(\alpha, t). \tag{34.28}$$

One may now use the last expression in (34.28) to extend analytically G_α, and hence G from real to complex α's. By inverting $z = e(\alpha, t)$, one may now construct $F(z, t)$ [invertibility follows from (2.4) which implies $e_\alpha \overline{e}_\alpha = 1$].

It is possible to prescribe $\eta(x, t)$ and then construct the associated function $r(\alpha, t)$. For it follows from (34.26) with $y = \eta\big(x(\alpha, t), t\big)$ that

$$\frac{\partial^2 x}{\partial t^2} + \eta_x\left[\eta_x\frac{\partial^2 x}{\partial t^2} + \eta_{xx}\left(\frac{\partial x}{\partial t}\right)^2 + 2\eta_{xt}\frac{\partial x}{\partial t} + \eta_{tt} + g\right] = 0. \tag{34.29}$$

Any set of solutions $x(\alpha, t)$ depending upon a parameter α yields a function $e(\alpha, t)$ defined by

$$e(\alpha, t) = x(\alpha, t) + i\,\eta\big(x(\alpha, t), t\big). \tag{34.30}$$

The function $r(\alpha, t)$ for real α is given by

$$r(\alpha, t) = \frac{e_{tt} + i\,g}{i\,e_\alpha} = -\frac{x_{tt}}{\eta_x\, x_\alpha}, \tag{34.31}$$

where (34.29) has been used in obtaining the last expression.

We shall suppose now that the motion is steady and make the following special choice of Lagrangian parameter α. Select some fixed point z_0 of the surface $y = \eta(x)$ and for any particle on the surface let $-\alpha$ be the time at which the particle was at z_0. Since the motion is steady, all particles take the same amount of time to travel from z_0 to any given point z and hence

$$e(\alpha, t) = e(0, t + \alpha) \equiv e(t + \alpha). \tag{34.32}$$

It then follows from (34.27) that also

$$r(\alpha, t) = r(\alpha + t). \tag{34.33}$$

Hence (34.22) becomes an ordinary differential equation in a single variable, say $\tau = t + \alpha$:

$$e''(\tau) - i\,r(\tau)\,e'(\tau) + i\,g = 0. \tag{34.34}$$

It follows next from (34.28) that $G(\alpha, t) = G(\alpha + t)$ and thus, if $e(\tau)$ is an analytic solution of (34.34), real for real τ,

$$G'(\tau) = \overline{e'(\overline{\tau})}\, e'(\tau). \tag{34.35}$$

In this case each choice of a function $r(\tau)$, real for real τ, results in a function $e(\tau)$ as a solution of (34.34), and then in a function $G(\tau)$ obtained by a quadrature of (34.35). One may invert $z = e(\alpha + t)$ and find F as a function of z as in the last paragraph or else regard

$$z = e(\tau), \qquad F = G(\tau) \tag{34.36}$$

as parametric equations with complex parameter τ.

Several examples are considered by John, two of which are time-dependent. A simple and interesting steady flow is obtained by taking $r(\tau) = \nu$, a constant, in (34.34). Then (34.34) and (34.35), after setting the constants of integration equal to zero, yield

$$z = \frac{g}{\nu}\,\tau + A\,e^{i\nu\tau}, \qquad F = \left(\frac{g^2}{\nu^2} + \nu^2 A^2\right)\tau - 2\,\frac{g}{\nu}\,A\cos\nu\tau. \tag{34.38}$$

The free surface, obtained by taking τ real in the first formula, is a trochoidal curve without self-intersections if $A < g/\nu^2$; the wavelength is $\lambda = 2\pi g/\nu^2$ and the amplitude is A. If $A < g/\nu^2$, then $|dF/dz| > 0$ and $A/\lambda < 1/2\pi$. However, dF/dz can become infinite if $dz/d\tau = 0$. Such points occur at

$$z = \left(n + \frac{1}{4}\right)\lambda + i\,\frac{\lambda}{2\pi}\left(1 - \log\frac{\lambda}{2\pi A}\right). \tag{34.39}$$

In order to avoid having them within the fluid, the bottom must be chosen as a streamline which passes above or through these points. Fig. 53, taken from John's paper, shows several profiles and the associated streamlines through the branch points (34.39) computed for various values of the constant A when

$\lambda = 2\pi$ (this is equivalent to graphing $2\pi z/\lambda$ for various values of $2\pi A/\lambda$). The surface profile and bottom come closer together as $A \to 1$ and draw further apart as $A \to 0$. For $A = 0.9$ they are so close that they cannot be conveniently separated in the figure; in such cases one may, of course have reservations about the applicability of the perfect-fluid model.

The surface profile in this example is exactly the same as in the Gerstner wave treated in the next section. However, the Gerstner wave is defined for infinite depth and is not irrotational. The flow described above may also be obtained by SAUTREAUX's method. VITOUSEK (1954) has studied it by this procedure.

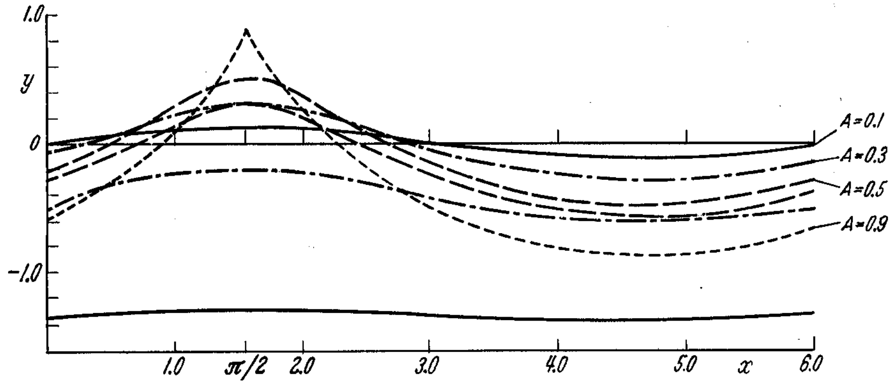


Fig. 53.

Methods of VILLAT and PONCIN. At the end of Sect. 32γ brief mention was made of a pair of integral equations derived by VILLAT (1915) for the determination of flows over some given bottom profile and with the top profile also given upstream of some point. The method seems to be chiefly useful as an inverse method in which the free surface is given and the other profiles sought. VILLAT has worked out one case, but not in complete detail, where the top cover is missing and the bottom has a declivity.

PONCIN (1932) has further elaborated VILLAT's method in the direction of starting with the fixed profile and finding the free profile. Actually, he does not really achieve this. Instead, he is able to construct a flow for a fixed profile of the same general behavior as the given one, but not identical with it. The method is applied to a number of interesting special cases, including flow over wavy bottoms and over bottoms with a declivity. The solutions are generally for large values of the velocity. The method and results do not lend themselves to a brief summary.

β) Gerstner's wave. GERSTNER'S wave (1802) is apparently the first flow to have been discovered which satisfies exactly the condition of constant pressure on the surface, and is, in fact, one of the earliest papers on water-wave theory. It was subsequently rediscovered by RANKINE (1863). As will be shown below, the motion is not irrotational. This fact itself would not be enough to rule it out as a mathematical model for real periodic waves. However, the direction of the vorticity is such that it is difficult to conceive of a scheme whereby such a wave could be generated in nature.

The motion is most easily described in Lagrangian coordinates. Each particle is associated with a pair of parameters (a, b), $b \leqq 0$. However, (a, b) does not represent a particular position of the particle at some time t_0, but instead a mean position. Hence, instead of (2.3) and (2.4) we need require instead only that the determinant D of those formulas be independent of t. The motion is described by the equations

$$x = a + A\,e^{mb}\sin(ma + \sigma t), \qquad y = b - A\,e^{mb}\cos(ma + \sigma t). \tag{34.40}$$

If $b=0$ is taken as the free surface, the motion evidently represents a wave moving to the left with velocity $c=\sigma/m$, while the particles themselves describe in a counter-clockwise direction circular paths about the points (a, b) associated with the particles. The surface $b=0$ is a trochoid and, in fact, each curve $b=$ const<0 is also a trochoid. In order that there shall be no self-intersections, one must have

$$A \leqq \frac{1}{m}. \tag{34.41}$$

In order to verify that the motion is kinematically possible, it is necessary to show, as noted above, only that the Jacobian $\partial(x, y)/\partial(a, b)$ is independent of t. An easy computation shows

$$\frac{\partial(x, y)}{\partial(a, b)} = 1 - m^2 A^2 e^{-2mb}, \tag{34.42}$$

so that the continuity condition is satisfied. Next one must show that the pressure is constant along the free surface. We shall, in fact, show more, namely, that it is constant along any line $b=$ const<0, provided $\sigma^2=gm$. To see this, introduce the Eq. (34.40) into the first of Eqs. (2.7). A straightforward computation yields

$$A(g\,m - \sigma^2)\,e^{mb}\sin(ma + \sigma t) = -\frac{1}{\varrho}\frac{\partial p}{\partial a}. \tag{34.43}$$

If the pressure is constant along the surface, then $\partial p/\partial a=0$. This can only hold if

$$\sigma^2 = g\,m. \tag{34.44}$$

However, if $\sigma^2=gm$, then $\partial p/\partial a=0$ for all b, so that each curve $b=$ const is an isobaric curve. Although we shall verify this fact directly, it now follows immediately from Burnside's theorem in subsection 32β that the motion cannot be irrotational. A direct computation of the vorticity vector is facilitated by noting that

$$\left.\begin{aligned} u &= \frac{\partial x}{\partial t} = A\,\sigma\,e^{mb}\cos(ma + \sigma t) = -\sigma(y - b), \\ v &= \frac{\partial y}{\partial t} = A\,\sigma\,e^{mb}\sin(ma + \sigma t) = \sigma(x - a). \end{aligned}\right\} \tag{34.45}$$

Hence

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \sigma\left(2 - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right). \tag{34.46}$$

The right-hand side of (34.46) may be computed from (34.40) by application of the rules of inversion for partial derivatives. The final result is

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{2\sigma\,m^2 A^2 e^{2mb}}{1 - m^2 A^2 e^{2mb}}; \tag{34.47}$$

the negative sign indicates that the vorticity is directed oppositely to the orbital motion of the particles. The relatively strong vorticity of Gerstner waves when mA is not quite small has been pointed out by Truesdell (1953), who measures it with a dimensionless vorticity number (see Sect. 27 of Serrin's article in Vol. VIII/1).

We shall omit a discussion of the construction of the curves $b=$const, streamlines in a coordinate system moving with the waves; it may be found in LAMB (1932, § 251), MILNE-THOMSON (1956, § 14.81) and in KOCHIN, KIBEL and ROZE (1948, Chap. 8, § 16) together with reproductions of GERSTNER'S original curves. It is, however, of interest to note that there is, according to (34.41), a "highest" wave of ratio $2A/\lambda = 1/\pi = 0.318$, a figure which may be compared with the value 0.142 for MICHELL'S wave. The highest Gerstner wave has a cusp at the crests, a further indication that the motion cannot be irrotational.

The pressure distribution may be found by substituting (34.40) in the second equation of (2.7), using (34.44), and integrating. The result is

$$p = p_0 - \varrho g b - \tfrac{1}{2}\varrho\sigma^2 A(1-e^{2mb}). \tag{34.48}$$

A computation of the potential and kinetic energies over a wave length yields

$$T = V = \frac{1}{4}\lambda g \varrho A^2\left[1 - \frac{2\pi^2}{\lambda^2}R^2\right]. \tag{34.49}$$

Finally, we note that nowhere in the preceding analysis have we made use of homogeneity of the fluid, i.e. the Gerstner wave also represents an exact solution for an arbitrary heterogeneous fluid (with ϱ constant along streamlines). Moreover, DUBREIL-JACOTIN (1935) has shown that the Gerstner wave is unique in this respect.

see errata

GERSTNER'S wave is defined only for infinite depth. One may ask if a similar wave exists for finite depth. "Similar", in this context, will be taken to mean a periodic wave which satisfies exactly the constant-pressure condition on the free surface and for which the particle orbits are closed. DUBREIL-JACOTIN (1934) has proved the existence of such a wave and showed that it is unique when the period is fixed. However, this motion cannot be given explicitly except in the case of infinite depth. Methods of approximate computation of the wave have been given by KRAVTCHENKO and DAUBERT (1957) and GOUYON (1958).

γ) Pseudo-exact solutions. Although the solutions of this section are not really solutions satisfying the exact boundary conditions formulated earlier, they are exact solutions to a closely related problem, also with a nonlinear boundary condition. Their interest derives from the fact that it is possible to encompass within one explicit formula waves of all amplitudes up to a highest wave analogous to MICHELL'S wave. The procedure also allows explicit construction of solitary and cnoidal waves. It is possibly a misnomer to call these solutions pseudo-exact, for one may also interpret them as the first term in a certain series solution of the correctly posed problem. In this sense they are analogous to HAVELOCK'S approximation procedure described in subsection 33β. The work to be described has appeared in a series of papers by DAVIES (1951, 1952), PACKHAM (1952) and GOODY and DAVIES (1957).

The motion will be described in terms of the variables introduced in (32.85), $\omega = \vartheta + i\tau$. The alteration in the boundary condition consists in replacing (32.89) by

$$\frac{\partial\vartheta}{\partial\psi} = l\frac{g}{c^3}e^{-3\tau}\sin 3\vartheta \quad \text{for } \psi = 0, \tag{34.50}$$

where l is some fixed constant chosen so that $l\sin 3\vartheta$ is a good approximation to $\sin\vartheta$. If one wishes to consider (34.50) as the first term in a series approximation to (32.89), one may expand $\sin\vartheta$ in a series in $\sin 3\vartheta$ and express (32.89) as

$$\frac{\partial\vartheta}{\partial\psi} = \frac{g}{c^3}e^{3\tau}\left[\frac{1}{3}\sin 3\vartheta + \frac{4}{81}\sin^3 3\vartheta + \cdots\right]. \tag{34.51}$$

In this case (34.50) with $l=\frac{1}{3}$ represents the approximation obtained by keeping only the first term of (34.51). However, we shall not pursue the approximation procedure and refer to DAVIES (1951) for further information. It will be convenient to reformulate the boundary condition (34.50) as follows:

$$\operatorname{Im}\left\{\frac{d\omega}{df}+l\frac{g}{c^3}\,\mathrm{e}^{i3\omega}\right\}=0 \quad \text{for } \psi=0, \tag{34.52}$$

or, after introducing the new variable $\chi(f)=\mathrm{e}^{-i3\omega}=\omega^3/c^3$, as

$$\operatorname{Im}\left\{\frac{1}{\chi}\left(i\frac{d\chi}{df}+3l\frac{g}{c^3}\right)\right\}=0 \quad \text{for } \psi=0. \tag{34.53}$$

In order to proceed further, we must further specify the nature of the wave motion. Let us suppose the motion to be periodic with wavelength λ and take the fluid to be infinitely deep. We again introduce the ζ-plane of (32.91) and take coordinates as in Fig. 50. The expression in curly brackets in (34.53) is a regular analytic function of ζ inside the unit disc of the ζ-plane with vanishing imaginary part on the boundary, hence is a constant. Since, for $\zeta=0$ (i.e. as $\psi\to-\infty$), $\chi=1$ and $d\chi/df=0$, the constant must be $3lg/c^3$. Thus χ must satisfy the differential equation

$$i\frac{d\chi}{df}-3l\frac{g}{c^3}\chi=-3l\frac{g}{c^3}. \tag{34.54}$$

The solution is easily found to be

$$\chi=1+A\,\mathrm{e}^{-i3lgf/c^3}. \tag{34.55}$$

Referring to Fig. 50, we see that, if $f=0$, $\chi=q_0^3/c^3$, where q_0 is the absolute velocity at a crest. Hence

$$A=\frac{q_0^3}{c^3}-1. \tag{34.56}$$

Since ϑ must also vanish at $\varphi=\pm\frac{1}{2}c\lambda$, i.e. the left-hand side of (34.55) must be real, we must also have $(3lg/c^3)\,\frac{1}{2}c\lambda=\pi$, or

$$c^2=3lg\,\lambda/2\pi. \tag{34.57}$$

Note that if $l=\frac{1}{3}$, the relation between c^2 and λ is the same as in the infinitesimal-wave theory. The solution (34.55) may now be put into the following form:

$$\chi=\frac{w^3}{c^3}=1-\left(1-\frac{q_0^3}{c^3}\right)\mathrm{e}^{-i2\pi f/c\lambda}, \tag{34.58}$$

where $0\leqq q_0\leqq c$. If $q_0=c$, then $w=c$ and the flow is uniform. If $q_0=0$, then

$$w^3=c^3\left[1-\mathrm{e}^{-i2\pi f/c\lambda}\right], \tag{34.59}$$

and near $f=0$, $\pm c\lambda$, $\pm 2c\lambda$, ... there is a corner in the wave profile with the two tangents making the same angle 120° as in STOKES' theorem [near $f=0$, (33.5) gives $w^3=i\,\frac{3}{2}gf$, (34.58) gives $w^3=i\,3lgf$]. Hence this wave corresponds to MICHELL's highest periodic wave. The ratio of amplitude to length of this wave may be computed from the following expression for the trough:

$$\frac{1}{2}\lambda-i\,a=\frac{1}{c}\int\limits_0^{\frac{1}{2}c\lambda}\left[1-\mathrm{e}^{-i2\pi\varphi/c\lambda}\right]^{-\frac{1}{3}}d\varphi.$$

By expanding in a series and integrating term by term, one finds

$$\frac{a}{\lambda} = 0.127. \tag{34.60}$$

We recall that the value for Michell's wave was 0.142.

If the depth of fluid is finite, one must add the additional boundary condition, $\mathrm{Im}\{\chi\}=0$ for $\psi=-Q$, as well as for $\varphi=0$ and $\pm\frac{1}{2}\lambda c$ if the motion is to be periodic. The determination of χ now becomes too difficult to carry through briefly. However, an explicit solution is still possible and has been worked out by Davies (1952) and further investigated by Goody and Davies (1957). Similarly, a "solitary wave" can be explicitly constructed which satisfies the boundary conditions $\mathrm{Im}\{\chi\}=0$ for $\psi=-Q$ and for $\varphi=0$, $0>\psi\geqq-Q$ and $\chi\to 1$ as $\varphi\to\pm\infty$. This has been done by Packham (1952). Either of these problems leads to the following differential-difference equation for $\chi(f)$:

$$\frac{1}{\chi(f+iQ)}\left[\chi'(f+iQ)-3l\frac{g}{c^3}i\right]+\frac{1}{\chi(f-iQ)}\left[\chi'(f-iQ)+3l\frac{g}{c^3}i\right]=0; \tag{34.61}$$

it may be established in a manner similar to that used in deriving (22.30) or (32.80).

δ) Pure capillary waves. The first investigation of periodic progressive capillary waves satisfying the exact boundary condition is apparently due to N. A. Slëzkin (1937). He formulated the boundary-value problem in the same manner as will be done below, reduced it to solution of a nonlinear integral equation analogous to Nekrasov's and proved existence and uniqueness of a solution. However, he apparently did not observe that an explicit solution was possible for infinite depth of fluid. This was discovered by Crapper (1957), following a different and, in fact, more elementary method.

We shall consider the motion as a steady one in which the fluid moves to the right with velocity c as $y\to-\infty$. The existence of a complex velocity potential $f(z)=\varphi+i\psi$ will be assumed and the free surface $y=\eta(x)$ will be taken to correspond to the streamline $\psi=0$ as usual. It will also be convenient to make use of the variable $\omega=\vartheta+i\tau$ introduced in (32.85). If p_0 is atmospheric pressure, then from Bernoulli's integral

$$p+\tfrac{1}{2}\varrho q^2=p_0+\tfrac{1}{2}\varrho c^2 \tag{34.62}$$

(we recall that gravity is being neglected). The dynamical condition at the free surface [see (3.8) and (3.9)] is

$$p-p_0=\frac{T}{R}=T\frac{\eta''}{[1+\eta'^2]^{\frac{3}{2}}}. \tag{34.63}$$

Before combining (34.62) and (34.63), we recall that the curvature of a streamline at any of its points is given by $d\vartheta/ds$ where s is arc length along the streamline. Hence, we may combine (34.62) and (34.63) to obtain the following boundary condition

$$\frac{1}{2}\varrho(c^2-q^2)=T\frac{d\vartheta}{ds}=T\frac{\partial\vartheta}{\partial\varphi}\frac{d\varphi}{ds}=Tq\frac{\partial\vartheta}{\partial\varphi}\quad\text{for}\quad\psi=0, \tag{34.64}$$

or

$$\frac{\varrho c}{2T}\left(\frac{c}{q}-\frac{q}{c}\right)=\frac{\partial\vartheta}{\partial\varphi}\quad\text{for}\quad\psi=0. \tag{34.65}$$

Since $q = c e^{\tau}$ and since $\partial\vartheta/\partial\varphi = \partial\tau/\partial\psi$ from the Cauchy-Riemann equations, (34.65) may be written

$$\frac{\partial\tau}{\partial\psi} = \frac{\varrho c}{2T}(e^{-\tau} - e^{\tau}) = -\frac{\varrho c}{T}\sinh\tau \quad \text{for} \quad \psi = 0. \tag{34.66}$$

The problem is now to find a function $\omega(f)$ analytic for $\psi \leqq 0$, such that $\omega \to 0$ as $\psi \to -\infty$ and such that the imaginary part τ satisfies (34.66). However, since the boundary condition (34.66) involves only τ, unlike its analogue (32.89) for pure gravity waves, it is possible to solve first for the harmonic function $\tau(\varphi, \psi)$ and then to find ϑ later.

We assume that a solution can be found which satisfies

$$\frac{\partial\tau}{\partial\psi} = -h(\psi)\sinh\tau, \qquad h(0) = \frac{\varrho c}{T}, \tag{34.67}$$

and proceed to verify the assumption. Integrating (34.67), we obtain

$$\log\tanh\tfrac{1}{2}\tau = -H(\psi) + G(\varphi), \tag{34.68}$$

where $H'(\psi) = h(\psi)$ and $G(\varphi)$ is an arbitrary function, or

$$\tau = \log\frac{e^{H} + e^{G}}{e^{H} - e^{G}} = \log\frac{X(\psi) + Y(\varphi)}{X(\psi) - Y(\varphi)}. \tag{34.69}$$

Since τ is a harmonic function of φ and ψ, LAPLACE'S equation must be satisfied by (34.69). This yields an equation to be satisfied by X and Y in which the two variables can be separated. We shall not repeat the detailed analysis, which is typical of that occurring in separation-of-variables problems. The final result is that X and Y must satisfy

$$\left.\begin{aligned} X'^2 &= a_1 + a_2 X^2 + a_3 X^4, \\ Y'^2 &= -a_1 - a_2 Y^2 - a_3 Y^4, \end{aligned}\right\} \tag{34.70}$$

where a_1, a_2, a_3 are arbitrary constants. CRAPPER states that the full equations may be used to construct a solution for fluid of finite depth, but that it is sufficient to set $a_3 = 0$ for infinite depth (in view of SLËZKIN'S result, this is presumably also necessary). Since τ is real, we shall also take X and Y to be real. If one does set $a_3 = 0$ and assumes $a_1 < 0$, $a_2 > 0$, the following give real solutions of (34.70):

$$X(\psi) = \sqrt{\frac{-a_1}{a_2}}\cosh\left(\sqrt{a_2}\,\psi + E\right), \quad Y(\varphi) = \sqrt{\frac{-a_1}{a_2}}\cos\left(\sqrt{a_2}\,\varphi + F\right), \tag{34.71}$$

where E and F are real constants. A glance at (34.69) shows that τ is independent of the choice of a_1. It will be convenient to let $a_2 = m^2/c^2$, where $m > 0$. One may determine E from (34.67), for

$$\frac{\varrho c}{T} = H'(0) = \frac{d}{d\psi}\log X|_{\psi=0} = \frac{m}{c}\tanh E$$

or

$$e^{2E} = \frac{mT/\varrho c^2 + 1}{mT/\varrho c^2 - 1} \equiv B^{-2}. \tag{34.72}$$

Since E is to be real, we must evidently have

$$\frac{mT}{\varrho c^2} \geqq 1. \tag{34.73}$$

Since F adds only a real constant to φ we may select it at our convenience; we take $F=0$. Substitution of (34.71) into (34.69) gives

$$\left.\begin{aligned}\tau &= \log\frac{\cosh(m\psi/c+E)+\cos(m\varphi/c)}{\cosh(m\psi/c+E)-\cos(m\varphi/c)}\\ &= \log\frac{\cos(i\,m\psi/c+i\,E)+\cos(m\varphi/c)}{\cos(i\,m\psi/c+i\,E)-\cos(m\varphi/c)}\\ &= \log\left[\cot\frac{1}{2}(m f/c+i\,E)\cot\frac{1}{2}(m\bar{f}/c-i\,E)\right].\end{aligned}\right\} \tag{34.74}$$

The analytic function ω, which has τ as imaginary part and which approaches zero as $\psi\to-\infty$, is given by

$$\omega = i\log\left[-\cot^2\tfrac{1}{2}(m f/c+i\,E)\right]. \tag{34.75}$$

We then have

$$\frac{df}{dz} = c\,e^{-i\omega} = -c\cot^2\frac{1}{2}(m f/c+i\,E) = c\coth^2\frac{1}{2}(i\,m f/c-E). \tag{34.76}$$

From this one may solve for z in terms of f:

$$\left.\begin{aligned}c\,z &= f-\frac{2c}{m}\tan\frac{1}{2}\left(\frac{m f}{c}+i\,E\right)+\text{const}\\ &= f-i\frac{4c}{m}\frac{1}{1+e^{(i m f/c-E)}}+\text{const}\\ &= f-i\frac{4c}{m}\frac{1}{1+B\,e^{i m f/c}}+i\frac{4c}{m},\end{aligned}\right\} \tag{34.77}$$

where the constant has been chosen so as to make cz reduce to f when $B=0$. It is evident that

$$z\left(f+\frac{2\pi c}{m}\right) = z(f)+\frac{2\pi}{m},$$

so that the streamlines are periodic in the x-direction with wavelength $\lambda=2\pi/m$.

The surface streamline is obtained by setting $\psi=0$. After separation of real and imaginary parts in (34.77) the equation for the surface becomes:

$$\left.\begin{aligned}x &= \frac{\varphi}{c}-\frac{4}{m}\frac{B\sin m\varphi/c}{1+B^2+2B\cos m\varphi/c},\\ y &= \frac{4}{m}-\frac{4}{m}\frac{1+B\cos m\varphi/c}{1+B^2+2B\cos m\varphi/c},\end{aligned}\right\} \tag{34.78}$$

with φ serving as a parameter. There is a crest when $\varphi=0$ and a trough when $\varphi=\pi c/m$. The difference in the values of y yields the following expression for the ratio of total amplitude to wavelength:

$$\frac{a}{\lambda} = \frac{4}{\pi}\frac{B}{1-B^2}. \tag{34.79}$$

Eq. (34.72) then provides a relation between A/λ and $mT/\varrho c^2$, which we may write, for example, as

$$\left.\begin{aligned}c &= \sqrt{\frac{Tm}{\varrho}}\left(1+\frac{1}{16}a^2m^2\right)^{-\frac{1}{4}}\\ &= \sqrt{\frac{Tm}{\varrho}}\left(1-\frac{1}{64}a^2m^2+\cdots\right).\end{aligned}\right\} \tag{34.80}$$

If this formula is compared with (27.29), it should be kept in mind that a is here the total amplitude and that in (27.29) A is a length associated with the half amplitude. The formulas are consistent.

As a/λ increases, the surface profile becomes steeper and steeper near the troughs until the two sides finally touch. This occurs for $a/\lambda = 0.730$. A wave of these proportions may be considered as a "highest" capillary wave, an analogue of MICHELL's wave, although the nature of the limitation is different. Fig. 54, reproduced from CRAPPER's paper, shows the profile of this wave together with other streamlines. It is a consequence of the form of the dependence in (34.77) that the other streamlines in Fig. 54 may also serve as surface profiles for different values of a/λ, i.e. for different values of B. It is not surprising, of course, that the profiles are similar to the middle three of Fig. 35.

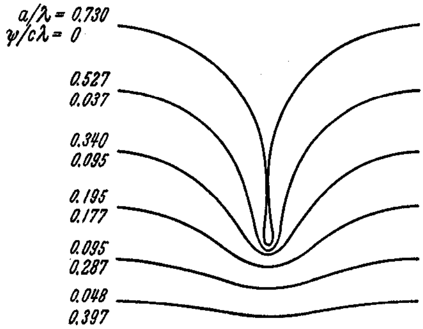


Fig. 54.

35. Existence theorems. In the various applications of the approximate theories of Chapt. D and E it is tacitly assumed that there is an exact solution which is being approximated. Without knowledge of conditions for existence and uniqueness of a solution to a particular problem, the status of an approximate solution is somewhat anomalous and one must rely upon comparison with experimental results for conviction concerning the correctness of the solution. However, such comparison is not a satisfactory criterion, for in the original formulation of a problem one will usually have already made a decision about the mathematical model of a fluid which will be used. Thus, if one has assumed a perfect fluid (as we usually have) and then made a further mathematical approximation in solving the problem at hand, the validity of this approximate solution must first be established before comparison of the predicted results with experimental measurements can be used as a criterion of the applicability of the fluid model. Without this additional knowledge, the comparison of approximate solutions with experimental results must be considered in some sense to be second best, even though valuable evidence may be provided by good agreement in a wide variety of situations.

Unfortunately, existence and uniqueness proofs in exact water-wave theory have generally been difficult to establish, and have usually been obtained for only rather restricted, although physically important, situations. Many of them are very recent and some rely upon methods of functional or topological analysis which cannot be briefly expounded. Although some proofs are so constructed that approximation methods are inherent in them, others are only able to assert the existence of a solution with no indication of how to obtain it approximately. Proofs are still lacking for many relatively simple but important problems, for example, MICHELL's highest wave and standing water waves.

No attempt will be made to give an exposition of the mathematical methods which have been used in establishing the various existing theorems. Instead only a discursive account will be given of the nature and limitations of the known theorems.

α) Irrotational waves—infinite depth. Proof of the existence of periodic waves of permanent type in infinitely deep water was first given by NEKRASOV (1921,

1922) in a journal of very restricted distribution. Shortly thereafter LEVI-CIVITA (1925) gave another proof along quite independent lines. Further proofs were given by NEUMANN (1929) and by LICHTENSTEIN (1931), these being more closely related to NEKRASOV's. A new treatment of LEVI-CIVITA's proof, due to LITTMAN and NIRENBERG, is contained in STOKER's *Water waves* (1957, § 12.2). Also, NEKRASOV (1951) has recently published his researches in a more accessible form.

NEKRASOV's method requires proving that there exists a solution $\vartheta(\gamma)$ to his nonlinear integral equation (32.104). His procedure, in brief, is to assume an expansion of $\vartheta(\gamma)$ in powers of the parameter $\mu' = \mu - 3 > 0$,

$$\vartheta(\gamma) = \mu' \vartheta_1 + \mu'^2 \vartheta_2 + \cdots, \tag{35.1}$$

then to derive equations relating each ϑ_k to ones of lower index, and finally to show that the series converges. $\mu = 3$ is chosen as a starting point because it is the first eigenvalue of the "linearized" equation (32.104), i.e. the one obtained by replacing the quotient containing $\sin\vartheta$ by simply $\mu\vartheta(\beta)$. This corresponds to the infinitesimal-wave theory. Proof of convergence requires that μ' be sufficiently small, and positive, but no estimate of radius of convergence is obtained. On the other hand, the method does allow computation of explicit approximate formulas for quantities of interest.

LEVI-CIVITA also works with the variable ω, treating it as a function of the variable ζ introduced in (32.90). Hence his formulation of the problem is essentially the same as NEKRASOV's, i.e. he is seeking a function $\omega(\zeta)$, regular in the disc $|\zeta| < 1$, vanishing at $\zeta = 0$ and satisfying (32.97) on $|\zeta| = 1$ and some further condition assuring that $|w/c - 1| < \beta < 1$. His procedure for finding such a function is to expand both ω and $k \equiv 1 - g\lambda/2\pi c^2$ in a power series in a parameter $\mu > 0$:

$$\omega = \sum_{n=1}^{\infty} \omega_n(\zeta)\, \mu^n, \qquad k = \sum_{n=1}^{\infty} k_n \mu^n, \tag{35.2}$$

where the functions $\omega_n(\zeta)$ and the constants k_n are to be determined by the boundary conditions. The first terms, $\omega_1 = -i\zeta$, $k_1 = 0$, correspond to infinitesimal waves, so that the parameter μ is essentially the amplitude/wavelength of this approximation. LEVI-CIVITA establishes the convergence of the series (35.2) for sufficiently small values of μ. No estimate of a radius of convergence is given, but HUNT (1953) has stated that an examination and refinement of LEVI-CIVITA's inequalities show that convergence is established for amplitude-wavelength ratios up to $\frac{1}{98}$. The procedure lends itself to explicit computation of higher-order computations, and, in fact, he carries them out through $n = 5$. LEVI-CIVITA further derives the interesting theorem that irrotational waves of permanent type must be symmetric about vertical lines through crest and trough. NEKRASOV assumed this at the outset.

NEUMANN and LICHTENSTEIN (the latter's approach is simpler) derive a coupled pair of nonlinear integral equations and put them into a form such that SCHMIDT's theory of nonlinear integral equations is applicable. Iterative methods of solution can be used to obtain approximate formulas.

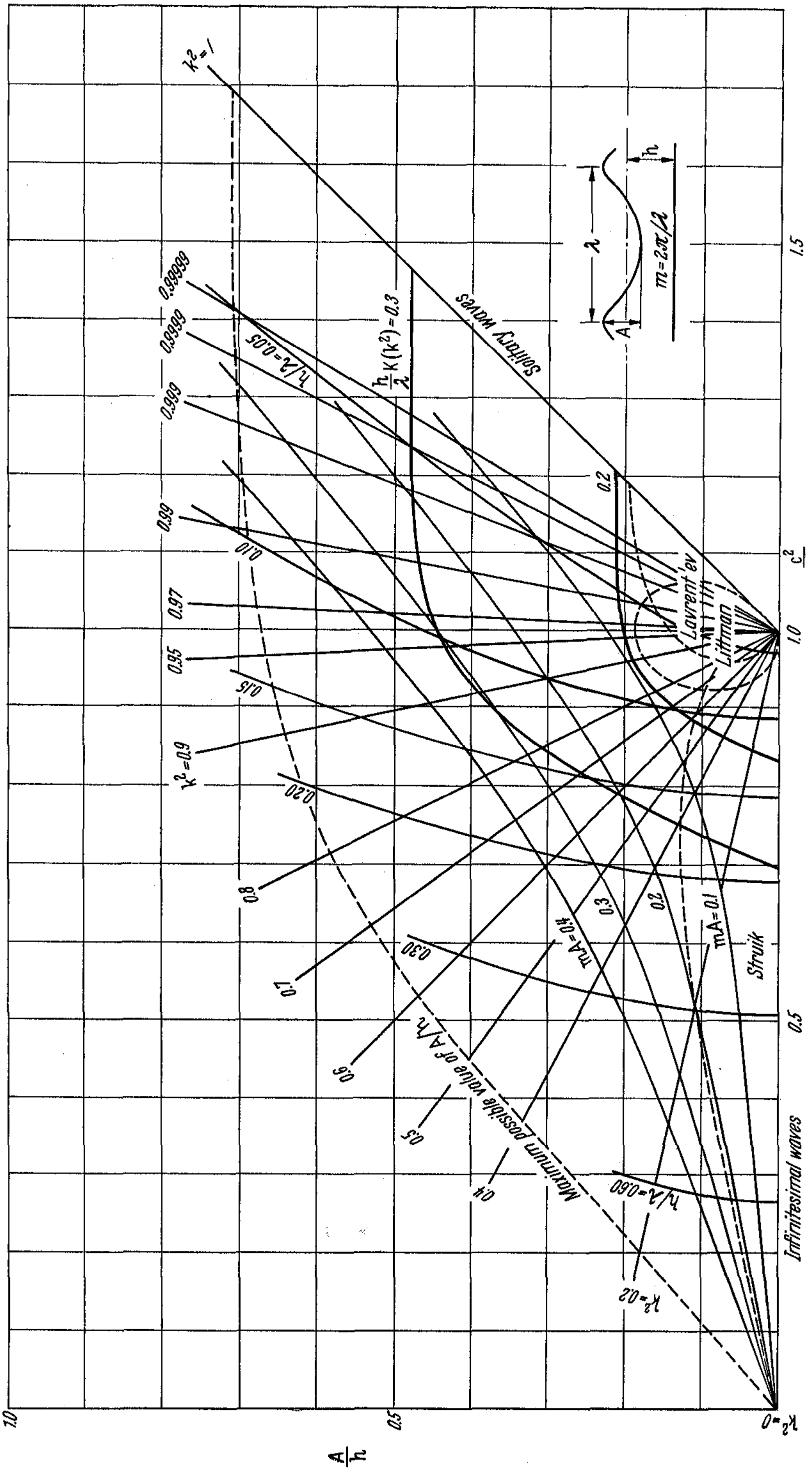
β) Irrotational waves—horizontal bottom. When the fluid is infinitely deep and the motion periodic, the only independent dimensionless parameter besides the amplitude-wavelength ratio is $c^2/g\lambda$. When the fluid is bounded below by a horizontal plane at mean depth h, then a new parameter, say c^2/gh, must enter into any solution. However, other independent sets of parameters may be used, and, in particular, different choices of a perturbation parameter have led earlier to different approximate solutions for finite depth. Thus, in Sects. 14β and 27 one finds the first and higher approximations for periodic waves of permanent type when A/λ is taken as a perturbation parameter, whereas in Sect. 31 one finds approximations to two further types of waves of permanent type, one of them periodic, corresponding to a different choice of parameter and a different method of approximation. In each of these cases there arises the question as to whether there exist waves of permanent type satisfying the exact boundary conditions for which these waves may be considered approximations. In each case the answer is affirmative.

Waves of small amplitude. The first proof of the existence of periodic progressive waves in fluid of finite depth is due to STRUIK (1926). His method of analysis is similar to LEVI-CIVITA'S for infinite depth. Existence of the desired wave is established for each value of $c^2/gh<1$ and for each sufficiently small value of A/λ, where the bound on A/λ depends upon c^2/gh. HUNT (1953) has recently corrected some errors in the proof which did not invalidate it but which resulted in incorrect approximate formulas.

NEKRASOV (1928, 1951) was also able to show that his integral equation for ϑ, as modified for finite depth [see (32.104) and (32.106)], had a solution, thus providing an independent proof. As was the case for infinite depth, NEKRASOV assumes that the waves are symmetric about verticals through trough and crest; STRUIK proves this. KRASNOSELSKII (1956) has recently applied topological methods of analysis to NEKRASOV'S equation and established not only existence of solutions for μ in the neighborhood of the eigenvalues of the linearized equation, but also their uniqueness and continuous dependence upon μ.

Solitary and cnoidal waves. LAVRENT'EV (1943, 1947) was the first one to establish the existence of cnoidal and solitary waves. Cnoidal waves are not mentioned by him by name, but, in fact, their existence for sufficiently large wavelength is established along with that of the solitary wave, the latter being obtained as a limiting case. The detailed exposition of the results (1947) is unfortunately both difficult of access and difficult to read, and relies upon earlier theorems of the author. Although the perturbation parameter appears at first glance to be taken as $\varepsilon^2=-1+gh^3/Q^2$, which for the solitary wave would be in contradiction with (32.52), the quantity h is not really mean depth but a related quantity which varies with the wavelength of the approximating periodic wave. FRIEDRICHS and HYERS (1954) by a completely different procedure have established the existence of the solitary wave. Their perturbation parameter is essentially $\varepsilon^2=1-gh^3/Q^2$ [actually it is $a^2=-\frac{1}{3}\log(gh^3/Q^2)$]. The point of departure is again the boundary condition (32.89) for the function ω. However, an integral equation is formulated, then altered by a change of variable $\hat{\varphi}=a\varphi$, $\hat{\psi}=\psi$. The different rates of stretching correspond to those of subsection 10β. (Something like this also occurs in LAVRENT'EV'S proof, but is disguised in his theorems on conformal mapping of narrow strip-like regions.) An iterative procedure is used to prove existence of a solution for sufficiently small values of ε^2.

LITTMAN (1957) has used a method somewhat similar to that of FRIEDRICHS and HYERS to establish the existence of cnoidal waves satisfying the exact boundary conditions. However, as a parameter he has used essentially h/λ, where h is the mean depth and λ the wavelength. It is demonstrated that solutions exist for values of c^2/gh which are both greater and less than 1. Fig. 55, modified slightly from one in LITTMAN'S paper, shows in a qualitative fashion the relation between the dimensionless parameters. The dotted lines enclose values of the parameters, again in a purely qualitative way, for which solutions have been demonstrated to exist. (Here k is the modulus of the elliptic functions



$k^2=1$
0.99999
0.9999
0.999
0.99
0.97
0.95
$k^2=0.9$
0.8
0.7
0.6
0.5
0.4
$k^2=0.2$
$h/\lambda=0.05$
0.10
0.15
0.20
0.30
$h/\lambda=0.60$
$\frac{h}{\lambda}K(k^2)=0.3$
Solitary waves
Laurent'ev
Littman
Struik
mA=0.4
0.3
0.2
mA=0.1
0.2
Maximum possible value of A/h
Infinitesimal waves
λ
A
h
$m=2\pi/\lambda$
1.0
0.5
0
$k^2=0$
1.5
1.0
0.5
$\frac{c^2}{gh}$
$\frac{A}{h}$

Fig. 55.

and K is the complete elliptic integral of the first kind. k serves as a parameter in certain approximate formulas.)

Fig. 55 was prepared by first computing the curves shown by means of both the cnoidal-wave theory and the theory of higher-order infinitesimal waves as developed in subsection 27α. [SKJELBREIA's tables (1959) facilitated the computation for the latter method.] Curves were then faired by eye in such a way as to pass smoothly from one set to the other. Hence, although they are claimed to be only qualitative, they have in fact a quantitative basis. An additional error has been introduced by taking the curves $k^2=$const as straight lines; they should, in fact, show some curvature as the radial distance from $c^2/gh=1$, $A/h=0$ increases. This additional complication of the computation did not seem necessary for the purpose at hand.

Although it is not strictly relevant to the material of the present section, it seems of interest to display the two sets of curves mentioned above, for they indicate in a rough way the ranges of validity of the two fundamental methods of approximation and show how they fit together. They are shown in Fig. 56. One expects the curves based on the infinitesimal-wave theory to be accurate near the horizontal axis, $A/h=0$, those based on cnoidal-wave theory to be accurate near $c^2/gh=1$, and the two to agree where these two regions overlap. The curves confirm this expected behavior. Computations based on the second-order cnoidal-wave theory of Eqs. (31.37) may be expected to produce better agreement over a wider range.

γ) Irrotational waves—other configurations. Flow over a wavy bottom. In connection with the study of inverse methods in subsection 34α an explicit example of a steady flow over a wave-shaped bottom was exhibited. However, there the surface profile was given and the bottom profile calculated. The direct problem, in which the bottom profile and other flow data are given, has also been considered by several persons. LAVRENT'EV (1943) announced theorems concerning this problem, but did not include them in his later (1947) exposition. GERBER (1955) has given a comprehensive treatment of the "supercritical" case and has announced further results for the "subcritical" case (1956). Let the bottom profile S be periodic and symmetric about vertical lines through the maxima and minima; let $\vartheta(s)$ be its intrinsic equation where s is arc length measured from a maximum and ϑ is the angle between the tangent and the x-direction. Let Q be the discharge rate for the fluid, and let q_0 be the velocity at a crest. In the first paper he considers flows in which the slope of the surface has the same sign as that of the bottom (we recall the two possible flows occurring in the linearized theory of subsection 20α). GERBER shows that there exists at least one solution of this type provided the following inequalities are satisfied in the interval between a maximum and the first minimum to the right:

$$\left.\begin{aligned} &\frac{gQ}{q_0^3}+\max|\vartheta|\leqq\pi-\varepsilon_1,\\ &-\frac{1}{2}\pi+\varepsilon_2\leqq\vartheta(s)\leqq 0,\end{aligned}\right\}\qquad(35.3)$$

where ε_1 and ε_2 are arbitrary small but positive quantities. If certain other inequalities, further limiting gQ/q_0^3, are satisfied, he is also able to prove uniqueness provided $\vartheta(s)\not\equiv 0$. In the second paper he announces that there exists at least one solution such that the profile has slope of opposite sign to that of the bottom if

$$\frac{gQ}{q_0^3}>(1+\varepsilon)\frac{\pi^3}{2}\qquad(35.4)$$

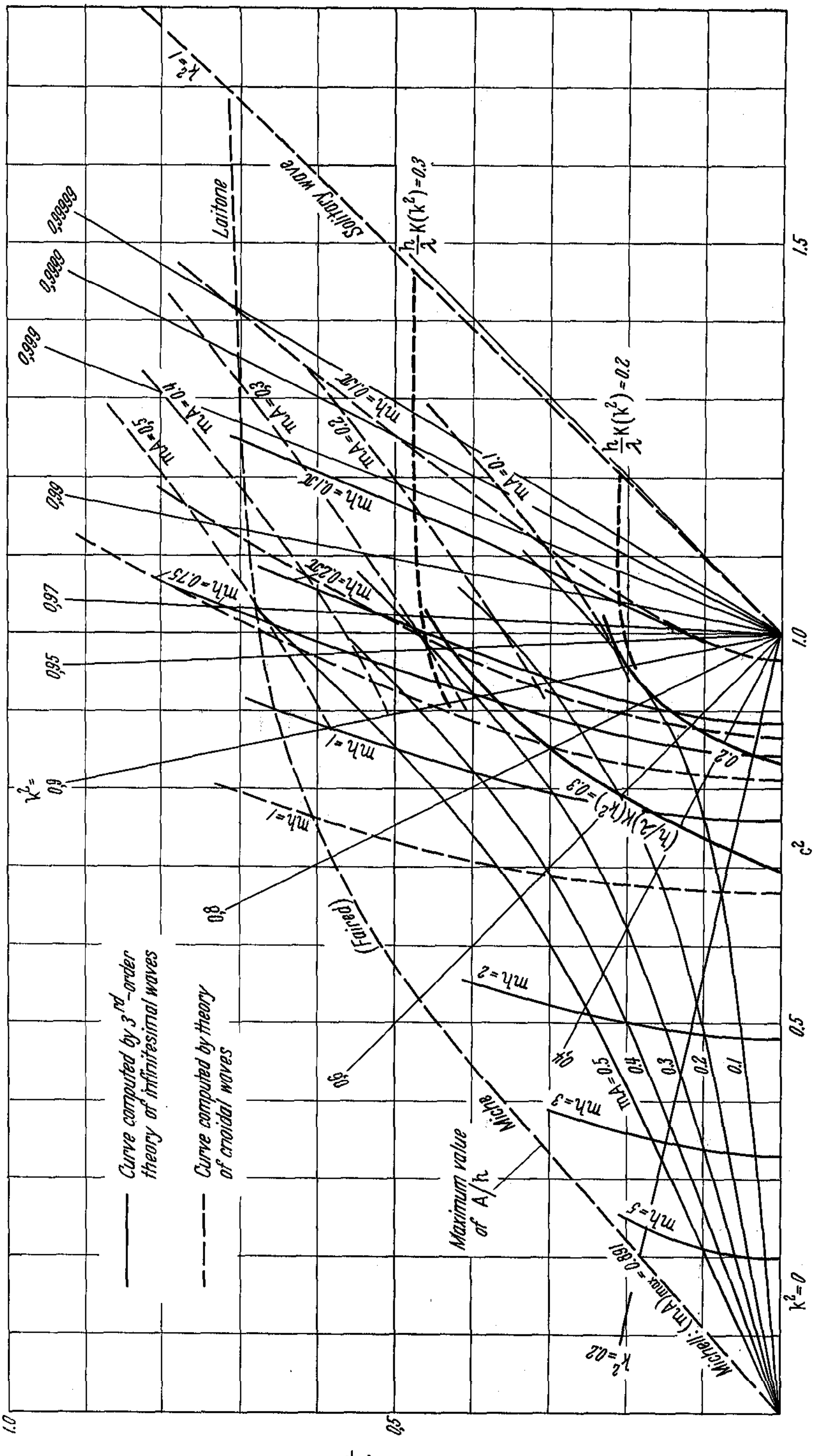

Solitary wave
Laitone
(Faired)
Curve computed by 3rd-order theory of infinitesimal waves
Curve computed by theory of cnoidal waves
Maximum value of A/h
$k^2=0$
$\frac{c^2}{gh}$
$\frac{A}{h}$
1.0
0.5
1.5

Fig. 56.

and $Q/L_0 q_0$ and $q_0 \Delta/Q$ are small enough; here L_0 is the arclength from a maximum to a minimum of S and Δ is the vertical distance. GERBER's methods are topological (Schauder-Leray theory) and do not yield effective methods of approximation.

MOISEEV (1957) has also considered this problem. By a modification of the method used to derive NEKRASOV's integral equation (32.104), he derives a pair of nonlinear integral equations to which the Lyapunov-Schmidt method is applicable. Let c be the average velocity defined by (7.5) for an allowable value of y (thus φ increases by $c\lambda$ over a wavelength), and let Q be the discharge rate. Then MOISEEV finds that there exists a sequence of velocities $c_1 > c_2 > \cdots > 0$ associated with the eigenvalues of a certain linear operator, such that, if $c \neq c_n$, there exists a unique flow provided the slope of the bottom is sufficiently small. Also, if $c > c_1$ or $c_{2n+1} < c < c_{2n}$ then the solution is such that the slopes of bottom and surface are of the same sign; if $c_{2n} < c < c_{2n+1}$, the slopes are of opposite sign.

see errata

Flow over a bottom with a declivity. Let the flow be from left to right and suppose the bottom profile to be asymptotic to horizontal lines as $x \to \pm\infty$, the one on the right being lower than that on the left. The discharge rate Q and velocity c at $x = -\infty$ should then be sufficient to determine the flow. The existence of a steady flow under these circumstances has been investigated by HAIMOVICI (1935) and GERBER (1955). The former derives a pair of nonlinear integral equations, similar to NEKRASOV's, relating ϑ and τ of (32.86). An iterative method is used to prove the existence of a solution. GERBER makes use again of the Schauder-Leray theory. The theorems established by each are very similar, but GERBER's is sharper. Let the bottom be given intrinsically by $\vartheta(s)$, measured from some fixed point. Then a solution exists if

$$\left.\begin{aligned} &\frac{gQ}{c^3} < 1, \qquad \max|\vartheta(s)| + \frac{gQ}{c^3} < 1, \\ &|\vartheta(s)| \leqq A\, e^{-\alpha|s|}, \qquad A, \alpha > 0. \end{aligned}\right\} \tag{35.5}$$

The last condition assures a rapid approach to the horizontal asymptotes. The case of subcritical flow does not appear to have been treated in the published literature.

Motion past a submerged vortex. TER-KRIKOROV (1958) has recently investigated steady flow past a submerged vortex of intensity Γ in a channel of depth h when the exact boundary conditions on the free surface are retained. If c is the velocity far upstream of the vortex, he proves existence and uniqueness of the flow provided that $c^2/gh > 1$ and Γ/ch is sufficiently small.

Interfacial waves. In subsection 14δ we considered the linearized theory of waves at an interface between two perfect fluids of different densities, bounded above and below by horizontal planes. The question naturally arises as to whether one can establish the existence of such waves when the exact boundary conditions at the interface are observed. KOCHIN (1927) extended the methods of LEVI-CIVITA and STRUIK to this problem and established the existence of (necessarily symmetric) interfacial waves of finite amplitude.

δ) Rotational waves. The explicit construction in subsection 34β of a periodic wave of permanent type which is rotational and the demonstrated existence of irrotational waves of this type which are of finite, if small, amplitude raises the question as to whether each of these waves is a special case of a more general type. This question has been discussed in a notable paper by DUBREIL-JACOTIN (1934) with results which include and generalize those of LEVI-CIVITA and STRUIK. We give a only a bare indication of the results.

Let us suppose that a coordinate system has been chosen so that we may treat the wave motion as a steady flow to the right. Although we do not assume the motion to be irrotational, there will still exist a stream function $\psi(x, y)$ by virtue of the continuity equation. The vorticity of the flow will be given by $-\Delta\psi$, and since by a classical theorem the vorticity is constant along a streamline, the following equation must be satisfied by ψ:

$$\Delta\psi = f(\psi), \tag{35.6}$$

where $f(\psi)$ is some unspecified function. The condition on the free surface $\psi = 0$ may still be derived from the special Bernoulli theorem [see Eq. (2.10″)]

$$g\,\eta(x) + \tfrac{1}{2}\left[\psi_x^2 + \psi_y^2\right] = \text{const}. \tag{35.7}$$

For irrotational waves the function $f \equiv 0$; for GERSTNER'S wave it is given by (34.47) after setting $b = \psi$, $\sigma = c m$. The question which DUBREIL-JACOTIN asked is whether a wave of finite amplitude exists for any distribution of vorticity $f(\psi)$. In order to encompass both of the known finite waves into her results, she limits f to functions of the following sort:

$$f(\psi) = -\mu\,\frac{Q}{h}\,m^2\,e^{2mh\psi/Q}\,F\left(e^{mh\psi/Q}\right), \qquad -Q \leqq \psi \leqq 0, \tag{35.8}$$

where Q is discharge rate, h the mean depth, and the function $F(\varrho)$ is bounded and satisfies a Hölder condition in ϱ; μ is a small parameter. If the depth is infinite, one must replace Q/h by c, the velocity at $\psi = -\infty$ (it is assumed that $\psi_y \to c$ as $\psi \to -\infty$).

DUBREIL-JACOTIN'S theorem is as follows. For any $m = 2\pi/\lambda$, h and $f(\psi)$ satisfying (35.8) there exists a $\delta > 0$ such that for $\mu < \delta$ there exists a unique corresponding progressive wave of permanent type with vorticity distribution $f(\psi)$. The waves are also shown to be symmetric about vertical lines through crest or trough. She also demonstrates that among this class of waves for finite depth there is a unique analogue of the Gerstner wave, in the sense that the trajectories of individual particles are all closed. This wave has recently been investigated by KRAVTCHENKO and DAUBERT (1957). The development of means of calculating rotational waves has been the subject of a recent investigation by GOUYON (1958).

ε) Waves in heterogeneous fluids—internal waves. It has been shown in subsection 32β that irrotational waves of permanent type are not possible in a heterogeneous fluid, but that GERSTNER'S rotational wave still provides a solution for infinite depth. DUBREIL-JACOTIN (1935) has shown that this is the only periodic wave of permanent type in infinitely deep fluid having this property. In a later paper (1937) she returned to this topic and made use of the methods developed by her for rotational waves to investigate the existence theory for the two problems described below. The first problem, a natural generalization of one investigated by KOCHIN and mentioned at the end of subsection 35γ, is the existence of periodic internal waves of permanent type in a heterogeneous fluid bounded both above and below by horizontal planes. In the second problem the upper surface is free.

The two problems may be formulated as follows. First we recall that in a steady flow of a heterogeneous fluid the density must be constant along streamlines. Hence, if ψ is the stream function, we may write $\varrho = \varrho(\psi)$. The equation analogous to (35.6) is now somewhat more complicated. It may be derived from (32.54) as follows. Apply the operators

$$\frac{\varrho'}{\varrho}\psi_y - \frac{\partial}{\partial y}, \qquad \frac{\varrho'}{\varrho}\psi_x - \frac{\partial}{\partial x}$$

to the two equations of (32.54), respectively, and subtract. This yields

$$\frac{\varrho'}{\varrho}\frac{\partial(E,\psi)}{\partial(x,y)} - \frac{\partial(\zeta,\psi)}{\partial(x,y)} = 0.$$

Since $\varrho = \varrho(\psi)$,

$$\frac{\varrho'}{\varrho}\frac{\partial(E,\psi)}{\partial(x,y)} = \frac{\partial(\varrho' E/\varrho, \psi)}{\partial(x,y)}$$

and hence

$$\frac{\partial(\varrho' E/\varrho - \zeta, \psi)}{\partial(x,y)} = 0$$

or, integrating and substituting $\zeta = -\Delta\psi$,

$$\Delta\psi + \frac{\varrho'}{\varrho}\left[g\,y + \frac{1}{2}(\psi_x^2 + \psi_y^2)\right] = f_1(\psi) \tag{35.9}$$

where $f_1(\psi)$ is an arbitrary function. This is the equation which ψ must satisfy. If $\psi = 0$ is the top streamline and $\psi = -Q$ the bottom streamline, then the boundary conditions are,

$$\psi = 0 \quad \text{for} \quad y = 0, \qquad \psi = -Q \quad \text{for} \quad y = -h \tag{35.10}$$

for the first problem, and

$$\left.\begin{aligned} \psi_x^2 + \psi_y^2 + 2g\,y = \text{const} \quad &\text{for} \quad \psi = 0, \\ \psi = -Q \quad &\text{for} \quad y = -h \end{aligned}\right\} \tag{35.11}$$

for the second problem [cf. (32.60)]. The function $\varrho(\psi)$ cannot be considered as an arbitrary given function in the same sense that $f_1(\psi)$ is arbitrary; it must be related to the density distribution when the fluid is at rest. DUBREIL-JACOTIN assumes that $\varrho(\psi)$ is the same as the density at the mean level of the streamline ψ when the fluid is at rest.

In order to obtain results analogous to those of subsection 35 δ, certain restrictions are placed upon the function $f_1(\psi)$ and the density distribution. Both problems are then reducible to integro-differential equations. In general there is no nontrivial solution. However, under certain conditions there are an infinite number of values of the parameter $\lambda g/2\pi c^2$ in the neighborhood of which there exist nontrivial symmetric waves of finite (but small) amplitude.

ζ) Waves with surface tension. It has already been mentioned in subsection 34 δ that SLËZKIN (1935b, 1937) had derived an integral equation for the motion of pure capillary waves and had proved both existence and uniqueness of solution under certain circumstances. The explicit solution for this problem derived by CRAPPER supersedes in a sense these earlier results.

SEKERZH-ZENKOVICH (1956) has formulated the exact boundary-value problem for combined gravity and capillary waves in terms of the function ω of (32.86) and announced that a proof of existence for sufficiently small amplitude-to-wavelength ratio can be carried out by LEVI-CIVITA's method for pure gravity waves.