where the last equation follows from (17.42) and where

$$P(\lambda) = \sum_{k=0}^{q-1} a_k \lambda^k. \tag{17.50}$$

From the assumptions originally made concerning $f(z)$ and from the method of selecting the $\{a_k\}$ it follows that $g(z)$ is regular everywhere in the wedge

$$-2\gamma \leqq \vartheta \leqq 0$$

except possibly at the origin, that

$$\mathrm{Im}\{g(z)\} = 0 \quad \text{for} \quad z = x > 0 \quad \text{and} \quad z = r\,\mathrm{e}^{-2i\gamma},$$

and finally, from the last of Eqs. (17.24), that

$$g(\beta z) = -g(z).$$

Since $f(z)$ is assumed bounded as $x \to \infty$, this is true also of $g(z)$. These various conditions imply that $g(z)$ must have the form

$$g(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{(2n+1)q}}, \qquad b_n \text{ real}. \tag{17.51}$$

We have thus shown that $f(z)$ satisfies the differential equation

$$P\left(\frac{d}{dz}\right)\left(\frac{d}{dz} + i\nu\right) f(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{(2n+1)q}}, \qquad b_n \text{ real}. \tag{17.52}$$

From the definition of $P(\lambda)$ it follows that

$$P(\lambda)(\lambda + i\nu) \equiv P(\beta\lambda)(-\beta\lambda + i\nu).$$

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Since the coefficients in $P(\lambda)$ are real, $\bar{\lambda}$ is a root or $P(\lambda) = 0$ if λ is a root. Furthermore, from the identity above also $\beta\lambda$ is a root providing $\beta\lambda \neq i\nu$. Since $\lambda = -i\nu$ is an obvious root of the left hand member, $-i\beta\nu$ is also a root and hence $-i\beta^2\nu$, $-i\beta^3\nu, \ldots$. Since $\beta^q = -1$, no new roots are added by going further than $-i\beta^{q-1}\nu$, and since $i\beta^{-k}\nu = -i\beta^{q-k}\nu$, a complete set of roots of $P(\lambda)(\lambda + i\nu)$ is

$$-i\nu, \quad -i\beta\nu, \quad -i\beta^2\nu, \ldots, \quad -i\beta^{q-1}\nu.$$

Thus the solution of the homogeneous equation can be expressed in the form

$$\sum_{k=0}^{q-1} A_k \exp(-i\nu\beta^k z). \tag{17.53}$$

This is, of course, exactly the form of KIRCHHOFF'S solution of (17.36). Since we have already determined the necessary form of the A_k in order to satisfy the boundary condition on the bottom, we need not pursue further the solution of the homogeneous equation.

The solution of the nonhomogeneous equation is straightforward. However, just as for the homogeneous equation, one must take care to satisfy the boundary condition on the bottom, i.e. $\mathrm{Im}\{\mathrm{e}^{-i\gamma} f'(r\,\mathrm{e}^{-i\gamma})\} = 0$. The detailed considerations may be found in the several cited papers; BRILLOUËT (1957) treats the matter thoroughly. If one considers (17.52) with the right-hand side replaced by only one of its summands, say $b_n z^{-(2n+1)q}$, then the complete solution can be put in the following form, as shown by BRILLOUËT:

$$f(z) = \sum_{k=0}^{q-1} A_k \exp(-i\nu\beta^k z)\left[c_n + \frac{1}{2}(-1)^{nq+q-1} \frac{b_n}{\sqrt{q}} \int_{\Gamma_k} \frac{e^t\,dt}{t^{(2n+1)q}}\right], \tag{17.54}$$

where c_n is an arbitrary real constant, B_0 of (17.37) has been set equal to 1, and where Γ_k indicates that the integral is to be carried out over each of the paths Γ_k^+ and Γ_k^- shown in Fig. 16. However, one may obtain a variety of other forms for the solution.

An asymptotic expression as $x\to\infty$ and for $y=0$ is given by

$$\left.\begin{aligned} f(x) &\sim \left[c_n + i b_n \frac{(-1)^{nq+q-1}\pi}{(2nq+q-1)!\sqrt{q}}\right] \exp\left(-i\nu x - i\pi\frac{q-1}{4}\right) \\ \text{or} \\ \varphi(x,0) &\sim c_n \cos\left(\nu x + \pi\frac{q-1}{4}\right) + b_n \frac{(-1)^{nq+q-1}\pi}{(2nq+q-1)!\sqrt{q}} \sin\left(\nu x + \pi\frac{q-1}{4}\right). \end{aligned}\right\} \quad (17.55)$$

In the neighborhood of $z=0$, $f(x)$ behaves like $\log z$ for $n=0$ and like z^{-2nq} for $n>0$.

It is not clear physically what type of singularity at $z=0$ most nearly describes the behavior of real waves. However, most writers have restricted their treatment to the weakest possible singularity, i.e., the logarithmic one.

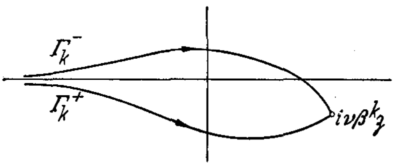

Fig. 16.

From the asymptotic expansion as $x\to\infty$ one sees that it is now possible to construct an incoming progressive wave by proper choice of the constants c_n and b_n. Thus, if we select

$$c_n = a\cos(\sigma t+\tau), \qquad b_n = -(-1)^{nq+q-1}\pi^{-1}(2nq+q-1)!\sqrt{q}\,a\sin(\sigma t+\tau),$$

then the resulting solution will behave like

$$a\cos(\nu x+\sigma t+\tau)$$

as $x\to\infty$ for $y=0$. In connection with (17.55) and the selection of b_n just made it is apparent that the formulas (17.54) and (17.55) will be more directly connected with parameters with a simple physical interpretation if we replace b_n by

$$d_n = b_n \frac{(-1)^{nq+q-1}\pi}{(2nq+q-1)\sqrt{q}}.$$

For $n=0$ companion singular solutions to the regular solutions (17.40) and (17.41) are not difficult to write out:

$\gamma=90°\ (q=1,\ n=0)$:

$$\varphi(x,y) = d_0 e^{\nu y}\sin\nu x - \frac{d_0}{\pi}\int_0^\infty e^{-\sigma x}\frac{\sigma\cos\sigma y+\nu\sin\sigma y}{\nu^2+\sigma^2}\,d\sigma; \qquad (17.56)$$

$\gamma=45°\ (q=2,\ n=0)$:

$$\left.\begin{aligned} \varphi(x,y) = \frac{d_0}{\pi} e^{\nu y}\Big[&\left(\frac{\pi}{2}+\mathrm{Si}(\nu x)\right)\sin\left(\nu x+\frac{\pi}{4}\right) + \\ &+\mathrm{Ci}(\nu x)\cos\left(\nu x+\frac{\pi}{4}\right)+\frac{1}{2}\sqrt{\nu}\,e^{-\nu x}\,\mathrm{Ei}(\nu x)\Big]. \end{aligned}\right\} \quad (17.57)$$

Further formulas for $\gamma=30°$ and $\gamma=6°$ may be found in Brillouët (1957, p. 93 ff.).

18. Three-dimensional progressive and standing waves in unbounded regions with fixed boundaries. The general remarks at the beginning of Sect. 17 apply here also. Although most of the solvable problems in the present category are such that they can be reduced to two-dimensional ones (however, see the end of Sect. 19β), the methods of complex-function theory are no longer applicable to the same extent. The division of topics is the same as in the last section, namely, diffraction of waves by obstacles and waves on beaches.

α) Diffraction of water waves. In a horizontally unbounded ocean of uniform depth h assume that an incoming wave is specified by

$$\Phi_I(x, y, z, t) = \frac{A g}{\sigma} \cosh m(y + h) \cos(m x + \sigma t + \alpha) \tag{18.1}$$

and that it is scattered by one or more obstacles in the water. We wish to find the velocity potential for the motion of the water in the form

$$\Phi(x, y, z, t) = \Phi_I + \Phi_S, \tag{18.2}$$

where Φ_S is the scattered wave and satisfies the radiation condition if the body is of bounded extent.

As usual, we may write Φ in the form

$$\Phi(x, y, z, t) = \operatorname{Re} \varphi(x, y, z)\, e^{-i\sigma t}, \qquad \varphi = \varphi_1 + i\varphi_2, \tag{18.3}$$

where φ must be a potential function satisfying

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$$\left.\begin{aligned} &\varphi_y(x, 0, z) - \nu\varphi(x, 0, z) = 0, \qquad \nu = \sigma^2/g,\\ &\varphi_n = \varphi_{In} + \varphi_{Sn} = 0 \quad \text{on the obstacles,}\\ &\lim_{R\to\infty} \sqrt{R}\left(\frac{\partial\varphi_S}{\partial R} - i\nu\varphi_S\right) = 0, \quad \sqrt{R}\,\varphi_S = O(1) \quad \text{as } R\to\infty. \end{aligned}\right\} \tag{18.4}$$

General obstructions. Consider a single submerged obstacle bounded by the surface S. We shall try to express the scattered wave $\Phi_S = \operatorname{Re}\varphi_S\, e^{-i\sigma t}$ by a distribution of sources over S. However, in order to satisfy the various boundary conditions, we take sources in the complex form (13.18) or, in the case of infinite depth, in the form (13.17''):

$$\varphi_S(x, y, z) = \frac{1}{4\pi} \iint_S \gamma(\xi, \eta, \zeta)\, G(x, y, z; \xi, \eta, \zeta)\, dS, \tag{18.5}$$

where we have written $G = G_1 + iG_2$ for the complex form of (13.18). The boundary condition on the body now becomes

$$\left.\begin{aligned} 0 = \frac{\partial\varphi_I}{\partial n} + \frac{\partial\varphi_S}{\partial n} &= \frac{\partial\varphi_I}{\partial n} - \frac{1}{2}\gamma(x, y, z) + \\ &+ \frac{1}{4\pi}\iint_S \gamma(\xi, \eta, \zeta) \frac{\partial}{\partial n} G(x, y, z; \xi, \eta, \zeta)\, dS \\ \text{or} \qquad \gamma(x, y, z) &= 2\frac{\partial\varphi_I}{\partial n} + \frac{1}{2\pi}\iint \gamma(\xi, \eta, \zeta) \frac{\partial G}{\partial n}\, dS. \end{aligned}\right\} \tag{18.6}$$

Since $\partial\varphi_I/\partial n$ is a known function, this is a Fredholm integral equation of the second kind for $\gamma(x, y, z)$. (We note in passing that if the motion of the surface S had been prescribed to be $\partial\varphi_I/\partial n$, then the same integral equation for γ would have been obtained.)

This equation has been considered by KOCHIN (1940) in the case of infinite depth, and he proves that a solution exists if $\nu = \sigma^2/g$ is large enough and the body is submerged. Iterative procedures for computing γ follow from the theory. HASKIND (1946) has extended the argument to finite depth.

John (1950) has treated both the uniqueness and existence problem in great detail and has shown that a unique solution exists for a body whose surface intersects the free surface perpendicularly and which can be represented as a single-valued function over the area enclosed in the intersection. His result holds for all values of m (or ν if the depth is infinite). He also reduces the existence problem to solution of an integral equation.

Vertical cylinders. When the obstacle or obstacles are vertical cylinders extending from above the free surface to the bottom, it is possible to reduce the problem to one in diffraction of sound waves for which many special solutions are known [see, e.g., Havelock (1940)]. In this case we may separate the y variable in the manner shown in Sect. 13α:

where

$$\left.\begin{aligned} \varphi(x,y,z) &= \varphi(x,z)\,Y(y) \\ Y(y) &= \cosh m(y+h)\,\varphi(x,z) \end{aligned}\right\} \tag{18.7}$$

and

$$\varphi_{xx}+\varphi_{zz}+m^2\varphi=0. \tag{18.8}$$

Here m must be the same as in (18.1) since the frequency is fixed by the incoming wave. $\varphi(x,z)$ must now satisfy (18.8) and the second two conditions of (18.4). This is exactly the same mathematical problem encountered in the diffraction of sound waves by a cylindrical body (in that case the air pressure replaces φ). Thus, any solutions known for sound diffraction by cylinders may be taken over immediately for water-wave diffraction. For example, if the obstacle is a vertical circular post of radius a, the velocity potential of the scattered wave is given by[1]

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$$\varphi_S(R,\vartheta,y) = \frac{-Ag}{\sigma}\cosh m(y+h)\,\Sigma(-i)^n\,\varepsilon_n\,e^{-i\gamma_n}\sin\gamma_n\cos\vartheta\,H_n^{(1)}(mR), \tag{18.9}$$

where

$$\tan\gamma_n = J_n'(ma)/Y_n'(ma)$$

and

$$\varepsilon_0=1,\qquad \varepsilon_n=2\quad\text{for}\quad n\geqq 1.$$

Various approximations for large and small values of ma are known. The maximum wave amplitude at any point is given by $\frac{\sigma}{g}|\varphi|$.

The diffraction of water waves by a vertical half-plane may also be treated by transferring known solutions due to Sommerfeld for sound and electromagnetic waves to the present context. This has been done by Haskind (1948) for normal incidence and by Penney and Price (1952a) for both normal and oblique incidence. Peters and Stoker (1954) [see also Stoker (1956) and (1957, pp. 109 to 133)] have also solved this problem by a new and rather easy method, following an investigation of boundary conditions which will ensure uniqueness. The solution has an obvious application in predicting the effect of breakwaters. Let the breakwater be the half-plane $z=0$, $x>0$ and the incoming wave be given by

$$\begin{aligned}\eta &= A\cos(mx\cos\alpha+mz\sin\alpha+\sigma t),\\ &= A\cos(mR\cos(\vartheta-\alpha)+\sigma t),\end{aligned}$$

where α is the angle between $-0x$ and the direction of propagation, measured clockwise. Then the solution given by Peters and Stoker is

$$\varphi(R,\vartheta,y)=\frac{Ag}{\sigma}\cosh m(y+h)\left[J_0(R)+2\sum_1^\infty e^{in\pi/4}J_{n/2}(R)\cos\frac{n\alpha}{2}\cos\frac{n\vartheta}{2}\right]. \tag{18.10}$$

[1] See P.M. Morse: Vibration and sound, 2nd ed., pp. 347ff., 449. New York 1948.

The result can also be expressed by means of integrals. In the case of normal incidence these reduce to Fresnel integrals, for which tables exist. Graphical representations of the behavior of the wave amplitudes may be found in PENNEY and PRICE (1952a).

PENNEY and PRICE also apply this analysis to an approximate treatment of diffraction by a breakwater of finite length and through a gap. The results are presumably applicable if the wavelength is small compared to the length of the breakwater or the gap.

Periodic solutions for horizontal cylindrical obstacles. In two physical situations the dependence upon z may be precipitated out, leaving a two-dimensional problem which in many cases can be solved by methods analogous to those used for the two-dimensional problems of Sect. 17.

Let the obstruction be an infinitely long horizontal cylinder parallel to Oz. This might be, for example, a semi-infinite dock or submerged plane barrier, say $y=-b$, $x<0$, a finite horizontal barrier, say $y=-b$, $|x|<a$, a vertical
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barrier, $x=0$, $-b<Y\leqq 0$, a beach, $y=-x\tan\gamma$, etc. Let an incoming plane wave at infinity propagate at an angle α to the x axis:

$$\eta_I(x, y, z, t) = A\cos[m(x\cos\alpha + z\sin\alpha) + \sigma t]. \tag{18.11}$$

Although one will not expect the velocity potential Φ to be periodic in x, it seems reasonable to assume that it will be periodic in z. In fact, we shall assume that

$$\Phi(x, y, z, t) = \varphi(x, y)\, e^{-i(mz\sin\alpha+\sigma t)}, \tag{18.12}$$

where $\varphi(x, y)$ must now satisfy, with $k=m\sin\alpha$,

$$\varphi_{xx}+\varphi_{yy}-k^2\varphi=0 \tag{18.13}$$

and the usual conditions on the free surface and rigid boundaries.

We should have come to the same conclusion if we had assumed an incoming wave at infinity of the form

$$\eta_I(x, y, z, t) = A\cos kz\cos(k_1 x + \sigma t), \qquad k^2+k_1^2=m^2, \tag{18.14}$$

a so-called short-crested wave (note that we assume $k^2<m^2$). That is, we shall now look for a solution in the form

$$\Phi(x, y, z, t) = \varphi(x, y)\cos kz\, e^{-i\sigma t} \tag{18.15}$$

satisfying Eq. (18.13) and the conditions on the free surface and rigid boundaries. Thus, a solution for one of these cases carries over easily to the other.

The problem is thus reduced to one almost identical with that of Sect. 17, with the exception that the two-dimensional Laplacian is replaced by (18.13). Many of the same methods may be carried over, e.g., the reduction method and the integral-equation method. HASKIND (1953) has considered some general aspects of the problem which will be outlined below, has derived the source solution of (18.13), and has treated the diffraction about a vertical barrier (an analogue of the problem treated in Sect. 17α) and a finite dock, all in infinitely deep water. MACCAMY (1957) has derived a source solution of (18.13) and treated the finite dock problem in water of finite depth. HEINS (1948, 1950, 1953) has given source solutions of (18.13) for finite depth and formulated and solved Wiener-Hopf integral equations for semi-infinite docks and submerged horizontal barriers. GREENE and HEINS (1953) treat the submerged barrier in water of infinite depth. The literature for beaches will be given in Sect. 18β.

Suppose the fluid infinitely deep and let a cross-section of the obstacle have contour C. We wish then to find a solution $\varphi(x, y) = \varphi_1 + i\varphi_2$ of (18.13) such that

$$\left.\begin{aligned} &\varphi_n = 0 \quad \text{on} \quad C, \\ &\varphi_y(x, 0) - \nu\,\varphi(x, 0) = 0 \quad \text{on the free surface}, \\ &\varphi \sim \frac{A\,g}{\sigma}\, e^{\nu y}\, e^{-i k_1 x} + \frac{B^+ g}{\sigma}\, e^{\nu y}\, e^{i k_1 x} \quad \text{as} \quad x \to +\infty, \\ &\varphi \sim \frac{A\,g}{\sigma}\, e^{\nu y}\, e^{-i k_1 x} + \frac{B^- g}{\sigma}\, e^{\nu y}\, e^{-i k_1 x} \quad \text{as} \quad x \to -\infty, \end{aligned}\right\} \tag{18.16}$$

where $k_1^2 < \nu^2$. HASKIND (1953) applies the reduction method in the following manner (we follow his presentation closely). Introduce the function $f(x, y)$ by

$$\frac{\partial f}{\partial y} = \frac{\partial \varphi}{\partial y} - \nu\,\varphi. \tag{18.17}$$

Then f also satisfies (18.13) and

$$f_y(x, 0) = 0 \quad \text{on the free surface}. \tag{18.18}$$

Consequently, f may be extended into the upper half-plane by defining $f(x, -y) = f(x, y)$ and f now satisfies (18.13) in the whole plane outside the contour C and its mirror image $\overline{C}$. Moreover, $|f| \to 0$ as $x^2 + y^2 \to \infty$. Assuming that f is known, one must now try to reconstruct φ from f in such a way that conditions (18.16) are satisfied. In order to do this, HASKIND differentiates (18.17) with respect to y, subtracts from

$$f_{xx} + f_{yy} - k^2 f = 0,$$

and after some easy manipulation obtains

$$\frac{\partial^2}{\partial x^2}(\varphi - f) + k_1^2(\varphi - f) = -\nu\left(\frac{\partial f}{\partial y} + \nu f\right). \tag{18.19}$$

Treating this as a differential equation for $\varphi - f$, he finds the following solution for φ:

$$\left.\begin{aligned} \varphi = f - \frac{\nu}{2 i k_1}\Big\{ & e^{i k_1 x}\int_{\infty}^{x} e^{-i k_1 \xi}(f_y + \nu f)\, d\xi - \\ & - e^{-i k_1 x}\int_{-\infty}^{x} e^{i k_1 \xi}(f_y + \nu f)\, d\xi\Big\} + \frac{A\,g}{\sigma}\, e^{\nu y}\, e^{-i k_1 x}, \end{aligned}\right\} \tag{18.20}$$

the integrals being taken along half-lines parallel to the x-axis and below C. One may verify without great difficulty that φ satisfies (18.13). The asymptotic form of φ as $x \to \pm\infty$ may be written down immediately, and gives

$$\frac{\nu}{2 i k_1}\, e^{\mp i k_1 x}\int_{-\infty}^{\infty} e^{\pm i k_1 \xi}(f_y + \nu f)\, d\xi + \frac{A\,g}{\sigma}\, e^{\nu y}\, e^{-i k_1 x}, \tag{18.21}$$

the path of integration being a line below the body. Consider now the region D bounded externally by this line and a large semicircle containing $C + \overline{C}$ and internally by $C + \overline{C}$. Application of GREEN's Theorem to f and $\chi = \exp(-\nu y + i k_1 x)$ shows that

$$e^{-\nu y}\int_{-\infty}^{\infty} e^{i k_1 \xi}(f_y + \nu f)\, d\xi = \int_{C+\overline{C}} (f \chi_n - \chi f_n)\, ds,$$

$$e^{-\nu y}\int_{-\infty}^{\infty} e^{-i k_1 \xi}(f_y + \nu f)\, d\xi = \int_{C+\overline{C}} (f \overline{\chi}_n - \overline{\chi} f_n)\, ds.$$

Hence, the asymptotic conditions are satisfied and, moreover,

$$\frac{B^+ g}{\sigma} = \frac{\nu}{2 i k_1} \int_{C+\bar{C}} (f \chi_n - \chi f_n)\, ds, \qquad \frac{B^- g}{\sigma} = \frac{\nu}{2 i k_1} \int_{C+\bar{C}} (f \bar{\chi}_n - \bar{\chi}_n f)\, ds. \tag{18.22}$$

By a similar application of GREEN's Theorem HASKIND shows that one may also write

$$\varphi = f + \nu\, e^{\nu y} \int_\infty^y f\, e^{-\nu \eta}\, d\eta + \frac{B^\pm g}{\sigma} e^{\nu y \mp i k_1 x} + \frac{A g}{\sigma} e^{\nu y - i k_1 x}, \tag{18.23}$$

where the plus sign is used for points to the right of C and the minus sign for points to the left. It is easy to verify directly that φ satisfies (18.13) and (18.17); however, (18.20) allows one to investigate the asymptotic behavior more simply. If φ has no singularities, then (17.3) must also hold here, i.e., $(B^+)^2 + (B^-)^2 + 2 A B^- = 0$.

This result may be used to find the source solutions giving outgoing waves at $\pm\infty$. For Eq. (18.13) the singular solutions for the whole plane are known to be the Bessel functions $K_n(kr)$, where $r^2 = (x-a)^2 + (y-b)^2$. To find the solution corresponding to (13.22), one assumes it may be expressed as

$$G(x, y; a, b) = \varphi_0 + K_0(k r) - K_0(k r_1),$$

with $r_1^2 = (x-a)^2 + (y+b^2)$, where φ_0 has no singularities for $y<0$. Then $f_{0y} = \varphi_{0y} - \nu\, \varphi_0$ may be extended as a regular solution of (18.13) to the whole plane. Also,

$$f_{0y}(x, 0) = 2 \frac{\partial}{\partial y} K_0(k r_1)\big|_{y=0}.$$

One may then show that this relation holds for all $y \leqq 0$:

$$f_{0y}(x, y) = 2 \frac{\partial}{\partial y} K_0(k r_1), \qquad y \leqq 0,$$

or

$$f_0(x, y) = 2 K_0(k r_1).$$

Substitution in (18.23) with $A=0$ and direct computation of $B^\pm$ from (18.22) by taking C as a small circle about the singularity gives

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$$G = K_0(k r) + K_0(k r_1) + 2\nu\, e^{\nu y} \int_\infty^{-y} e^{-\nu y} K_0(k r_1)\, dy - 2\pi i \frac{\nu}{k_1} e^{\nu (y+b) \mp i k_1 (x-a)}. \tag{18.24}$$

For HASKIND's application of this method to the diffraction about a vertical and a horizontal barrier we refer to the original paper. Force and moment are obtained in terms of Mathieu functions. For the horizontal barrier in water of finite depth we refer to MACCAMY's paper (1957) where a formula analogous to (18.24) is derived.

β) Waves on beaches. Much of the immediately preceding discussion of diffraction of plane waves approaching at an angle or of short-crested waves approaching normally applies also to this case. One is led to the following boundary-value problem for $\varphi(x, y) = \varphi_1 + i \varphi_2$:

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$$\left.\begin{aligned}
&1.\quad \varphi_{xx} + \varphi_{yy} - k^2 \varphi = 0, \qquad k^2 < \nu^2,\\
&2.\quad \varphi_y(x, 0) - \nu\, \varphi(x, 0) = 0,\\
&3.\quad \varphi_x \sin\gamma + \varphi_y \cos\gamma = 0 \quad \text{for } y + x \tan\gamma = 0,\\
&4.\quad \varphi \sim \frac{A g}{\sigma} e^{\nu y} e^{-i k_1 x} \quad \text{as} \quad x \to \infty, \quad k = \nu^2 - k^2,\\
&5.\quad \varphi_x^2 + \varphi_y^2 \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \quad \text{along} \quad y + x \tan\gamma = 0.
\end{aligned}\right\} \tag{18.25}$$

Many of the authors cited in Sect. 17β considered this problem along with the two-dimensional one. In particular, we refer to HANSON (1926), MICHE (1944), STOKER (1947), WEINSTEIN (1949), ROSEAU (1952), and PETERS (1952). Both PETERS and ROSEAU solve the problem for arbitrary angle γ, $0<\gamma\leqq\pi$ [thus including the semi-infinite dock problem treated differently by HEINS (1948)]. The use of the reduction method limits one here, as in the two-dimensional case, to angles $\gamma=p\pi/2q$. We shall illustrate the procedure briefly for $\gamma=\pi/4$ and $\gamma=\pi/2$, following essentially WEINSTEIN'S (1949) treatment [see also BRILLOUËT (1957, Chaps. I, II)].

Since the boundary condition on the free surface and bottom is the same in the two- and three-dimensional cases, we may make use of the auxiliary function g constructed in (17.49) by using only the real part of the complex potential. Thus, for $\gamma=\pi/4$ one finds from (17.48) that $a_1=a_0/\nu$. Hence, from (17.50)

$$p(\lambda)=a_0(1+\lambda/\nu),$$

and

$$g(z)=\frac{a_0}{\nu}\left(\frac{d}{dz}+\nu\right)\left(\frac{d}{dz}+i\nu\right)(\varphi+i\psi),$$

$$\operatorname{Im} g(z)=\frac{a_0}{\nu}\left(\frac{\partial}{\partial x}+\nu\right)\left(-\frac{\partial}{\partial y}+\nu\right)\varphi.$$

Thus, the boundary conditions 2 and 3 of (18.25) imply that

$$h(x,y)\equiv\left(\frac{\partial}{\partial x}+\nu\right)\left(\frac{\partial}{\partial y}-\nu\right)\varphi(x,y)=0\quad\left.\begin{array}{ll}\text{on} & y=0,\ x>0\\ \text{and} & x=0,\ y<0.\end{array}\right\}\tag{18.26}$$

We recall that the definition of $\varphi(x,y)$ has been extended from the original wedge by reflection in the bottom. One must now find a function $h(x,y)$ satisfying equation 1 of (18.25) and the boundary conditions (18.26) and which is regular everywhere in the extended wedge, $0\geqq\vartheta\geqq\frac{1}{2}\pi$, except possibly at the origin, bounded as $x^2+y^2\to\infty$, and symmetric about the line $y=-x$. It is known that the general solution of this problem is given by

$$h(x,y)=\left(\frac{\partial}{\partial x}+\nu\right)\left(\frac{\partial}{\partial y}-\nu\right)\varphi(x,y)=\sum_{n=0}^{\infty}A_n K_{2(2n+1)}(kr)\sin 2(2n+1)\vartheta.\tag{18.27}$$

A similar analysis for waves approaching a vertical cliff ($\gamma=\frac{1}{2}\pi$) leads to

$$h(x,y)\equiv\left(\frac{\partial}{\partial y}-\nu\right)\varphi(x,y)=\sum_{n=0}^{\infty}A_n K_{2n+1}(kr)\sin(2n+1)\vartheta.\tag{18.28}$$

Let us take the weakest possible singularity in each case, i.e., K_1 for the 90° cliff and K_2 for the 45° beach. Consider first the vertical cliff. Taking account of the relation $K_0'(u)=-K_1(u)$, we have

$$\left(\frac{\partial}{\partial y}-\nu\right)\varphi(x,y)=-\frac{A_0}{k}\frac{\partial}{\partial y}K_0(kr).$$

We may then identify $-A_0K_0/k$ with f and from (18.23), with $B^{\pm}=0$, we have

$$\varphi=-\frac{A_0}{k}K_0(kr)-A_0\frac{\nu}{k}e^{\nu y}\int_{\infty}^{y}e^{-\nu\eta}K_0\left(k\sqrt{x^2+y^2}\right)d\eta+\frac{Ag}{\gamma}e^{\nu y-ik_1x},$$

where A_0 must still be determined so that $\varphi_x(0, y) = 0$, $y < 0$. In computing φ_x as $x \to 0$, one must remember that $K_0(u) \sim \ln(2/u)$ as $u \to 0$. Hence, one finds

$$\varphi_x(0, y) = -\frac{A_0}{k} \nu e^{\nu y} \lim_{\varepsilon\to 0} \lim_{x\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{-x}{x^2+y^2} dy - i \frac{A g k_1}{\sigma} e^{\nu y}$$
$$= \frac{A_0}{k} \nu \pi e^{\nu y} - i \frac{A g k_1}{\sigma} e^{\nu y}.$$

Setting this equal to zero, one finds

$$\frac{A_0}{k} = \frac{A g}{\sigma} \cdot \frac{i k_1}{\pi \nu}.$$

Substituting above and separating the real and imaginary parts of $\varphi = \varphi_1 + i\varphi_2$, we obtain an everywhere regular solution φ_1 and a solution φ_2 with a singularity at the origin and 90° out of phase at $x = \infty$:

$$\left.\begin{aligned} \varphi_1(x, y) &= \frac{A g}{\sigma} e^{\nu y} \cos k_1 x, \\ \varphi_2(x, y) &= -\frac{A g}{\sigma} \frac{k_1}{\pi \nu} \left[K_0(k r) + \nu e^{\nu y} \int_{\infty}^{y} e^{-\nu\eta} K_0\left(k\sqrt{x^2+\eta^2}\right) d\eta\right] + \\ &\quad + \frac{A g}{\sigma} e^{\nu y} \sin k_1 x. \end{aligned}\right\} \tag{18.29}$$

The corresponding equation for (18.27) can be written in the form

$$\left(\frac{\partial}{\partial x} + \nu\right)\left(\frac{\partial}{\partial y} - \nu\right) \varphi(x, y) = A_0 K_2(k r) \sin 2\vartheta = \frac{2A_0}{k^2} \frac{\partial^2}{\partial x \partial y} K_0(k r). \tag{18.30}$$

One can find its integration discussed in ROSEAU (1952, Chap. IV). A solution for the next simplest case, $\gamma = 30°$, does not seem to have been published. For $\gamma = 45°$ the regular solution φ_1, and singular solution φ_2 as given by ROSEAU, but corrected according to personal communications from ROSEAU and LEHMAN, are

$$\left.\begin{aligned} \varphi_1 &= A_1\{e^{\nu y}[k_1 \cos k_1 x - \nu \sin k_1 x] + e^{-\nu x}[k_1 \cos k_1 y + \nu \sin k_1 y]\}, \\ \varphi_2 &= A_2\{e^{\nu y}[\nu \cos k_1 x + k_1 \sin k_1 x] + e^{-\nu x}[\nu \cos k_1 y - k_1 \sin k_1 y]\} + \\ &+ A_2 \frac{\nu^2 + k_1^2}{\pi \nu} \Big\{-K_0\left(k\sqrt{x^2+y^2}\right) + \nu e^{-\nu x} \int_{-\infty}^{x} e^{\nu\xi} K_0\left(k\sqrt{\xi^2+y^2}\right) d\xi + \\ &+ \nu e^{\nu y} \int_{y}^{\infty} e^{-\nu\eta} K_0\left(k\sqrt{x^2+\eta^2}\right) d\eta - \nu^2 e^{\nu y} \int_{y}^{\infty} d\eta\, e^{-\nu\eta} \left(e^{-\nu x} \int_{-\infty}^{x} d\xi\, e^{\nu\xi} K_0\left(k\sqrt{\xi^2+\eta^2}\right)\right)\Big\}. \end{aligned}\right\} \text{(18.}$$

In order to satisfy condition 4 of (18.25) one must take

$$A_1 = \frac{A g}{\sigma} \cdot \frac{k_1 + i\nu}{k_1^2 + \nu^2}, \qquad A_2 = \frac{A g}{\sigma} \cdot \frac{\nu - i k_1}{k_1^2 + \nu^2}, \qquad \varphi = \varphi_1 + \varphi_2.$$

Edge waves. In the investigation of diffraction of waves on horizontal cylindrical obstacles and of waves on beaches, it was specifically assumed that $k^2 < m^2$. This was automatically fulfilled for plane waves approaching at an angle, but needed to be assumed for short-crested waves. For the short-crested waves there also exist standing-wave solutions which can be exhibited in certain cases for $k^2 > m^2$. Such solutions were apparently first noticed by STOKES (1846, p. 7 = 1880, p. 167) in connection with the propagation of waves in a canal of non-rectangular cross-section. Certain peculiarities of these solutions have been pointed out by URSELL (1951, 1952).

Consider the first three conditions of (18.25) for waves on a sloping beach, but with $k^2 > \nu^2$. Then one may verify directly that

$$\varphi(x, y) = e^{k[y \sin\gamma - x\cos\gamma]}$$

is a solution. This gives a velocity potential for standing waves:

$$\Phi(x, y, z, t) = e^{k[y\sin\gamma - x\cos\gamma]} \cos(kz+\varepsilon)\cos(\sigma t+\tau), \tag{18.32}$$

where

$$k\sin\gamma = \sigma^2/g.$$

The wave amplitude is bounded at the origin and drops off very quickly as x increases. Clearly, one must have $\gamma < \frac{1}{2}\pi$. URSELL has pointed out other interesting aspects. For a given γ and σ there is only one allowable k, i.e., it is a discrete point of the spectrum. In the case discussed earlier with $k^2 < \nu^2$ all values of k between 0 and ν were allowable. In addition, the total energy per unit length in the z direction is finite for the Stokes edge wave.

From (18.29) one may construct a progressive wave moving in the direction Oz with velocity.

$$c = \frac{g\sin\gamma}{\sigma}.$$

There is evidence that such waves have been observed in nature (cf. MUNK, SNODGRASS and CARRIER 1956; DONN and EWING 1956).

URSELL (1952) has shown that (18.32) is only the first in a sequence of solutions of this nature for a sloping beach. He shows, in fact, that the following velocity potential also satisfies the condition:

see errata

$$\left.\begin{aligned}\Phi(x, y, z, t) = \Big\{ & e^{-k[x\cos\gamma - y\sin\gamma]} + \sum_{m=1}^{n} A_{mn}\left[e^{-k[x\cos(2m-1)\gamma + y\sin(2m-1)\gamma} + \right. \\ & \left. + e^{-k[x\cos(2m+1)\gamma - y\sin(2m+1)\gamma]}\right]\Big\}\cos(kz+\varepsilon)\cos(\sigma t+\tau),\end{aligned}\right\} \tag{18.33}$$

where

$$A_{mn} = (-1)^m \prod_{r=1}^{n} \frac{\tan(n-r+1)\gamma}{\tan(n+r)\gamma}, \qquad \sigma^2 = g\,k\sin(2n+1)\gamma.$$

It follows from the last condition that one must have

$$(2n+1)\gamma \leqq \frac{\pi}{2} \quad \text{or} \quad n < \frac{\pi}{4\gamma} + \frac{1}{2},$$

where $n=0$ will be taken to indicate the Stokes edge wave. Thus, for fixed wave number k, the above formula gives one frequency σ if $\frac{1}{2}\pi > \gamma > \frac{1}{6}\pi$, two if $\frac{1}{6}\pi > \gamma > \frac{1}{10}\pi$, etc. An experiment carried out by URSELL confirms the existence of these other modes of motion. The solutions (18.33) for $\gamma = \pi/2(2n+1)$ have also been given by MACDONALD (1896). At these critical angles the solution (18.33) does not vanish as $x\to\infty$. MACDONALD apparently discarded the other solutions as being of little interest, not "being sensible at a distance from the edge". ROSEAU (1958) has recently carried through a systematic investigation of edge waves, including ones with singular behavior at the edge.

KELDYSH (1936) has stated without proof that for $\gamma = 45°$ the Stokes edge wave and the function φ_1 from (18.31) constitute a complete set of bounded solutions in the sense that for any absolutely integrable function $f(x, y)$, $x=0$, the following Fourier-integral-like theorem holds [cf. formula (16.5)]:

see errata

$$f(x,z)=\frac{2}{\pi^2}\int_0^\infty\int_0^\infty\frac{dk\,dk_1}{k^2+2k_1^2}\int_{-\infty}^{\infty}d\zeta\cos(z-\zeta)\int_0^\infty d\xi\, f(\xi,\zeta)\times$$
$$\times\{[k_1\,\mathrm{e}^{-\nu x}+k_1\cos k_1 x-\nu\sin k_1 x]\,[k_1\,\mathrm{e}^{-\nu\xi}+k_1\cos k_1\xi-\nu\sin k_1\xi]+$$
$$+2k^2\exp\left(-k(x+\xi)/\sqrt{2}\right)\}.$$

It is possible to construct other types of edge waves. First we rederive the Stokes wave from the third formula in (13.5) with $a=0$. A surface satisfying $\Phi_n=0$ is defined by

$$\frac{dy}{dx}=\frac{\Phi_y}{\Phi_x}=-\frac{\nu}{\sqrt{k^2-\nu^2}},$$

or

$$y=-x\tan\gamma+C,\qquad \tan\gamma=\nu/\sqrt{k^2-\nu^2},$$

where we may set $C=0$ since it does not provide essentially different solutions for the bottom. This is just STOKES' solution.

One may expect to find a different type of solution by using the third equation of (13.6) with $a=0$. Here the corresponding solution is

$$-\log\frac{\sinh m_0(y+h)}{\sinh m_0 h}=\frac{m_0^2}{\sqrt{k^2-m_0^2}}\,x, \tag{18.34}$$

where again we have dropped an added constant. This describes a bottom which starts as a sloping beach and approaches, as $x\to\infty$, a flat bottom at depth h. The initial slope of the beach is $\sigma^2/g\sqrt{k^2-m_0^2}$. The velocity potential describes edge waves for such a configuration.

One may proceed in the same fashion with the last formulas of (13.5) and (13.6). They turn out to give identical bottoms:

$$\log\frac{\sin m_i(y+h)}{\sin m_i h}=\frac{m_i^2}{\sqrt{k^2+m_i^2}}\,x. \tag{18.35}$$

This corresponds to edge waves along an overhanging cliff in water of finite depth. The initial backward slope of the cliff is $\sigma^2/g\sqrt{k^2+m_i^2}$.

A particularly interesting sort of edge wave, although the name is now a misnomer since there is no edge, has been discovered by URSELL (1951). He has shown the existence of standing waves of the form

$$\varphi(x,y)\cos kz\cos\sigma t$$

in the neighborhood of a fixed submerged cylinder of radius a if ka is small enough. The waves are symmetric about the vertical plane through the axis of the cylinder and decay exponentially as $|x|$ increases. One can, of course, also construct waves progressing along the cylinder.

URSELL calls such modes of motion "trapping modes" since, if they occur in a canal with sides given by $z=0$ and $z=n\pi/k$, no energy is radiated away, even though there is a path of escape. In fact, the motion is similar in this respect to standing waves in a basin of finite extent. The edge waves considered above also can be used to construct trapping modes.

γ) Waves in canals. The propagation of periodic waves along a canal leads to problems similar to those occurring in the propagation of waves parallel to a beach. Let the canal be parallel to Oz with cross-sectional contour C. We wish to find

$$\Phi(x, y, z, t) = \varphi(x, y) \cos(kz - \sigma t)$$

where $\varphi(x, y)$ satisfies

$$\varphi_{xx} + \varphi_{yy} - k^2 \varphi = 0, \tag{18.36}$$

$$\varphi_y(x, 0) - \nu \varphi(x, 0) = 0, \quad \nu = \sigma^2/g,$$

on the free surface,

$$\varphi_n = 0 \quad \text{on } C.$$

It will also be assumed that $\varphi_x^2 + \varphi_y^2$ is bounded.

Clearly the same equations arise in searching for standing-wave solutions in a horizontal cylindrical basin with cross-sectional contour C bounded at either end by vertical walls at a distance l apart. In this case k is restricted to the values $n\pi/l$. For progressive waves solutions with $k=0$ are, of course, of no interest.

The special case when C is a rectangle has already been discussed in Sect. 14γ. The configuration for C which seems to have attracted the next most attention is a triangular one in which the two sides are inclined at the same angle. KELLAND (1844) was apparently the first to consider this problem for infinitesimal waves, limiting his treatment to angles of 45°. The matter was treated systematically by MACDONALD (1894) who states that a solution with the properties of (18.36) exists only for angles $\gamma = 45°$ and $\gamma = 30°$. This does not exclude the possibility of the existence for other angles of a periodic progressive wave with a curved wave front, for these would not be described by the assumed form of Φ.

The solutions for $\gamma = 45°$ can be obtained from the fundamental solutions of (13.6), but it is nearly as easy to find them directly. In the third formula of (13.6) let $a = b = \frac{1}{2}A$, $k^2 = 2m_0^2$. This gives the velocity potential, after forming a progressive wave,

$$\Phi(x, y, z, t) = A \cosh \frac{k}{\sqrt{2}} (y + h) \cosh \frac{k}{\sqrt{2}} x \cos(kz - \sigma t). \tag{18.37}$$

Let the sides of the canal be given by $y = \pm x - h$. Then it is easy to verify that

$$\Phi_n|_{y=x-h} = -\Phi_x + \Phi_y|_{y=x-h} = 0, \qquad \Phi_n|_{y=-x-h} = \Phi_x + \Phi_y|_{y=-x-h} = 0,$$

so that the boundary conditions are all satisfied. Since

$$\sigma^2 = g m_0 \tanh m_0 h = \frac{1}{\sqrt{2}} g k \tanh \frac{1}{\sqrt{2}} kh,$$

the wave velocity is given by

$$c^2 = \frac{g}{k\sqrt{2}} \tanh \frac{kh}{\sqrt{2}}. \tag{18.38}$$

If $m_0^2 > m_i^2$ [in the notation of Eq. (13.6)], there will be i further symmetric modes. In (13.6), formula 4, set $a = b = \frac{1}{2}A$ and add this to formula 1 with $a = A$, $b = 0$. This gives

$$\Phi(x, y, z, t) = A\left[\cos m_i(y+h) \cosh \sqrt{k^2 + m_i^2}\, x + \right.$$
$$\left. + \cosh m_0(y+h) \cos \sqrt{m_0^2 - k^2}\, x\right] \cos(kz - \sigma t).$$

One may again verify easily that $\Phi_n = 0$ on the two sides of the canal if $k^2 = m_0^2 - m_i^2$. Hence this mode of motion will exist only if $m_0^2 > m_i^2$. For given σ there will be no modes of this sort if h is small enough, for then $m_0^2 < m_1^2$. The number gradually increases as h increases. If h and k are fixed and σ allowed to increase, there will be an infinite sequence $\sigma_1, \sigma_2, \ldots$ for which $k^2 = m_0^2 - m_i^2$ will be satisfied; $\sigma_n^2 h/g \to (n + \frac{3}{4})\pi$ as $n \to \infty$. The situation is easily visualized by plotting on one graph $\tanh mh$, $-\tan mh$ and $(\sigma^2 h/g)/mh$. One may write the potential function in the form

$$\Phi(x, y, z, t) = A\left[\cos m_i(y+h)\cosh m_0 x + \cosh m_0(y+h)\cos m_i x\right]\cos(kz - \sigma t),$$

where

$$m_0 \tanh m_0 h = \nu, \qquad m_i \tan m_i h = -\nu; \qquad k^2 = m_0^2 - m_i^2. \tag{18.39}$$

The velocity is given by

$$c^2 = \frac{g\, m_0 \tanh m_0 h}{m_0^2 - m_i^2}. \tag{18.40}$$

Asymmetric modes of motion also exist, having first been noticed by GREENHILL (1886). These cannot be deduced from (13.6) but must be found directly. The velocity potential corresponding to (18.37) is

$$\Phi(x, y, z, t) = A \sinh \frac{k}{\sqrt{2}}(y+h) \sin \frac{k}{\sqrt{2}} x \cos(kz - \sigma t) \tag{18.41}$$

The wave velocity is

$$c^2 = \frac{g}{k\sqrt{2}} \coth \frac{kh}{\sqrt{2}}, \tag{18.42}$$

which approaches infinity as $kh \to 0$. In addition to this mode, other asymmetric modes may exist under conditions similar to those required for (18.39). The velocity potential for these modes is

$$\left.\begin{aligned}\Phi(x, y, z, t) = A\left[\sin n_i(y+h)\sinh n_0 x + \sinh n_0(y+h)\sin n_i x\right] \times \\ \times \cos(kz - \sigma t),\end{aligned}\right\} \tag{18.43}$$

where

$$n_0 \coth n_0 h = \nu, \qquad n_i \cot n_i h = \nu, \qquad k^2 = n_0^2 - n_i^2.$$

The velocity of propagation is given by

$$c^2 = \frac{n_0 \coth n_0 h}{n_0^2 - n_i^2}. \tag{18.44}$$

The solution for the angle $\gamma = 30°$ will not be discussed here. It can be found in LAMB's Hydrodynamics (1932, p. 449) as well as in MACDONALD's paper cited above.

One may construct other possible contours for the canal cross-section by starting from one of the solutions (13.5) or (13.6) and finding surfaces for which $\Phi_n = 0$. Thus, from the third equation of (13.5) form

$$\Phi = A\, e^{\nu y} \sinh x\sqrt{k^2 - \nu^2} \cos(kz - \sigma t).$$

Solution of the differential equation $dy/dx = \Phi_y/\Phi_x$ leads easily to

$$y + h = \frac{\nu}{k^2 - \nu^2} \log\cosh x\sqrt{k^2 - \nu^2}$$

as an equation for the contour of a possible canal. The contour is reasonably shaped but varies with the choice of k. Also, the method is unsatisfactory in that it gives no information about other possible modes of motion.

19. Problems with steadily oscillating boundaries. Such problems include waves resulting from forced oscillation of a submerged body and the waves associated with steady oscillations of a freely floating body in oncoming waves.

In this section we shall assume the fluid of infinite extent. Waves in an oscillating bounded basin will be discussed later. The mathematical treatment has much in common with that of the last two sections, the scattered wave of those sections becoming the forced wave of this one.

α) Forced oscillations. Suppose that the surface of the oscillator in its equilibrium position is represented by $F(x, y, z) = 0$. Let us take it, for example, to be oscillating vertically with amplitude ε. Then the equation of the oscillating surface S may be written $F(x, y, z, t) = F(x, y + \varepsilon a \sin \sigma t, z) = 0$ where a is some length dimension of the oscillator. This ε will be taken as the expansion parameter in the perturbation procedure. In the perturbation theory of Sect. 10, we were concerned only with the functions $\Phi(x, y, z, t)$ and $\eta(x, y, t)$. However, we must similarly expand F before substituting it into the boundary condition satisfied on the surface of the oscillator, namely,

$$F_x \Phi_x + F_y \Phi_y + F_z \Phi_z + F_t = 0 \quad \text{on} \quad F(x, y, z, t) = 0. \tag{19.1}$$

The expansion for this case is

$$\left.\begin{aligned} &F(x, y + \varepsilon a \sin \sigma t, z) \\ &\quad = F(x, y, z) + \varepsilon a \sin \sigma t\, F_y(x, y, z) + \tfrac{1}{2} \varepsilon^2 a^2 \sin^2 \sigma t\, F_{yy}(x, y, z) + \cdots. \end{aligned}\right\} \tag{19.2}$$

We don't wish to restrict ourselves to this one mode of motion for the oscillator, but an examination of the form of this and similar expansions indicates that we may assume in general that the surface of the oscillator can be represented by the series

$$\left.\begin{aligned} F(x, y, z, t) &= F^{(0)}(x, y, z) + \varepsilon\,[F_1^{(1)}(x, y, z) \cos \sigma t + F_2^{(1)}(x, y, z) \sin \sigma t] + \\ &\quad + \text{time-periodic terms in higher powers of } \varepsilon = 0, \end{aligned}\right\} \tag{19.3}$$

where $F^{(0)}(x, y, z) = 0$ is the equilibrium position of the oscillator. We may now assume either that Φ is periodic, i.e.,

$$\Phi(x, y, z, t) = \sum \varphi_{1n}(x, y, z) \cos n \sigma t + \varphi_{2n}(x, y, z) \sin n \sigma t \tag{19.4}$$

or, more simply, that it is simple harmonic,

$$\Phi(x, y, z, t) = \varphi_1(x, y, z) \cos \sigma t + \varphi_2(x, y, z) \sin \sigma t, \tag{19.5}$$

where each function φ_{in} or φ_i is still to be expanded in a perturbation series. The two assumptions are not quite equivalent, even for the first-order theory, but since under certain conditions (19.4) leads to the same first-order equations as (19.5), we shall assume the latter form, together with

$$\eta(x, z, t) = \eta_1(x, z) \cos \sigma t + \eta_2(x, z) \sin \sigma t. \tag{19.6}$$

Substitution of the perturbation series into the exact equations and boundary conditions, as in Sect. 10, then leads to the following first-order equation and boundary conditions:

$$\left.\begin{aligned} &1.\quad \Delta \varphi_k^{(1)} = 0, \quad k = 1, 2, \\ &2.\quad \varphi_{ky}^{(1)}(x, 0, z) - \frac{\sigma^2}{g} \varphi_k^{(1)}(x, 0, z) = 0, \quad k = 1, 2, \\ &3.\quad \operatorname{grad} F^{(0)} \cdot \operatorname{grad} \varphi_1^{(1)} + \sigma F_2^{(1)} = 0 \quad \text{on} \quad F(x, y, z) = 0, \\ &4.\quad \operatorname{grad} F^{(0)} \cdot \operatorname{grad} \varphi_2^{(1)} - \sigma F_1^{(1)} = 0 \quad \text{on} \quad F(x, y, z) = 0. \end{aligned}\right\} \tag{19.7}$$

One should note that it is a natural consequence of the method that the boundary condition on the oscillator is to be satisfied at its equilibrium position.

If we let

$$A_1(x, y, z) = \frac{-\sigma F_2^{(1)}}{|\operatorname{grad} F^{(0)}|}, \quad A_2(x, y, z) = \frac{\sigma F_1^{(1)}}{|\operatorname{grad} F^{(0)}|} \quad \text{for } F^{(0)}(x, y, z) = 0, \tag{19.8}$$

then conditions 3. and 4. of (19.7) may be written

$$\varphi_n^{(1)} = A(x, y, z) \quad \text{on} \quad F^{(0)} = 0, \tag{19.9}$$

where

$$\varphi^{(1)} = \varphi_1^{(1)} + i\,\varphi_2^{(1)} \quad \text{and} \quad A = A_1 + i A_2.$$

We shall henceforth drop the superscripts and consider only the first-order equations. In addition to Eqs. (19.7) the functions φ_i must also satisfy the usual conditions on fixed solid boundaries, $\varphi_{in} = 0$, and, if the fluid is infinitely deep, $|\operatorname{grad} \varphi| \to 0$ as $y \to -\infty$. Finally, one needs a boundary condition to ensure only outgoing waves at infinity. As has been pointed out by URSELL (1951), the foregoing conditions are not always sufficient to guarantee uniqueness of solution.

KOCHIN (1939, 1940) has considered the general mathematical problem in water of infinite depth for both two and three dimensions. HASKIND (1942b, 1944, 1946) has extended KOCHIN'S methods to water of finite depth. The frequently-cited paper by JOHN (1950) treats the theoretical aspects of the problem in a thorough manner and includes many of the results of KOCHIN and HASKIND. Special problems have been considered by numerous authors. HAVELOCK (1929b) considers the waves generated by oscillation of a vertical plate extending to the bottom in water of infinite depth for both two and three dimensions, and in water of finite depth for two dimensions; he also considers waves generated by oscillations of a vertical cylinder. MACCAMY (1957) has treated the three-dimensional problem in water of finite depth. KENNARD (1949) has treated the two-dimensional problem as an initial-value problem. URSELL (1948) has considered waves generated by oscillation of a vertical strip with finite depth of immersion in water of infinite depth; the treatment is two dimensional. ALBAS (1958) treats a similar three-dimensional problem in which the motion is periodic along the length of the strip. In a later paper URSELL (1949b) considered the waves generated by the rolling of long cylinders of ship-like cross-section. In addition, URSELL has treated the waves generated by a heaving half-submerged circular cylinder (1949a, 1953c, 1954) and by a pulsing submerged cylinder (1950). HAVELOCK (1955) has treated the wave motion generated by a half-submerged sphere. Certain mathematical aspects of this last problem have been examined in more detail by MACCAMY (1954). Because of its interest in connection with the heaving motion of a ship there exist many papers attempting to compute approximately the force and moment on a heaving shiplike body resulting from wave formation. We mention particularly one by GRIM (1953). In the cited papers by KOCHIN and HASKIND certain special problems are solved approximately; by improving the approximation, LEVINE (1958) has clarified certain anomalous results of KOCHIN for an oscillating horizontal plate. In addition, HASKIND (1942, 1943b) has considered the motion resulting from forced oscillation of a plate, or a system of plates, on the surface. In a more recent paper HASKIND (1953a) has developed a method for finding solutions, and in particular the force and moment on the body, for a wide class of two-dimensional contours of ship-like cross-section. One should also consult a recent expository paper by MARUO (1957). A general survey of methods of generating waves in the laboratory, including some account of theoretical results, may be found in a recent paper by BIESEL and SUQUET (1951, 1952).

This brief summary of papers on forced water waves is by no means complete but lists many of the important papers and indicates the richness of the literature.

As was stated in the introductory remarks, the theory of forced water waves is mathematically almost identical with the diffraction theory. If one interprets the value of $-\partial\Phi_I/\partial n$ on the body as the function describing the motion of the oscillator, then it is clear that the problems are the same. Hence, the general remarks about existence of solutions, uniqueness, and special methods carry over directly and will not be repeated. However, we wish to consider here one further topic in the general theory.

KOCHIN's *H*-function. The H-function was apparently first introduced by KOCHIN (1937) in connection with the theory of wave resistance. He later extended it (1939, 1940) to waves generated by oscillating bodies, and it has become a standard device among other Russian workers in this field, especially HASKIND, who has extended its definition to other situations.

Each potential function φ satisfying the boundary conditions has associated with it an H-function which is related to it much in the same way that the Fourier transform of a function is related to the function. One of its chief virtues is that it allows one to give compact formulas for force and moment on an oscillating body (in the present context) as well as certain other quantities. It is also sometimes helpful in suggesting approximate solutions.

Let us suppose that the surface S of a body of bounded extent is oscillating in some manner in fluid of infinite depth and let $\varphi=\varphi_1+i\varphi_2$ be the solution to the potential-theory problem formulated earlier. Let S_1 and S_2 be two closed surfaces lying below $y=0$ with S_2 enclosing S_1 and S_1 enclosing S. Let us denote the source potential introduced in (13.17″) by $G(x,y,z;\xi,\eta,\zeta)$, where (ξ,η,ζ) are the coordinates of the singularity, and let us write it as a contour integral:

$$G(x,y,z;\xi,\eta,\zeta)=\frac{1}{r}+\frac{1}{2\pi}\int_{-\pi}^{\pi}d\vartheta\int_{0\,(L)}^{\infty}dk\,\frac{k+\nu}{k-\nu}\,e^{k(y+\eta-i(x-\xi)\cos\vartheta-i(z-\zeta)\sin\vartheta)}, \tag{19.10}$$

where the path L passes below the singularity at $k=\nu=\sigma^2/g$. [The residue at this point gives exactly the imaginary part of (13.17″).]

Now apply GREEN's Theorem to the region between S_1 and S_2 (the following argument is very similar to a two-dimensional one used in Sect. 17α in discussing the integral-equation method):

see errata

$$\left.\begin{aligned}\varphi(x,y,z)&=-\frac{1}{4\pi}\iint_{S_1}\left[\frac{1}{r}\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma+\frac{1}{4\pi}\iint_{S_2}\left[\frac{1}{r}\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma\\&=\varphi^{(1)}+\varphi^{(2)}.\end{aligned}\right\} \tag{19.11}$$

Then $\varphi^{(1)}$ may be extended to a function harmonic in the whole space exterior to S_1. $\varphi^{(2)}$ is harmonic in the whole interior of S_2, but since S_2 may be indefinitely enlarged as long as it remains below $y=0$, $\varphi^{(2)}$ may be extended to be harmonic in the whole half-space, $y\leqq 0$. Consider now the function

$$\psi(x,y,z)=-\frac{1}{4\pi}\iint_{S_1}\left[G\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial G}{\partial n}\right]d\sigma. \tag{19.12}$$

ψ satisfies the free-surface condition and the condition at infinity. Moreover, since $G=r^{-1}+$a function harmonic in the lower half-space, $\varphi-\psi$ is harmonic in the lower half-space and satisfies the other boundary conditions. But then $\varphi-\psi\equiv 0$, as follows from a uniqueness theorem proved by KOCHIN (1940, Sect. 1).

Hence, we may write

$$\varphi(x, y, z) = \frac{1}{4\pi} \iint\limits_{S_1} [\varphi(\xi, \eta, \zeta)\, G_n(x, y, z; \xi, \eta, \zeta) - G\varphi_n]\, d\sigma. \tag{19.13}$$

Now define

$$\left.\begin{aligned} H(k, \vartheta) &= \iint\limits_{S_1} e^{k[\eta + i\xi\cos\vartheta + i\zeta\sin\vartheta)]} \{\varphi_n(\xi, \eta, \zeta) - \\ &\quad - k\varphi[\cos(n, \eta) + i\cos\vartheta\cos(n, \xi) + i\sin\vartheta\cos(n, \zeta)]\}\, d\sigma, \\ &= \iint\limits_{S_1} e^{k[\eta + i\xi\cos\vartheta + i\zeta\sin\vartheta]} \{\varphi_n(\xi, \eta, \zeta) + \\ &\quad + i\cos\vartheta[\varphi_\eta\cos(n, \xi) - \varphi_\xi\cos(n, \eta)] + \\ &\quad + i\sin\vartheta[\varphi_\eta\cos(n, \zeta) - \varphi_\zeta\cos(n, \eta)]\}\, d\sigma. \end{aligned}\right\} \tag{19.14}$$

Then, after some manipulation with (19.13), one can show that

$$\left.\begin{aligned} \varphi(x, y, z) &= \frac{1}{4\pi} \iint\limits_{S_1} \left[\varphi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\varphi}{\partial n}\right] d\sigma - \\ &\quad - \frac{1}{8\pi^2} \int\limits_{-\pi}^{\pi} d\vartheta \int\limits_{0\,(L)}^{\infty} dk\, \frac{k+\nu}{k-\nu}\, e^{k(y - ix\cos\vartheta + iz\sin\vartheta)}\, H(k, \vartheta). \end{aligned}\right\} \tag{19.15}$$

We give a few of KOCHIN'S derived formulas. The asymptotic form of the free surface in a direction α is given by

$$\eta(R, \alpha, t) \cong \mathrm{Re}\left[\frac{\sigma}{g}\sqrt{\frac{\nu}{2\pi R}}\, \overline{H}(\nu, \alpha)\, e^{i\left(\nu R - \sigma t - \frac{\pi}{4}\right)}\right] \quad \text{as} \quad R \to \infty. \tag{19.16}$$

The rate at which energy is being carried off by the waves (and hence also the power input) is given by

$$N = \frac{1}{8\pi} \frac{\varrho\sigma^3}{g} \int\limits_0^{2\pi} |H(\nu, \vartheta)|^2\, d\vartheta. \tag{19.17}$$

The force components on the oscillating body, averaged over a period, are given by

$$\left.\begin{aligned} X_{\mathrm{av}} &= \frac{\varrho\nu^2}{8\pi} \int\limits_{-\pi}^{\pi} |H(\nu, \vartheta)|^2 \cos\vartheta\, d\vartheta, \\ Y_{\mathrm{av}} &= \varrho g V + \frac{\varrho}{16\pi^2} \int\limits_{-\pi}^{\pi} \mathrm{PV} \int\limits_0^{\infty} k\, \frac{k+\nu}{k-\nu} |H(k, \vartheta)|^2\, dk\, d\vartheta, \\ Z_{\mathrm{av}} &= \frac{\varrho\nu^2}{8\pi} \int\limits_{-\pi}^{\pi} |H(\nu, \vartheta)|^2 \sin\vartheta\, d\vartheta. \end{aligned}\right\} \tag{19.18}$$

The formulas can be derived from (8.4), (9.4), and asymptotic expressions for φ.

In formulas (19.14) and (19.15) the surface S_1 over which the integrals are taken may be contracted to S. This sometimes makes it possible to express H directly in terms of known boundary values. If φ can be expressed by means of a source distribution, say

$$\varphi(x, y, z) = \frac{1}{4\pi} \iint\limits_S \gamma(\xi, \eta, \zeta)\, G(x, y, z; \xi, \eta, \zeta)\, d\sigma, \tag{19.19}$$

then one has

$$H(k,\vartheta) = -\iint_S \gamma(\xi,\eta,\zeta)\, e^{k[\eta + i\xi\cos\vartheta + i\zeta\sin\vartheta]}\, d\sigma. \tag{19.20}$$

In order to find approximate answers, Kochin frequently uses the distribution γ which would be proper in an infinite fluid without free surface, substitutes this in (19.20) and then uses the resulting approximation to H in (19.17) and (19.18) above. The procedure may be looked upon as the first two steps in an alternating type of approximation in which one first satisfies the boundary condition on the body, neglecting the free surface, next corrects this so as to satisfy the free-surface condition, but now disturbing the condition on the body, then corrects again to satisfy the condition on the body, etc. This method of approximation has frequently been used by HAVELOCK (e.g., 1929a).

KOCHIN (1939) has also defined the H-function for two-dimensional wave motion excited by an oscillating body. We simply reproduce the formulas. Let, as usual, $f(z,t) = f_1(z)\cos\sigma t + f_2(z)\sin\sigma t$ be the complex potential and let C_1 and C_2 be two contours in the lower half-plane containing C, C_1 inside C_2. Define

$$H_s(k) = \int_{C_1} e^{-ik\zeta} f_s'(\zeta)\, d\zeta, \qquad s = 1, 2. \tag{19.21}$$

Then

$$\left.\begin{aligned} f_s'(z) &= \frac{1}{2\pi i}\int_{C_1} \frac{f_s'(\zeta)}{z-\zeta}\, d\zeta - \frac{1}{2\pi}\int_0^\infty \overline{H}_s(k)\, e^{-ikz}\, dk - \\ &\quad - \frac{\nu}{\pi}\,\mathrm{PV}\int_0^\infty \frac{\overline{H}_s(k)}{k-\nu}\, e^{-ikz}\, dk + (-1)^{s+1} H_{s+1}(\nu)\, e^{-i\nu z}, \end{aligned}\right\} \tag{19.22}$$

where $H_3 \equiv H_1$. This follows immediately from a formula similar to (17.15). For the asymptotic form of the waves one gets

$$\left.\begin{aligned} \eta(x,t) &\cong \mathrm{Re}\,\frac{i\nu}{\sigma}\,[\overline{H}_1(\nu) - i\overline{H}_2(\nu)]\, e^{-i(\nu x - \sigma t)} \quad \text{as} \quad x \to +\infty, \\ \eta(x,t) &\cong \mathrm{Re}\,\frac{-i\nu}{\sigma}\,[\overline{H}_1(\nu) + i\overline{H}_2(\nu)]\, e^{-i(\nu x + \sigma t)} \quad \text{as} \quad x \to -\infty. \end{aligned}\right\} \tag{19.23}$$

The rate of dissipation of energy is

see errata

$$N = \tfrac{1}{2}\varrho\sigma[|H_1(\nu)|^2 + +|H_2(\nu)|^2]. \tag{19.24}$$

The mean values of the force and moment, averaged over a period, are

$$\left.\begin{aligned} X_{\mathrm{av}} &= \varrho\nu\,\mathrm{Im}\{\overline{H}_1(\nu)\, H_2(\nu)\}, \\ Y_{\mathrm{av}} &= \frac{\varrho}{4\pi}\,\mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\{|H_1(k)|^2 + |H_2(k)|^2\}\, dk, \\ M_{\mathrm{av}} &= -\frac{\varrho}{4\pi}\,\mathrm{Im}\left\{\mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\,[H_1'\overline{H}_1 + H_2'\overline{H}_2]\, dk\right\} + \\ &\quad + \tfrac{1}{2}\nu\varrho\,\mathrm{Im}\{H_1'(\nu)\,\overline{H}_2(\nu) - H_2'(\nu)\,\overline{H}_1(\nu)\}. \end{aligned}\right\} \tag{19.25}$$

Roughly the same remarks apply to the use of the two-dimensional formulas as of the three-dimensional ones.

Waves from an oscillator in a wall. In order to illustrate the use of the H-function, we consider the following problem. Let the (y, z)-plane be a rigid wall expect for a certain bounded area S in which there is a membrane oscillating according to a given law

$$x = F(y, z) \sin \sigma t, \qquad (y, z) \text{ in } S. \tag{19.26}$$

The boundary condition which has to be satisfied on the plane $x=0$ is then

$$\varphi_x(0, y, z) = \left\{ \begin{matrix} \sigma F(y, z), & (y, z) \text{ in } S \\ 0, & (y, z) \text{ not in } S, \end{matrix} \right\} \tag{19.27}$$

where we still have $\varphi = \varphi_1 + i\varphi_2$.

This boundary condition, as well as the ones at infinity, can be satisfied by distributing "modified" sources (13.17″) or (19.10) over S with density $-\sigma F(y, z)/2\pi$:

$$\varphi(x, y, z) = \frac{-\sigma}{2\pi} \iint_S F(\eta, \zeta)\, G(x, y, z; 0, \eta, \zeta)\, d\eta\, d\zeta. \tag{19.28}$$

In order to compute the H-function, we shall interpret the source distribution as representing a thin body making symmetric pulsations in an infinite fluid. Hence, we may assume that the wall is removed and the membrane replaced by a doubled one. (That the requisite motion is physically impossible doesn't invalidate the considerations; a more realistic model can easily be devised.) In (19.14) we take S_1 to be both sides of the thin body. Then, remembering that

$$\varphi_n(+0, \eta, \zeta) = \varphi_x(0, \eta, \zeta) = \sigma F, \qquad \varphi_n(-0, \eta, \xi) = -\varphi_x(0, \eta, \zeta) = \sigma F,$$

$$\cos(n, \xi) = 1 \quad \text{for } x > 0 \quad \text{and} \quad \cos(n, \xi) = -1 \quad \text{for } x < 0,$$

one finds easily that

$$H(k, \vartheta) = 2\sigma \iint_S F(\eta, \zeta)\, e^{k(\eta + i\zeta \sin\vartheta)}\, d\eta\, d\zeta. \tag{19.29}$$

From (19.17) one then finds immediately, after carrying out the ϑ integration, that the rate of dissipation of energy to one side is given by

$$N = \frac{\varrho\sigma^5}{4\pi g} \iint_S dy\, dz \iint_S d\eta\, d\zeta\, F(y, z)\, F(\eta, \zeta)\, e^{\nu(y+\eta)}\, J_0(\nu(z-\zeta)). \tag{19.30}$$

Expressions for Y_{av} and Z_{av} may also be written down. The result $X_{av}=0$ is not really significant because the integral is over both sides of the thin pulsing body.

The theory for generation of two-dimensional waves in a semi-infinite channel by a vertical wave maker in the end-wall is easily derived in the same way. If the motion of the wave-maker is described by

$$x = F(y) \sin \sigma t, \qquad a \leqq y \leqq b \leqq 0 \tag{19.31}$$

then

$$H_1(k) = \int_a^b e^{k\eta} F(\eta)\, d\eta, \qquad H_2(k) = 0, \tag{19.32}$$

and, for example, the rate of dissipation of energy is given by

$$N = \varrho\sigma^3 \left[\int_a^b e^{\nu y} F(y)\, dy\right]^2. \tag{19.33}$$

The generation of short-crested waves is subject to the limitations described in Sect. 14γ. Suppose, for example, that the water is of depth h, the channel of breadth b, and that the motion of the wave-maker is described by

$$x = F(y) \cos kz \sin \sigma t, \qquad k = n\pi/b, \qquad -h \leqq y \leqq 0. \tag{19.34}$$

Then, since $\cos m_i(y+h)$, $\cosh m_0(y+h$ form a complete set of functions in $-h \leqq y \leqq 0$, there is no difficulty in representing $F(y)$ by a series of the fundamental solutions (13.6), but if $k^2 > m_0^2$, no progressive waves will move down the tank (within the limits of applicability of the linearized theory, of course). The analysis of the filtering effect of the tank on more complicated wave-maker motions can easily be carried through by Fourier analysis.

Waves from an oscillator not in a wall. Let us now suppose that we have a two-dimensional oscillator in infinitely deep water moving according to the law

$$x = F(y) \sin \sigma t, \qquad a < y < b \leqq 0, \tag{19.35}$$

but with no wall present. This small change complicates the solution of the problem in a substantial way, the complication being associated with the now possible flow under (and over if $b < 0$) the oscillator. In addition, in order to ensure a unique solution some further condition analogous to the Kutta-Joukowski condition in airfoil theory is required; here the last two conditions of (19.36) play this role. Then the boundary conditions to be satisfied on the oscillator by the velocity potential

$$\Phi(x, y, t) = \varphi_1 \cos \sigma t + \varphi_2 \sin \sigma t$$

are

$$\left.\begin{aligned} \Phi_x(0, y, t) &= \sigma F(y) \cos \sigma t, \quad a < y < b \leqq 0, \\ \Phi_y(0, a, t) &= 0, \\ \Phi_y(0, b, t) &= 0 \quad \text{if} \quad b < 0, \end{aligned}\right\} \tag{19.36}$$

The problem is clearly closely related to that of diffraction of plane waves by a vertical barrier and could be treated by a modification of the method used in Sect. 17α for that problem. It may also be solved by the integral-equation method discussed in Sect. 17α. A modification of this method has been used by URSELL (1948).

Introduce the complex potential

$$\left.\begin{aligned} \Phi + i\Psi &= \mathrm{Re}_j\{f(z)\, e^{-j\sigma t}\}, \\ f(z) &= f_1(z) + j f_2(z) = (\varphi_1 + j\varphi_2) + i(\psi_1 + j\psi_2). \end{aligned}\right\} \tag{19.37}$$

where

We try to construct a solution by means of a distribution of vortices of the form (13.28)

$$\left.\begin{aligned} f_v(z;\zeta) &= \frac{1}{2\pi i} \log(z-\zeta)(z-\bar{\zeta}) + \frac{1}{\pi i} \mathrm{PV} \int_0^\infty \frac{e^{-ik(z-\bar{\zeta})}}{k-v}\, dk - j i\, e^{-iv(z-\bar{\zeta})} \\ &= f_{v1} + j f_{v2} \end{aligned}\right\} \tag{19.38}$$

with intensity

$$\gamma(\eta) = \gamma_1 + j\gamma_2, \qquad a < y < b, \tag{19.39}$$

along the oscillator:

$$f(z) = \int_a^b \gamma(\eta)\, f_v(z; i\eta)\, d\eta. \tag{19.40}$$

An analysis almost identical with that in Sect. 17α leads quickly to the integral equation

$$\mathrm{Re}_i \int_a^b \gamma(\eta)\, f_v'(iy; i\eta)\, d\eta = \sigma F(y) + j\cdot 0, \qquad a < y < b. \tag{19.41}$$

Separating γ_1 and γ_2 and noting that $f'_v(iy; i\eta)$ is real with respect to i, one finds

$$\left.\begin{aligned}\int_a^b [\gamma_1(\eta)\, f'_{v1}(iy; i\eta) - \gamma_2(\eta)\, f'_{v2}(iy; i\eta)]\, d\eta &= \sigma F(y),\\ \int_a^b [\gamma_1(\eta)\, f'_{v2}(iy; i\eta) + \gamma_2(\eta)\, f'_{v1}(iy; i\eta)]\, d\eta &= 0.\end{aligned}\right\} \tag{19.42}$$

The equations can be uncoupled by applying the operator $[\partial/\partial y - \nu]$ to each (so that the reduction method enters after all!). Introducing

$$\mu_k = \gamma'_k - \nu\gamma_k, \qquad G(y) = F' - \nu F, \tag{19.43}$$

one finally obtains the pair of equations

$$\left.\begin{aligned}\int_a^b \mu_1(\eta)\frac{d\eta}{y^2-\eta^2} &= \frac{\gamma_1(b)}{y^2-b^2} - \pi\sigma\frac{G(y)}{y},\\ \int_a^b \mu_2(\eta)\frac{d\eta}{y^2-\eta^2} &= \frac{\gamma_2(b)}{y^2-b^2},\end{aligned}\right\} \tag{19.44}$$

where we have taken advantage of the fact that $\varphi_y(\pm 0, y) = \mp\gamma(y)$ and hence $\gamma(a) = 0$; if $b<0$ also $\gamma(b) = 0$. The integral equations are easily reduced to a known type occurring in airfoil theory[1] by the transformation

$$r = y^2 - \tfrac{1}{2}(a^2+b^2), \qquad \varrho = \eta^2 - \tfrac{1}{2}(a^2+b^2).$$

The solution may be written in the form

$$\left.\begin{aligned}\mu_1(\eta) &= \frac{2\eta}{\pi\sqrt{(a^2-\eta^2)(\eta^2-b^2)}}\times\\ &\times\left[\nu\int_a^b \gamma_1(y)\,dy + 2\varepsilon\sigma\,\mathrm{PV}\int_a^b G(y)\sqrt{(a^2-y^2)(y^2-b^2)}\,\frac{dy}{\eta^2-y^2}\right],\\ \mu_2(\eta) &= \frac{2\eta\nu}{\pi\sqrt{(a^2-\eta^2)(\eta^2-b^2)}}\int_a^b \gamma_2(y)\,dy.\end{aligned}\right\} \tag{19.45}$$

It is evident that the solution is not uniquely determined without some statement about the total circulation. Fixing the total circulation is equivalent to fixing $\gamma(b)$, as follows easily from the form of $\mu(\eta)$ and the relation

$$\gamma(\eta) = e^{\nu\eta}\int_a^\eta e^{-\nu s}\mu(s)\,ds. \tag{19.46}$$

It is possible to compute the H-function directly in terms of $\mu(s)$. First, we note that

$$\begin{aligned}H(\lambda) &= \oint_{C_1} e^{-i\lambda\zeta} f'(\zeta)\,d\zeta = \oint_{C_1} e^{-i\lambda\zeta}\,d\zeta\int_a^b \gamma(y)\, f'_v(\zeta; iy)\,dy\\ &= \int_a^b \gamma(y)\,dy \oint_{C_1} e^{-i\lambda\zeta} f'_v(\zeta; iy)\,d\zeta = \int_a^b \gamma(y)\, e^{\lambda y}\,dy.\end{aligned}$$

It then follows from (19.46) that

$$H(\lambda) = \frac{e^{\lambda b}}{\lambda+\nu}\gamma(b) - \frac{1}{\lambda+\nu}\int_a^b \mu(y)\, e^{\lambda y}\,dy. \tag{19.47}$$

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[1] See, e.g., W. SCHMEIDLER: Integralgleichungen ..., pp. 55—56. Leipzig: Akademische Verlagsgesellschaft 1950, or S. G. MIKHLIN: Integral'nye uravneniya..., pp. 149—154, Gostekhizdat, Moscow 1949.

One may now apply formulas (19.23) to (19.25) to find the quantities indicated there (note that $\overline{H}=H$).

One notes again that the function $H(\lambda)$ is determined only after $\gamma(b)$ is fixed. Taking $\gamma(b)\neq 0$ is equivalent to having a singularity at the end. If the oscillator is totally submerged, it seems reasonable to set $\gamma(b)=0$, as we assumed in (19.36), for then the vertical velocity is continuous at the end, i.e., $\varphi_y(+0, b)=\varphi_y(-0, b)$, as has already been assumed for the lower end at $y=a$. It is not clear what is the proper assumption if $b=0$, i.e., if the oscillator extends through the surface. In the similar problem of diffraction by a vertical plate, treated by the reduction method in Sect. 17α, the assumption of no singularity at the surface is equivalent to assuming $\gamma(0)=0$. We note that if $\gamma(b)=0$, then it follows from (19.46) and the form of μ_2 in (19.45) that $\mu_2\equiv 0$, and hence that $\gamma_2\equiv 0$. This is not true, of course, for γ_1.

Waves generated by a heaving hemisphere. We describe briefly a procedure used by HAVELOCK (1955) and MACCAMY (1954), and before them also by URSELL (1949a) for an analogous two-dimensional problem. Let a hemisphere of radius a have its center on the free surface in its undisturbed position and let it undergo forced vertical oscillations described by

$$x^2+(y-b_0\sin\sigma t)^2+z^2=a^2. \tag{19.48}$$

Then the boundary condition to be satisfied by $\varphi(x, y, z)=\varphi_1+i\varphi_2$ on the hemisphere is

$$\frac{\partial\varphi_1}{\partial r}=b_0\sigma\frac{y}{a}=b_0\sigma\cos\vartheta,\quad \frac{\partial\varphi_2}{\partial r}=0 \quad\text{on}\quad x^2+y^2+z^2=a^2,\quad y\leqq 0. \tag{19.49}$$

φ must, of course, also satisfy the free-surface condition and the radiation condition, as stated in (19.7).

The method of the above-named authors is to represent φ as a series in which the first term is a source at the center, say (13.17), and the remaining terms represent only local disturbances of the sort shown in (13.21), with $m=0$ since we have radial symmetry. The source term is actually taken in the form (13.17‴). Since the source is at (0, 0, 0), $r=r_1$ in the formulas and certain terms cancel and others double. Let

$$\left.\begin{aligned}\varphi^{(0)}&=\frac{1}{r}-\frac{2\nu}{\pi}\int_0^\infty[\nu\cos ky-k\sin ky]\frac{K_0(kR)}{k^2+\nu^2}\,dk-\\&\quad-\pi\nu\,e^{\nu y}Y_0(\nu R)+i\pi\nu\,e^{\nu y}J_0(\nu R),\\\varphi^{(n)}&=\frac{-\nu}{2n}\,\frac{P_{2n-1}(\cos\vartheta)}{r^{2n}}+\frac{P_{2n}(\cos\vartheta)}{r^{2n+1}}.\end{aligned}\right\} \tag{19.50}$$

Then the assumed form for φ is

$$\varphi(x, y, z)=\sum_{n=0}^{\infty}a^{n+2}(A_n+iB_n)\,\varphi^{(n)}(x, y, z). \tag{19.51}$$

Substitution in the boundary condition (19.49) leads to an infinite set of linear equations for the coefficients A_n, B_n. Numerical methods may then be used to find any desired number of terms.

Having found φ approximately, one may proceed to compute the vertical hydrodynamic force on the sphere by integrating the pressure $p=-\varrho\,\partial\Phi/\partial t$

over the hemisphere. HAVELOCK carried through an approximate calculation, expressing the result in the form

$$\left.\begin{aligned} Y &= \frac{2}{3}\pi\varrho\, a^3 b_0 \sigma^2 \left[k \sin\sigma t - 2h\cos\sigma t\right] \\ &= -M\cdot k\cdot\frac{d^2 y_0}{dt^2} - M\cdot 2h\sigma\cdot\frac{dy_0}{dt}, \end{aligned}\right\} \tag{19.52}$$

where M is the mass of displaced fluid and y_0 the coordinate of the center. The parameter k is usually called the added-mass coefficient; h is called the damping parameter. Fig. 17 from HAVELOCK's paper shows k and $2h$ as functions of νa. As $\nu a \to \infty$, $2h \to 0$ and $k \to \frac{1}{2}$; $k(0) = 0.828\ldots$. The average rate at which work is being done by the sphere is $\frac{2}{3}\pi\varrho a^3 b_0^2 \sigma^3 h$ and does not involve k.

see errata

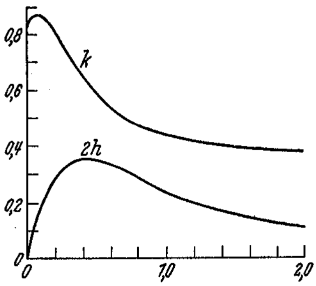

Fig. 17.

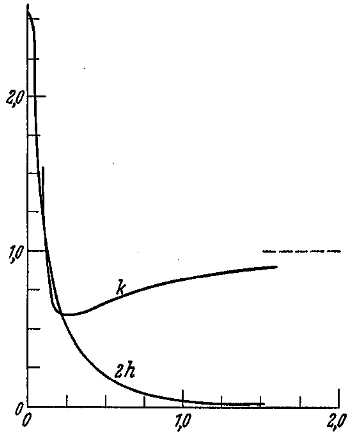

Fig. 18.

It is of interest to compare the same parameters as computed by URSELL (1949a) for a circular cylinder (per unit length). They are shown in Fig. 18. The asymptotic behavior of k is given by

$$\left.\begin{aligned} k(\nu a) &= \frac{8}{\pi^2}\left[\log\frac{1}{\nu a} + \frac{3}{2} - 2\log 2 - \gamma\right] + o(\nu a) \\ &= \frac{8}{\pi^2}\left[\log\frac{1}{\nu a} - 0.46\right] + o(\nu a) \quad \text{as} \quad \nu a \to 0, \\ k(\nu a) &= 1 - \frac{4}{3\pi\nu a} + o\left(\frac{1}{\nu a}\right) \quad \text{as} \quad \nu a \to \infty. \end{aligned}\right\} \tag{19.53}$$

β) Steady oscillations of a freely floating body in waves. Let us suppose that a rigid body is floating in an infinite ocean with prescribed plane waves approaching from a fixed direction, say from $x = +\infty$. If the motion has persisted for some time, we may suppose that the body is moving with a simple periodic motion of the same frequency as the waves. With this assumption the proper formulation of the linearized equations and boundary conditions has been derived by JOHN (1949).

Suppose the body is at rest in still water and let (x_0, y_0, z_0) be the coordinates of its center of gravity. Let $\bar{O}\bar{x}\bar{y}\bar{z}$ be a coordinate system fixed in the body with $\bar{O}$ at the center of gravity and the axes parallel to the space axes $Oxyz$. When the body is displaced, one may describe its position by giving the new position of the center of gravity $(\xi, \eta, \zeta) = (x_0 + \varepsilon x_1, y_0 + \varepsilon y_1, z_0 + \varepsilon z_1)$ and the Eulerian angles $\varepsilon\alpha$, $\varepsilon\beta$, $\varepsilon\gamma$ (we change notation from the customary φ, ϑ, ψ to avoid confusion with our other use of these letters). Thus the choice of ε implies that the amplitude of motion is small compared to some typical body length. The assumption of Sect. 10α that $\Phi = \varepsilon\,\Phi^{(1)} + \cdots$ implies that the amplitude of the prescribed incoming waves is also small compared to this length. The relationship between the two sets of coordinates may be easily found from the usual formulas concerning Eulerian angles to be of the form

$$\left.\begin{aligned}\bar{x} &= x - x_0 - \varepsilon\,[x_1 - \gamma(y-y_0) + \beta(z-z_0)] + \varepsilon^2[\cdots] + \cdots,\\ \bar{y} &= y - y_0 - \varepsilon\,[\gamma(x-x_0) + y_1 - \alpha(z-z_0)] + \cdots,\\ \bar{z} &= z - z_0 - \varepsilon\,[-\beta(x-x_0) + \alpha(z-z_0) + z_1] + \cdots.\end{aligned}\right\} \quad (19.54)$$

Let the surface of the body be described by

$$F(\bar{x}, \bar{y}, \bar{z}) = 0 \quad (19.55)$$

in body coordinates. To find the position in space coordinates one must substitute from (19.54) in (19.55). The kinematic boundary condition [see Eq. (19.1)] then becomes

$$\left.\begin{aligned}&\varepsilon\{\operatorname{grad} F(x-x_0, y-y_0, z-z_0)\cdot\operatorname{grad}\Phi^{(1)} + F_x(x-x_0, y-y_0, z-z_0)\times\\ &\quad\times[-\dot{x}_1 - \dot{\gamma}(y-y_0) - \dot{\beta}(z-z_0)] + F_y[-\dot{y}_1 + \dot{\alpha}(z-z_0) - \dot{\gamma}(x-x_0)] +\\ &\quad + F_z[-\dot{z}_1 + \dot{\beta}(x-x_0) - \dot{\alpha}(y-y_0)]\} + \varepsilon^2\{\cdots\} + \cdots = 0.\end{aligned}\right\} \quad (19.56)$$

Letting n_x, n_y, n_z be the components of the unit inward normal vector to the surface at rest, i.e.,

$$F(x-x_0, y-y_0, z-z_0) = 0 \quad (19.57)$$

(we shall call this surface S_0), and $\boldsymbol{q} = (\boldsymbol{r} - \boldsymbol{r}_0)\times\boldsymbol{n}$, i.e.

$$\left.\begin{aligned}&q_x = (y-y_0)\,n_z - (z-z_0)\,n_y, \quad q_y = (z-z_0)\,n_x - (x-x_0)\,n_z,\\ &q_z = (x-x_0)\,n_y - (y-y_0)\,n_x,\end{aligned}\right\} \quad (19.58)$$

we may express the first-order term in (19.56), after dropping the superscript, in the form

$$\Phi_n = \dot{x}_1 n_x + \dot{y}_1 n_y + \dot{z}_1 n_z + \dot{\alpha}\,q_x + \dot{\beta}\,q_y + \dot{\gamma}\,q_z \quad \text{for} \quad (x, y, z) \text{ on } S_0. \quad (19.59)$$

We call attention to the fact that it follows as a natural consequence of the linearization that the boundary condition is to be satisfied on the surface in its undisturbed position.

In order to state the dynamical conditions on the body we introduce the following notations. Let M be its mass and I_x, I_y, I_z, I_{xx}, I_{xy}, ... its moments and moments and products of inertia about the body axes selected above. Let V be the volume bounded by the plane $y=0$ and the submerged part of the surface in its rest position, and let I^V, I^V_x, I^V_y, I^V_z, I^V_{xx}, I^V_{xy}, ... be the volume, the moments and the moments and products of inertia of this volume about the body axes in their rest position. Let A be the intersection of the body in its rest position with the surface $y=0$, and let I^A, I^A_x, I^A_z, I^A_{xx}, I^A_{xz}, I^A_{zz} denote the area, the moments and the moments and products of inertia of A with respect to axes through $(x_0, 0, z_0)$ and parallel to the body axes; e.g.,

$$I^A_{xz} = \iint_A (x-x_0)(z-z_0)\,dx\,dz.$$

The exact dynamical equations are

$$\left.\begin{aligned} M\ddot{\xi} &= \iint_S p\cos(n,x)\,d\sigma,\\ M\ddot{\eta} &= \iint_S p\cos(n,y)\,d\sigma - Mg,\\ M\ddot{\zeta} &= \iint_S p\cos(n,z)\,d\sigma, \end{aligned}\right\} \tag{19.60}$$

where S is the wetted surface of the body in its (to-be-determined) position at time t and

$$p = -\varrho g y - \varrho \Phi_t - \tfrac{1}{2}\varrho\,|\mathrm{grad}\,\Phi|^2;$$

and three similar equations for $\ddot{\alpha}, \ddot{\beta}, \ddot{\gamma}$. Substitution of the perturbation series gives for the zero-order terms

$$M = \varrho I^V, \qquad I^V_x = I^V_y = 0, \tag{19.61}$$

i.e., ARCHIMEDES' law and the statement that the center of buoyancy and center of gravity are on the same vertical line. The first-order equations, after dropping superscripts, are

$$\left.\begin{aligned} M\ddot{x}_1 &= -\varrho\iint_{S_0}\Phi_t n_x\,d\sigma,\\ M\ddot{y}_1 &= -\varrho\iint_{S_0}\Phi_t n_y\,d\sigma + \varrho g\,(-I^A y_1 - I^A_x\gamma + I^A_z\alpha),\\ M\ddot{z}_1 &= -\varrho\iint_{S_0}\Phi_t n_z\,d\sigma,\\ -(I_{yy}+I_{zz})\ddot{\alpha} + I_{xy}\ddot{\beta} + I_{xz}\ddot{\gamma} &= \varrho\iint_{S_0}\Phi_t q_x\,d\sigma - \varrho g\,[I^A_z y_1 + I^A_{xz}\gamma - I^A_{zz}\alpha - I^V_y\alpha],\\ I_{xy}\ddot{\alpha} - (I_{xx}+I_{zz})\ddot{\beta} + I_{yz}\ddot{\gamma} &= \varrho\iint_{S_0}\Phi_t q_y\,d\sigma,\\ I_{xz}\ddot{\alpha} + I_{yz}\ddot{\beta} - (I_{xx}+I_{yy})\ddot{\gamma} &= \varrho\iint_{S_0}\Phi_t q_z\,d\sigma + \varrho g\,[I^A_x y_1 + I^A_{xx}\gamma - I^A_{xz}\alpha + I^V_y\gamma]. \end{aligned}\right\} \tag{19.62}$$

We note that the boundary conditions have been derived for general motions of the body and fluid, not just for the simply periodic ones for which they will be used below.

JOHN (1949) has used the equations to investigate the stability of a floating body. We shall not reproduce the results but remark that he shows that the usual condition for stability, namely that the metacenter lie above the center of gravity, derived from purely hydrostatic considerations, is in fact still a sufficient condition for stability when the hydrodynamic equations are considered (within the limitations of the linearized theory).

It is also shown by JOHN that the above equations have a unique solution for an initial-value problem, i.e., if at some instant the position and velocity of body and fluid are prescribed. However, for the problem with which we are concerned in this section, steady simple harmonic motion with a prescribed incoming wave, he proves uniqueness only for sufficiently large values of σ and for bodies such that a vertical line intersects the immersed surface only once (e.g., a floating sphere with its center at or above the free surface).

Knowledge of the motion of a floating body in surface waves is obviously of great importance to ship designers, and, as might be expected, there is a large amount of specialized literature. However, most of this literature may be considered irrelevant to this article for it is based upon the assumption that one may neglect the kinematic boundary condition (19.59) completely and, in the dynamic boundary condition (19.62), that one may take for Φ simply the velocity potential for the oncoming wave, thus neglecting the effect of the diffracted waves and the waves generated by the ship's own motion. This assumption is usually called the Froude-Krylov Hypothesis. W. FROUDE (1861) introduced it in connection with an investigation of ship rolling in waves and A. N. KRYLOV (1896, 1898) investigated its implications rather thoroughly for general motions. In spite of its apparent crudeness this assumption has been useful in elucidating many aspects of ship motions.

In recent years there have appeared a number of papers in which an attempt has been made to take account of the proper boundary conditions, but no attempt will be made to summarize this literature. The most systematic investigation of the matter has been made by JOHN (1949, 1950), HASKIND (1946a), and PETERS and STOKER (1957). The papers by JOHN consider the proper formulation of the linearized problem for a body with no average forward speed and the uniqueness and existence of solutions. Both HASKIND and PETERS and STOKER are primarily concerned with ships having a constant average forward speed. PETERS and STOKER treat carefully the proper formulation of the linearized problem and conclude that HASKIND'S fundamental equations are not properly formulated in that some of his terms really belong with the second-order terms and should have been discarded. The objection applies also to part of his results for a stationary ship. The other part will be summarized below.

The motion of a ship in waves when it has a nonzero translational velocity will not be considered in this article. For this theory one should refer to the cited papers, to STOKER'S *Water waves* (1957, Chap. 9), or to a recent survey by MARUO (1957). The transient oscillatory motion of a floating body in calm water will be considered later.

Let us return to the problem of steady oscillation of a floating body in oncoming waves. Since we assume steady oscillation, we shall write

$$\left.\begin{aligned} \Phi = \operatorname{Re}\{\varphi\, e^{-i\sigma t}\}, \qquad (x_1, y_1, z_1) &= \operatorname{Re}\{(a_0, b_0, c_0)\, e^{-i\sigma t}\}, \\ (\alpha, \beta, \gamma) &= \operatorname{Re}\{(\alpha_0, \beta_0, \gamma_0)\, e^{-i\sigma t}\}, \end{aligned}\right\} \tag{19.63}$$

where $\varphi = \varphi_1 + i\varphi_2$, $a_0 = a_0' + i a_0''$, etc. The unknown function φ and the constants $a_0, \ldots, \gamma_0$ are to be determined from the equations and boundary conditions.

We shall assume that Φ can be expressed as the sum of the velocity potentials of the incoming wave, say

$$\Phi^e = \frac{A\sigma}{g}\, e^{\nu y} \cos(\nu x + \sigma t) \tag{19.64}$$

if the fluid is infinitely deep, a diffracted wave $\Phi^0 = \varphi^0\, e^{-i\sigma t}$ and a forced wave $\Phi_f = \varphi_f\, e^{-i\sigma t}$ resulting from the body's own motion:

$$\Phi = \Phi^e + \Phi^0 + \Phi_f. \tag{19.65}$$

We shall express Φ_f in the following form (following HASKIND):

see errata

$$\Phi_f = \operatorname{Re}\{\varphi^1 \dot{x}_1 + \varphi^2 \dot{y}_1 + \varphi^3 \dot{z}_1 + \varphi^4 \dot{\alpha} + \varphi^5 \dot{\beta} + \varphi^6 \dot{\gamma}\}. \tag{19.66}$$

Then the kinematic boundary condition (19.59) implies:

$$\left.\begin{aligned} &\varphi_n^0 = -\varphi_n^e, \\ &\varphi_n^1 = n_x, \qquad \varphi_n^2 = n_y, \qquad \varphi_n^3 = n_z, \\ &\varphi_n^4 = q_x, \qquad \varphi_n^5 = q_y, \qquad \varphi_n^6 = q_z, \end{aligned}\right\} \tag{19.67}$$

all to be satisfied on S_0, the rest position of the body. The functions φ^k, $k = 0, 1, \ldots, 6$, are to satisfy also the radiation condition and the condition at $y = -\infty$ (or at $y = -h$ for a flat bottom). The dynamical condition (19.62) will be used to determine the amplitudes and phases (i.e., the complex amplitudes), but first we introduce some notation. Let

$$\mu_{jk} + \frac{i}{\sigma} \lambda_{jk} = \varrho \iint\limits_{S_0} \varphi^j \frac{\partial \varphi^k}{\partial n} \, d\sigma. \tag{19.68}$$

The constants μ_{jk} and λ_{jk} depend only upon the geometry of the body. It may be shown by an application of GREEN'S Theorem that $\mu_{kj} = \mu_{jk}$ and $\lambda_{kj} = \lambda_{jk}$.

Let us now substitute the expanded expression for Φ into, say, the first of Eqs. (19.62) (the others may be treated similarly), remembering that $n_x = \varphi_n^1$ on S_0:

$$M \ddot{x}_1 = -\varrho \iint\limits_{S_0} (\Phi^e + \Phi^0)_t \, n_x \, d\sigma - \varrho \iint\limits_{S_0} (\varphi^1 \ddot{x}_1 + \cdots + \varphi^6 \ddot{\gamma}) \, \varphi_n^1 \, d\sigma. \tag{19.69}$$

Consider, for example, the second term of the second integral:

$$\varrho \iint\limits_{S_0} \varphi^2 \varphi_n^1 \ddot{y}_1 \, d\sigma = \left(\mu_{21} + \frac{i}{\sigma} \lambda_{21}\right) \ddot{y}_1 = \mu_{21} \ddot{y}_1 + \lambda_{21} \dot{y}_1, \tag{19.70}$$

where we have used the special form of $y_1 = b_0 \, e^{-i\sigma t}$. Thus, (19.69) may be written

$$(M + \mu_{11}) \ddot{x}_1 + \mu_{21} \ddot{y}_1 + \cdots + \mu_{61} \ddot{\gamma} + \lambda_{11} \dot{x}_1 + \lambda_{21} \dot{y}_1 + \cdots + \lambda_{61} \dot{\gamma} = F_{ex} + F_{0x}, \tag{19.71}$$

where $F_{ex} = f_{ex} \, e^{-i\sigma t}$ and $F_{0x} = F_{0x} \, e^{-i\sigma t}$ represent the x-components of the forces resulting from the incoming and diffracted waves and are to be computed from the first integral in (19.69). The form of (19.71) explains the names given to the μ_{ij} and λ_{ij}: the μ_{ij} are called *added masses*, the λ_{ij}, *damping coefficients*. If one now writes $x_1, \ldots, \gamma$ in their assumed forms in (19.63) and substitutes in (19.71), one obtains

$$\left.\begin{aligned} -\sigma^2 (M + \mu_{11}) \, a_0 - \sigma^2 \mu_{21} \, b_0 - \cdots - \sigma^2 \mu_{61} \gamma_0 - \\ - i \sigma \lambda_{11} a_0 - \cdots - i \sigma \lambda_{61} \gamma_0 = f_{ex} + f_{0x} \end{aligned}\right\} \tag{19.72}$$

and five similar equations. Since the amplitudes $a_0, \ldots, \gamma_0$ are complex, this gives twelve equations to determine the twelve unknown quantities. It is thus clear that, providing these equations can be uniquely solved, the problem of finding the steady oscillatory motion of a freely floating body can be reduced to the solution of several problems of the type studied in Sects. 18 and 19α. From the form of (19.72) and the similar equations, it is clear that the complex amplitudes are all proportional to the amplitude A of the incoming wave as would be expected.

HASKIND has applied the method outlined above to a body symmetric with respect to the (x, y)-plane, e.g., a ship heading into waves. The only possible motions are heaving, pitching and surging. In carrying out some numerical computations he makes a further approximation that the kinematic boundary condition on the body may be satisfied on its plane of symmetry rather than on the surface. Although this approximation is perfectly consistent with the linearized theory in certain contexts, as will be seen in Sect. 21, it is not consistent with the theory as formulated here and must be considered to be a further approximation of some sort.

Freely floating sphere. Computation of the motion of a freely floating sphere with its center at the undisturbed water level can be carried through without an unreasonable amount of numerical work. The procedure for the heaving motion has been carried up to the point of numerical computation by BARAKAT (1958) [in an earlier investigation by MACCAMY (1954) the multipole terms in the potential for the diffracted wave were omitted]. Part of the problem has already been solved in Sect. 19α, i.e., the waves resulting from the forced motion.

Since the phase at infinity must be kept arbitrary, one must replace (19.48) by

$$x^2 + (y - b_0' \cos\sigma t - b_0'' \sin\sigma t)^2 + z^2 = a^2 \tag{19.72}$$

However, the solution of that problem may be taken over with practically no change, for the velocity potential φ^2 in the notation of (19.66) must satisfy

$$\frac{\partial}{\partial r}\varphi^2 = \frac{y}{a} = \cos\vartheta \quad \text{for } x^2 + y^2 + z^2 = a^2, \quad y \leq 0. \tag{19.73}$$

Thus we need only set $b_0\sigma = 1$ in (19.49) and later. In fact, from formula (19.52)

$$\mu_{22} = \tfrac{2}{3}\pi\varrho a^3 \cdot k, \qquad \lambda_{22} = \tfrac{2}{3}\pi\varrho a^3 \cdot 2h. \tag{19.74}$$

Finding the diffracted wave requires finding an outgoing wave satisfying

$$\left.\frac{\partial\varphi^0}{\partial r}\right|_{r=a} = -\frac{Ag}{\sigma}\nu\left[\cos\vartheta - i\sin\vartheta\cos\alpha\right] e^{\nu a\cos\vartheta} e^{-i\nu a\sin\vartheta\cos\alpha}, \tag{19.75}$$

where $x = r\sin\vartheta\cos\alpha$, $y = r\cos\vartheta$, $z = r\sin\vartheta\sin\alpha$. BARAKAT shows that φ^0 can be found as a series in functions of the form (13.21), with $b = 0$ and account taken of certain symmetries, and functions of the form (13.20) with $b = 0$ and $m = n$. Let

$$\left.\begin{aligned}
G_{2k}^{2m} &= \left[\frac{P_{2k}^{2m}(\cos\vartheta)}{r^{2k+1}} - \frac{\nu}{2k-2m}\,\frac{P_{2k-1}^{2m}(\cos\vartheta)}{r^{2k}}\right]\cos 2m\alpha, \\
&\qquad\qquad k = 1, 2, \ldots; \quad m = 0, \ldots, k-1; \\
G_{2k}^{2m-1} &= \left[\frac{P_{2k+1}^{2m-1}(\cos\vartheta)}{r^{2k+2}} - \frac{\nu}{2k-2m+2}\,\frac{P_{2k}^{2m-1}(\cos\vartheta)}{r^{2k+1}}\right]\cos(2m-1)\alpha, \\
&\qquad\qquad k = 1, 2, \ldots; \quad m = 1, \ldots, k; \\
\Phi_n &= \left[\frac{P_n^n(\cos\vartheta)}{r^{n+1}} + (-1)^n\,\mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\,k^n e^{ky} J_n(kR)\,dk + \right. \\
&\qquad \left. + 2\pi i(-1)^n \nu^{n+1} e^{\nu y} J_n(\nu R)\right]\cos n\alpha, \quad n = 1, 2, \ldots.
\end{aligned}\right\} \tag{19.76}$$

Then φ^0 may be expressed as follows

$$\varphi^0 = \sum_{k=1}^{\infty}\sum_{m=0}^{k-1} A_{2k}^{2m} a^{2k+2} G_{2k}^{2m} + i\sum_{k=1}^{\infty}\sum_{m=1}^{k} B_{2k}^{2m-1} a^{2k+3} G_{2k}^{2m-1} + \sum_{n=0}^{\infty} C_n\Phi_n, \tag{19.77}$$

where the complex coefficients A_{2k}^{2m}, B_{2k}^{2m-1}, C_n are to be determined from (19.75). No numerical computations seem to be available.

20. Motions which may be treated as steady flows. In this section we shall consider several problems which are time-independent, either by their formulation or by introduction of moving coordinates. The flow associated with a constant discharge rate through a canal is of the first type; the waves associated with a ship which has moved with constant velocity C over a long period is typical of the second.

The boundary conditions at the free surface have been derived in Sects. 10 and 11. For three-dimensional motion the velocity potential must satisfy [see Eq. (11.3)]

$$\varphi_y(x, 0, z) + \frac{c^2}{g}\varphi_{xx}(x, 0, z) = 0; \tag{20.1}$$

the equation of the free surface is

$$y = \eta(x, z) = \frac{c}{g}\varphi_x(x, 0, z). \tag{20.2}$$

In two-dimensional motion, if the complex potential $f = \varphi + i\psi$ is used, then the boundary condition may be written

$$\operatorname{Re}\left\{f''(x + i\,0) + i\frac{g}{c^2}f'(x + i\,0)\right\} = 0. \tag{20.3}$$

If the potential has been taken in the form $F(z) = -cz + f(z)$ with $\Psi = 0$ as the free-surface streamline, then

$$\operatorname{Re}\left\{f'(x + i\,0) + i\frac{g}{c^2}f(x + i\,0)\right\} = 0; \tag{20.3'}$$

the surface is given by

$$y = \eta(x) = \frac{1}{c}\psi(x, 0). \tag{20.4}$$

On obstructions, which are now all fixed, one has as usual

$$\varphi_n = 0 \quad \text{or} \quad \psi = \text{const}. \tag{20.5}$$

Far ahead of, or far upstream of, the obstruction the motion must approach either rest, or a uniform flow, respectively.

The general theory of steady free-surface flow about a submerged obstacle in infinitely deep fluid has been considered by KOCHIN (1937) for both two and three dimensions. HASKIND (1945a, b) has extended KOCHIN's treatment to fluid of constant finite depth. The methods used for waves generated by oscillating bodies carry over with only slight change, so that we shall not consider here the general aspects of the theory but consider instead several special problems.

α) *Flow over an uneven bottom.* Let us first derive the proper boundary condition on the bottom. We shall assume that the bottom may be represented in the form

$$y = -h + \varepsilon b^{(1)}(x) \tag{20.6}$$

and that the fluid flows from the right with discharge rate $q = ch$. As in the derivation of (10.19) we take

$$F(z) = -cz + \varepsilon f^{(1)}(z) + \varepsilon^2 f^{(2)}(z) + \cdots. \tag{20.7}$$

Then the condition that the bottom be a streamline is

$$-c(-h + \varepsilon b^{(1)} + \cdots) + \varepsilon\psi^{(1)}(x, -h + \varepsilon b^{(1)} + \cdots) + \cdots = ch. \tag{20.8}$$

Expansion in the manner of Sect. 10 and grouping of coefficients leads to the boundary condition for $\psi^{(1)}$:

$$\psi^{(1)}(x, -h) = c b^{(1)}. \tag{20.9}$$

We may hereafter write ψ for $\varepsilon\psi^{(1)}$ and b for $\varepsilon b^{(1)}$. We note that the choice of ε indicates that the amplitude of unevenness of the bottom must be small compared with h for the linearized theory to be applicable.

Consider now a bottom of the form [see LAMB (1932, p. 409), Wien (1900, p. 200)]

$$y = -h + b_0 \cos kx. \tag{20.10}$$

We look for a solution in the form

$$f(z) = A \cos kz + B \sin kz, \tag{20.11}$$

where A and B are complex. Substitution in (20.9), with $b^{(1)} = b_0 \cos kx$, shows that A must be pure imaginary, say iA', and B real, and further that

$$A' \cosh kh - B \sinh kh = c b_0. \tag{20.12}$$

Substitution in (20.3), i.e., $\psi_y(x, 0) - g c^{-2} \psi(x, 0) = 0$ yields

$$kB = \frac{g}{c^2} A'. \tag{20.13}$$

One then finds easily that

$$f(z) = \frac{\nu \sin kz + i k \cos kz}{k \cosh kh - \nu \sinh kh} c b_0, \qquad \nu = \frac{g}{c^2}, \tag{20.14}$$

$$\eta(x) = \frac{k b_0}{k \cosh kh - \nu \sinh kh} \cos kx.$$

An interesting consequence is that when $c^2/gh < 1$, i.e., when the flow is subcritical, the crests and troughs just oppose those of the bottom, whereas, if $c^2/gh > 1$, they occur together. If $c^2/gh = 1$, there is no steady flow. Also, when c^2/gh is close to 1, it is clear that the assumption of small perturbations is no longer satisfied.

By use of the Fourier Integral Theorem one may now construct solutions for an arbitrarily shaped bottom, within the limitations of the theorem. For from

$$b(x) = \frac{1}{\pi} \int_0^\infty dk \int_{-\infty}^\infty b(\xi) \cos k(x - \xi)\, d\xi \tag{20.15}$$

one may derive

$$\left.\begin{aligned} f(z) &= \frac{c}{\pi} \,\mathrm{PV} \int_0^\infty dk \int_{-\infty}^\infty b(\xi) \frac{\nu \sin k(z-\xi) + i k \cos k(z-\xi)}{k \cosh kh - \nu \sinh kh}\, d\xi, \\ \eta(x) &= \frac{1}{\pi} \,\mathrm{PV} \int_0^\infty \frac{k}{k \cosh kh - \nu \sinh kh}\, dk \int_{-\infty}^\infty b(\xi) \cos k(x-\xi)\, d\xi. \end{aligned}\right\} \tag{20.16}$$

An examination of the asymptotic properties of these integrals as $x \to +\infty$ shows that they do not vanish if $\nu h = gh/c^2 > 1$. Conditions for the validity of the Fourier Integral Theorem, e.g., that $b(x)$ is of bounded variation and absolutely integrable, indicate that it applies to situations in which the bottom unevenness is somewhat localized. Hence, it is reasonable to require the additional boundary condition

$$\lim_{x \to \infty} \eta(x) = 0.$$

see errata

Thus we must amend the solutions (20.10) if $gh/c^2 > 1$ by adding, respectively,

$$\left.\begin{aligned} &\frac{-\nu c}{\cosh^2 k_0 h - \nu h} \int_{-\infty}^\infty b(\xi) \cos k_0(z - \xi + ih)\, d\xi, \\ &\frac{+\nu \sinh k_0 h}{\cosh^2 k_0 h - \nu h} \int_{-\infty}^\infty b(\xi) \sin k_0(x - \xi)\, d\xi, \end{aligned}\right\} \tag{20.17}$$

where k_0 is the real solution of $k \cosh kh - \nu \sinh kh = 0$. We note that the other boundary conditions are not spoiled, for the first expression in (20.17) satisfies (20.3) and its imaginary (stream-function) part vanishes for $y = -h$ so that (20.3) is still satisfied.

Thus, if $c^2 > gh$ there is a local disturbance of the fluid in the region of unevenness which eventually smooths out. If $c^2 < gh$ there is also a local disturbance given by (20.16), but as $x \to -\infty$ there remains a disturbance given by twice the expressions in (20.17).

We remark in passing that we might have obtained this solution by distributing along the bottom dipoles of the form (13.48) with $\alpha = 0$ and with moment density $c\,b(x)$.

Various special cases of $b(x)$ may be considered. LAMB (1932, p. 410) replaces the unevenness by a single dipole. WIEN (1900, p. 202) takes $b(x) = \arctan e x$ and in the limit lets $e \to \infty$ in order to find the flow over a small step. However, KOCHIN (1938) has treated this problem by a different method and finds that WIEN has made an error by a factor of two in the downstream waves (he had not satisfied the upstream condition) [see also LAMB (1934)]. The flow about a vertical plate in the bottom may be treated by distributing vortices (13.47) along the plate with the intensity to be determined by solving an integral equation.

One will find an attractive discussion of the subject in four papers of W. THOMSON (Lord KELVIN) (1886, 1887). EKMAN (1906) has applied the same method to three-dimensional flow. First he finds the form of the free surface over a doubly periodic bottom, then applies the double Fourier integral theorem to construct flows over irregular bottoms. He analyzes the asymptotic behavior of the surface for the case of an isolated dipole on the bottom and presents graphs showing the change in wave amplitude for different radial sections. The method of analysis may also be extended to superposed fluids of different densities (see KOCHIN (1937a, b, 1938c), LONG (1953, § 4)].

β) Flow about submerged obstacles. Linearization. The procedure for linearizing may be carried through in at least two ways, leading to somewhat different boundary conditions for the body. Consider a body moving in a fluid. For the time being, in order to achieve somewhat greater generality, we shall not restrict the velocity to be constant. If the dimensions of the body are sufficiently small compared with the depth of submersion, it will not disturb the surface appreciably, and one will expect to be able to use the infinitesimal-wave approximation. However, the same end is obtained if the body approximates to a flat disc moving in its plane, a line segment moving along its line, a piece of a cylindrical surface moving along the cylinder, etc., various combinations being easily visualized. We consider the two situations separately.

Let $F(x, y, z, t) = 0$ describe the surface of a bounded body at time t, and let a be some typical dimension of the body, say its maximum diameter, and let h be the depth of submersion measured to some point $(x_0, -h, z_0)$ of the body. Now, consider the family of flows associated with the family of surfaces

$$F^{(\varepsilon)}(x, y, z, t) = F\left(\frac{x - x_0}{\varepsilon} + x_0, \frac{y + h}{\varepsilon} - h, \frac{z - z_0}{\varepsilon} + z_0, t\right) = 0 \qquad (20.18)$$

where $\varepsilon = a/h$. As $\varepsilon \to 0$ the surface $F^{(\varepsilon)} = 0$ contracts to a point and the fluid approaches a state of rest. Hence, as in Sect. 10α, it is allowable to expand Φ and η in a perturbation series

$$\Phi = \varepsilon \Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + \cdots, \qquad \eta = \varepsilon \eta^{(1)} + \varepsilon^2 \eta^{(2)} + \cdots. \qquad (20.19)$$

The boundary condition to be satisfied on the body, namely,

$$\operatorname{grad} F^{(\varepsilon)} \cdot \operatorname{grad} \Phi + F_t^{(\varepsilon)} = 0, \tag{20.20}$$

becomes

$$\operatorname{grad} F \cdot \operatorname{grad} \Phi^{(1)} + F_t + \varepsilon \operatorname{grad} F \cdot \operatorname{grad} \Phi^{(2)} + \cdots = 0,$$

and $\Phi^{(1)}$ must satisfy

$$\operatorname{grad} F \cdot \operatorname{grad} \Phi^{(1)} + F_t = 0 \quad \text{on} \quad F^{(\varepsilon)} = 0.$$

Thus, one finds that, in this method of linearizing, the boundary condition to be satisfied on the body is the exact one

$$\operatorname{grad} F \cdot \operatorname{grad} \Phi + F_t = 0 \quad \text{on} \quad F(x, y, z, t) = 0. \tag{20.21}$$

The boundary condition satisfied by Φ on the free surface is, of course, the linearized one. The approximation to the exact solution is better, the deeper the relative submergence.

The second method of linearization will be illustrated with the so-called *thin-ship* approximation. Let the equation of a ship hull be given in the form

$$\bar{z} = \pm F(\bar{x}, \bar{y}). \tag{20.22}$$

in coordinates fixed in the ship. Let us write this in the form

$$\bar{z} = \pm \varepsilon F^{(1)}(\bar{x}, \bar{y}) \tag{20.23}$$

where ε is, say, the beam/length. Suppose the ship moves in direction OX with velocity $c(t)$ and consider the family of flows generated by the motion of such bodies for different ε. Let the velocity potential be $\Phi(x, y, z, t; \varepsilon)$. Then, since as $\varepsilon \to 0$ the hull approaches a flat disc S_0, the ship's centerplane section, the motion of the fluid will also approach a state of rest and we may expand

$$\Phi = \varepsilon \Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + \cdots \tag{20.24}$$

and similarly for η. The assumed forms for Φ and η lead immediately as in Sect. 10α to the linearized free-surface boundary condition for $\Phi^{(1)}$. The exact condition on the hull is

$$F_x\left(x - \int^t c(\tau)\, d\tau, y\right) \Phi_x\left(x, y, F\left(x - \int^t c\, d\tau, y\right), t\right) + F_y \Phi_y - \Phi_z - c(t) F_x = 0. \tag{20.25}$$

After substituting (20.23) and (20.24) in (20.25), one finds that $\Phi^{(1)}$ must satisfy

$$\Phi_z^{(1)}(x, y, \pm 0, t) = \mp c(t) F_x^{(1)}\left(x - \int^t c(\tau)\, d\tau, y\right), \tag{20.26}$$

$\Phi^{(2)}$ must satisfy

$$\Phi_z^{(2)}(x, y, \pm 0, t) = \pm \left[F_x^{(1)}(x, y)\, \Phi_x^{(1)}(x, y \pm 0, t) + F_y^{(1)} \Phi_y^{(1)} - F^{(1)} \Phi_{zz}^{(1)}\right], \tag{20.27}$$

and $\Phi^{(i)}$ a relation of the form

$$\Phi_z^{(i)}(x, y, \pm 0, t) = \pm C_i\{F^{(1)}, \Phi^{(1)}, \ldots, \Phi^{(i-1)}\}, \tag{20.28}$$

where C_i is a functional of the functions in braces. We note especially that it is a consequence of the linearization that the boundary condition imposed by the presence of the body is to be satisfied on the centerplane section and not on the actual surface. One will expect this linearized theory to be more accurate the smaller ε is, i.e., the smaller the beam-to-length ratio.

It is clear that one may proceed similarly in the situations mentioned earlier. We record the results in several cases for reference.

First consider the *thin-wing approximation* for two-dimensional hydrofoils. In a coordinate system $\bar{O}\bar{x}\bar{y}$ fixed in the hydrofoil let the trailing edge of the hydrofoil be at $(-a, -h)$, and let the upper and lower surfaces be given by

$$\bar{y} = -h + u(\bar{x}) \quad \text{and} \quad \bar{y} = -h + b(\bar{x}), \qquad -a \leqq \bar{x} \leqq a, \tag{20.29}$$