that is, by taking α as defined by the average line between the two characteristics in the physical plane, or the line normal to the midpoint of the corresponding characteristic epicycloid arc in the hodograph plane (see Fig. 44). The close approximation of the characteristic epicycloid as given by f from (30.26) and Table 1, to the oblique hydraulic-jump relations (30.61) is strikingly illustrated when they are both plotted in the hodograph plane as in Fig. 45d [see also PREISWERK (1938)]. The trace of the endpoint of the velocity vector for the oblique shock wave of gas dynamics, generally called the "shock polar", is also shown in Fig. 45d for a specific heat ratio $\gamma = 2$.

As long as f from (30.26) is in close agreement with the oblique hydraulic jump relations (30.61), then the problems involving the interaction and reflection of hydraulic jumps can be closely approximated by the same procedure as detailed previously for the characteristic epicycloids involving compression waves (see Figs. 43 and 44). Whenever the required flow deflection ϑ is greater than that provided by the epicycloid passing through F_1, as shown in Fig. 43, then subcritical flow follows the curved or normal hydraulic jump as indicated by $N=1$ in Fig. 46. Similarly, $N=2$ defines the maximum flow reflection angle (ϑ) that can occur without ending in subcritical flow with a curved or normal hydraulic jump. In both cases two curves are shown for the oblique hydraulic jump: one shows the turning angle ϑ that will make the flow critical $(F_2=1)$, and the other one is the maximum possible turning angle $\vartheta_{\max}$ for any oblique hydraulic jump at the given value of F_1. The latter always produces subcritical flow $(F_2<1)$ as indicated in Fig. 43.

All of the preceding results primarily hold for hydraulic jumps in rectangular cross-section channels with a nearly horizontal bottom. BAKHMETEFF (1932) shows experimentally the various effects of steepening bottom slopes. He also generalizes (30.51) so that it will apply to any constant cross-section shape. However, it must be noted that our Eq. (29.3) shows conclusively that (30.51) which completely neglects the w velocity component, cannot be applicable to channel walls that are not nearly vertical. Sloping sides on a channel would increase the vertical velocity gradients, make a normal hydraulic jump impossible, and induce unsteady vortex motions.

It must also be noted that all of the preceding results are valid only for relatively small bottom slopes, as indicated by the direct comparison of (30.50) and (30.51) with (28.1) and (29.3). When the flow is rapidly varying because of large changes in the bottom slope, then the change in surface profile curvature is so pronounced that the pressure variation can no longer be considered as hydrostatic. For example, over the spillway of a dam the centrifugal force due to the streamline curvature can actually exceed the hydrostatic pressure, thereby leading to a pressure less than atmospheric resulting in flow separation or violent oscillations. At present spillway design is based on semi-empirical methods or model tests since no satisfactory mathematical analysis is available.

31. Higher-order theories and the solitary and cnoidal waves. It will now be shown that many of the preceding methods and results based on the shallow-water approximation are valid only if the local variations in water depth are not too large, and the average or undisturbed water depth is sufficiently small. The first requirement implies that the solutions of the first-order nonlinear shallow-water equations (28.1) do not greatly differ (at least for Froude numbers not near unity) from the linearized solutions given by (29.3) or (29.7). The second requirement essentially demands that the depth h be much less than the effective wavelength λ in any application, say $\frac{h}{\lambda} < \frac{1}{10}$, in order to reduce the effects associated with the infinitesimal-wave approximation.

As already discussed, the infinitesimal-wave approximation predicts that the fluid particle motion varies with the distance below the free surface, and also that the propagation velocity depends upon the wavelength, as shown in Sect. 15. There it was proved that the velocity defined by

$$c = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi h}{\lambda}\right)} = \sqrt{gh}\left[1 - \frac{1}{6}\left(\frac{2\pi h}{\lambda}\right)^2 + \cdots\right] \tag{31.1}$$

can only be considered a phase velocity while the actual rate of propagation of energy is associated with the group velocity defined by

$$c - \lambda\frac{dc}{d\lambda} = \frac{1}{2}c\left(1 + \frac{4\pi h/\lambda}{\sinh 4\pi h/\lambda}\right) = c\left[1 - \frac{1}{3}\left(\frac{2\pi h}{\lambda}\right)^2 + \cdots\right] \tag{31.2}$$

[see also LAMB (1932, p. 381)]. Any such variation will directly interfere with the applicability of the shallow-water results. However, as long as $\frac{h}{\lambda} < \frac{1}{10}$ it is seen that the phase velocity and the group velocity are both satisfactory approximations to the shallow-water first-order result that $c = \sqrt{gh}$ and is independent of the effective wave length.

This means that if small-scale model tests are used to simulate results appropriate to the shallow-water theory, then the undisturbed water depth should be less than $\frac{1}{10}$ the principal model dimensions. Consequently, if models less than 10 cm in effective dimensions are used, the depth of test water should be less than 1 cm, so the capillary ripples produced by surface tension must be considered. As shown in Sects. 15 and 24 the effect of the surface tension T is to increase the phase velocity for the short wavelength capillary ripples so that (31.1) is replaced by

$$c = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\varrho}\right)\tanh\frac{2\pi h}{\lambda}}. \tag{31.1'}$$

For ordinary water (at 20° C, $T = 72.8$ dynes/cm, $\varrho = 0.998$ gm/cm³) this gives the interesting result that both the phase velocity and the group velocity are closely approximated by $\sqrt{gh}$ for all $\lambda > 2$ cm if $h \approx \frac{1}{2}$ cm. However, in any small-model tests the surface wave patterns formed by the capillary ripples must be ignored since they are short-wavelength surface waves that are never in accord with the long-wavelength shallow-water theory.

Except for the section on hydraulic jumps the preceding shallow-water results have all been based entirely on (28.1), the first approximation to shallow-water theory, and this will now be shown to be limited to relatively small wave amplitudes even though the complete nonlinear equation (28.1) be used, and even though the bottom surface be flat and horizontal. The second approximation to shallow-water theory will be shown to immediately yield particular solutions corresponding to continuous permanent wave profiles of finite amplitude that can be propagated without a change in form or shape if viscosity effects are neglected. These permanent, finite-amplitude wave forms are the cnoidal waves discovered by KORTEWEG and DE VRIES (1895) which reduce, in the limiting case of essentially infinite wavelength, to the solitary wave of RUSSELL (1837, 1844) which was first analyzed theoretically by BOUSSINESQ (1871, 1872) and RAYLEIGH (1876).

The second approximation to shallow-water theory will show that the limitation of the nonlinear first approximation to relatively small amplitudes is primarily due to the fact that the variation in the vertical velocity cannot be neglected as the wave amplitude is increased. This of course invalidates even the rectangular channel hydraulic analogy to compressible gas flow, since, as previously discussed, the principal assumption of the hydraulic analogy is that the vertical acceleration be negligible.

The third approximation to shallow-water theory will then be presented to obtain new relations which will predict the limiting heights of the continuous finite-amplitude steady-state wave forms and give, for the first time, the complete second approximation to the cnoidal and solitary waves. It will be found that the pressure is no longer hydrostatic, thereby violating the remaining principal assumption of the hydraulic analogy and the ordinary classical shallow-water theory.

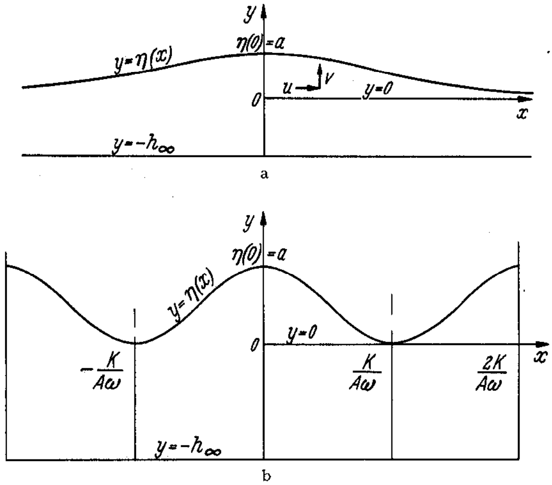

Fig. 47a and b. (a) Solitary wave over a flat horizontal bottom. (b) Cnoidal wave $\eta(x) = a\,\mathrm{cn}^2(A\omega x, k)$.

α) The first and second approximations to the cnoidal and solitary waves. We will now extend the perturbation method of FRIEDRICHS (1948), which was used to derive the nonlinear first-order approximation (28.1) to shallow-water theory, to obtain the second and higher orders of approximations for the special case of the steady-state propagation of a wave independent of z and t over a flat horizontal bottom described by $y = -h_\infty = \text{const}$ as in Fig. 47.

First we will show that the only steady-state finite-amplitude solution of the first-order equation (28.2) is $y^{(0)} = \eta_0 = \text{const}$ and $u^{(0)} = u_0 = \text{const}$. This is most easily proved by substituting the solution of the zeroth-order terms in (10.24) for steady water flow over a flat horizontal bottom, namely

$$u^{(0)} = u^{(0)}(x), \qquad v^{(0)} = 0, \qquad p^{(0)} = 0, \qquad \eta^{(0)} = \eta^{(0)}(x), \tag{31.3}$$

into the first-order terms in (10.27) to obtain

$$u^{(0)} = u_0 = \text{const}, \qquad v^{(1)} = 0, \qquad p_y^{(0)} = -\varrho g, \qquad \eta^{(0)} = \eta_0 = \text{const} \tag{31.4}$$

since $\eta_x^{(0)} = 0 = p_x^{(0)}$. Consequently the only finite-amplitude first-order steady-state solution must have $\eta_x^{(0)} = 0$, which would permit only the hydraulic jump as a solution since $\eta_x^{(0)} = 0$ and $u^{(0)} = \text{const}$ on each side of the finite discontinuity defining the hydraulic jump. This is in agreement with the well-known fact that the gas-dynamics equation or (28.2), predicts that any finite amplitude disturbance must form a finite discontinuity which is a shock wave, or hydraulic jump [see, e.g., LAMB (1932, pp. 278, 481)]. However, the second-order approximation of shallow-water theory (10.33) does yield a permanent finite-amplitude, steady-state wave profile that does not form a discontinuity. These are called the cnoidal waves, discovered by KORTEWEG and DE VRIES (1895), and the solitary waves of RUSSELL (1837, 1844), BOUSSINESQ (1871, 1872), and RAYLEIGH (1876). In order to obtain the higher-order approximations and limiting heights of these waves, it is more convenient to use exactly the same non-dimensional variables introduced by FRIEDRICHS (1948), and also used by KELLER (1948), namely,

$$\left.\begin{aligned} &\varepsilon=\omega^2 h^2, \quad \alpha=\omega x, \quad \beta=y/h, \quad H=h_\infty/h,\\ &u(\alpha,\beta)=u(x,y)/\sqrt{g h}, \qquad v(\alpha,\beta)=\frac{v(x,y)}{\sqrt{g h}}\,\omega h,\\ &Y(\alpha)=\eta(x)/h, \quad \pi(\alpha,\beta)=p(x,y)/\varrho g h,\\ &Y(\alpha)=Y^{(0)}+\varepsilon Y^{(1)}+\varepsilon^2 Y^{(2)}+\cdots,\\ &\eta(x)=h Y^{(0)}+\omega^2 h^3 Y^{(1)}+\omega^4 h^5 Y^{(2)}+\cdots, \end{aligned}\right\} \tag{31.5}$$

the only difference in notation being that x and y are now defined as in Fig. 47; consequently, the flat horizontal bottom is given by $y=-h_\infty$ or $\beta=-h_\infty/h=-H$, and the expansion parameter $\varepsilon=\omega^2 h^2$ is used as defined in (10.23), with (31.5) replacing (10.21).

Introducing the transformation defined by (31.5) into (31.4) and into the corresponding equivalent of (10.33), we obtain

$$\left.\begin{aligned} &v^{(0)}=0=v^{(1)}\\ &u^{(0)}(\alpha,\beta)=u_0=\text{const}, \qquad Y^{(0)}(\alpha)=Y_0=\text{const}=\eta_0/h,\\ &\pi_\beta^{(0)}=-1, \quad \pi_\beta^{(1)}=0, \quad u_\beta^{(1)}=v_\alpha^{(1)}=0, \quad u_\alpha^{(1)}=-v_\beta^{(2)},\\ &u_0 u_\alpha^{(1)}+\pi_\alpha^{(1)}=0, \qquad v^{(2)}(Y_0)=u_0 Y_\alpha^{(1)},\\ &v^{(2)}(-H)=0, \qquad \pi^{(1)}(Y_0)=-Y^{(1)}\pi_\beta^{(0)}=Y^{(1)}(\alpha). \end{aligned}\right\} \tag{31.6}$$

These expressions may be integrated to obtain

$$\left.\begin{aligned} u^{(1)}(\alpha,\beta)&=f(\alpha)=u^{(1)}(\alpha),\\ v^{(2)}(\alpha,\beta)&=v^{(2)}(\alpha,\beta)-v^{(2)}(\alpha,-H)\\ &=-\int_{-H}^{\beta} f_\alpha\,d\beta=-(\beta+H)\,f_\alpha,\\ \pi^{(1)}(\alpha,\beta)&=-\int u_0 f_\alpha\,d\alpha=-(u_0 f+C)=\pi^{(1)}(\alpha)=Y^{(1)}(\alpha),\\ Y_\alpha^{(1)}&=\pi_\alpha^{(1)}=-u_0 f_\alpha=\frac{v_2(Y_0)}{u_0}=-\left(\frac{Y_0+H}{u_0}\right)f_\alpha. \end{aligned}\right\} \tag{31.7}$$

The identities in the last equation show that the solution for constant u_0 is restricted to the unique value defined by

$$u(x,y)=u_0\sqrt{g h}=\sqrt{g h\,(Y_0+H)}=\sqrt{g\,(\eta_0+h_\infty)}=\text{const} \tag{31.8}$$

which corresponds to the infinitesimal-wave propagation velocity (28.3) and shows that the steady-state solution will be in the neighborhood of the critical speed defined by a Froude number of unity. However, $u^{(1)}=f(\alpha)$ now provides a finite-amplitude steady-state solution that does not form a discontinuity; consequently, the behavior of the second-order shallow-water theory is mathematically completely different from the first-order (28.2) shallow-water theory or the gas-dynamics equations. The pressure variation is still hydrostatic, since $\pi^{(1)}$ does not depend upon β, and only $v^{(2)}$ has a direct dependence upon $\beta\,(=y/h)$.

Now in order to continue the solution and determine $f(\alpha)$ we must introduce some ε^3 terms. By following the same procedure as used in collecting the ε^2 terms for (10.33) we obtain for the particular case of steady flow over a flat horizontal bottom the following additional terms that are required for completing the second order solution:

$$\left.\begin{aligned}
&u_\beta^{(2)} = v_\alpha^{(2)}, \quad u_\alpha^{(2)} = - v_\beta^{(3)},\\
&u_0\, u_\alpha^{(2)} + u^{(1)} u_\alpha^{(1)} + \pi_\alpha^{(2)} = 0, \qquad u_0\, v_\alpha^{(2)} + \pi_\beta^{(2)} = 0,\\
&v^{(3)}(Y_0) = u_0\, Y_\alpha^{(2)} + u^{(1)}\, Y_\alpha^{(1)} - v_\beta^{(2)}\, Y^{(1)}, \qquad v^{(3)}(-H) = 0,\\
&\pi^{(2)}(Y_0) = -\, Y^{(2)}\, \pi_\beta^{(0)} = Y^{(2)}.
\end{aligned}\right\} \tag{31.9}$$

These expressions were first given by KELLER (1948) and they may be directly integrated to give the following:

$$\left.\begin{aligned}
u^{(2)} &= \int v_\alpha^{(2)}\, d\beta = - f_{\alpha\alpha} \int (\beta + H)\, d\beta\\
&= -\tfrac{1}{2}(\beta^2 + 2H\beta)\, f_{\alpha\alpha} + R(\alpha) = u^{(2)}(\alpha, \beta)\\
\pi^{(2)}(\alpha, \beta) &= Y^{(2)}(\alpha) - [\tfrac{1}{2}(Y_0^2 - \beta^2) + H(Y_0 - \beta)]\, u_0\, f_{\alpha\alpha},\\
v^{(3)}(\alpha, \beta) &= v^{(3)}(\alpha, \beta) - v^{(3)}(\alpha, -H) = \int_{-H}^{\beta} - u_\alpha^{(2)}\, d\beta\\
&= [\tfrac{1}{6}(\beta^3 + H^3) + \tfrac{1}{2} H(\beta^2 - H^2)]\, f_{\alpha\alpha\alpha} - (\beta + H)\, R_\alpha,\\
v^{(3)}(Y_0) &= u_0\, Y_\alpha^{(2)} - u_0\, f\, f_\alpha - (u_0\, f + C)\, f_\alpha\\
&= \tfrac{1}{6}(Y_0^3 + 3H Y_0^2 - 2H^3)\, f_{\alpha\alpha\alpha} - (Y_0 + H)\, R_\alpha.
\end{aligned}\right\} \tag{31.10}$$

The last equation for $v^{(3)}$ gives the following expression for the ε^2 term in the surface profile

$$u_0\, Y^{(2)}(\alpha) = u_0\, f^2 + C\, f + \tfrac{1}{6}(Y_0^3 + 3H\, Y_0^2 - 2H^3)\, f_{\alpha\alpha} - (Y_0 + H)\, R + \text{const}, \tag{31.11}$$

while a similar expression may be obtained directly from $\pi^{(2)}(Y_0)$ by equating its relation in (31.9) and (31.10) so as to obtain

$$\left.\begin{aligned}
u_0\, Y^{(2)}(\alpha) &= u_0\, \pi^{(2)}(Y_0) = -\, u_0\, [u_0\, u^{(2)} + \tfrac{1}{2} u^{(1)2}]_{\beta = Y_0}\\
&= \tfrac{1}{2} u_0^2 (Y_0^2 + 2H\, Y_0)\, f_{\alpha\alpha} - u_0^2\, R(\alpha) - \tfrac{1}{2} u_0\, f^2 + \text{const}.
\end{aligned}\right\} \tag{31.12}$$

Since (31.11) and (31.12) must be identical, we may equate them and find that $f(\alpha)$ must satisfy the ordinary differential equation

$$f_{\alpha\alpha} - \frac{9}{2u_0^5} f^2 - \frac{3C}{u_0^6} f + C_0 = 0 \tag{31.13}$$

after having introduced (31.8) to eliminate Y_0. Eq. (31.13) may be integrated to

$$\tfrac{1}{3}\, u_0^6\, f_\alpha^2 - u_0\, f^3 - C\, f^2 + \tfrac{2}{3} u_0^6\, C_0\, f = \text{const}. \tag{31.14}$$

Upon noting from (31.7) that $f(\alpha) = u^{(1)}(\alpha)$ and

$$\frac{u_0}{C} f = -\left[1 + \frac{Y^{(1)}(\alpha)}{C}\right],$$

it is evident that (31.13) and (31.14) are the same equations as obtained by BOUSSINESQ (1871, 1872), RAYLEIGH (1876), KORTEWEG and DE VRIES (1895), LAVRENT'EV (1943), and KELLER (1948). The physical significance of each term in (31.14) was first pointed out by BENJAMIN and LIGHTHILL (1954), who derived (31.14) in an entirely different manner, starting with the same series expansion of the stream function as was introduced by RAYLEIGH (1876). BENJAMIN and LIGHTHILL (1954) use the continuity equation (30.49), the specific-energy equation (30.50), and the specific-momentum equation (30.51) to derive the equivalent of (31.14), and then they give a very useful discussion of the mathematical and physical behavior of its solutions.

The appropriate solution of (31.13) for the boundary conditions shown in Fig. 47 is given by the square of the Jacobian elliptic function "cn" having the modulus $0<k\leqq 1$ and the real period

$$4K(k) = 4\int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} > 2\pi .$$

Substituting

$$f(\alpha) = -B\,\mathrm{cn}^2(A\,\alpha, k) \tag{31.15}$$

into (31.13) we find that (31.15) is a solution if, and *only* if, $0<k\leqq 1$ and

$$B = \frac{4}{3} u_0^5 A^2 k^2 = \frac{C}{u_0}\frac{k^2}{2k^2-1} = \frac{C_0}{2A^2(1-k^2)}\,. \tag{31.16}$$

Substituting (31.15) into (31.7) and (31.5) we obtain

$$\eta(x) = \eta_0 + \omega^2 h^3 B u_0 \left[\mathrm{cn}^2(A\,\omega\, x, k) - \frac{2k^2-1}{k^2}\right] + O(\varepsilon^2)\,.$$

The boundary conditions in Fig. 47 then yield

$$\left.\begin{aligned}
\eta(0) &= \eta_0 + \omega^2 h^3 B u_0 \frac{1-k^2}{k^2} = a,\\
\eta\left(\frac{K}{A\,\omega}\right) &= \eta_0 - \omega^2 h^3 B u_0 \frac{2k^2-1}{k^2} = 0,\\
a &= \omega^2 h^3 B u_0, \qquad \eta_0 = a\frac{2k^2-1}{k^2},\\
\eta(x) &= a\,\mathrm{cn}^2(A\,\omega\, x, k)\,.
\end{aligned}\right\} \tag{31.17}$$

Then upon introducing (31.8) and (31.16) into (31.17) we obtain

$$\left.\begin{aligned}
A\,\omega\, x &= \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\frac{a/h_\infty}{(1+\eta_0/h_\infty)^3}} = \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\frac{a/h_\infty}{[1+(a/h_\infty)(2k^2-1)/k^2]^3}}\\
&= \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\frac{a}{h_\infty}}\left\{1-\frac{3}{2}\frac{a}{h_\infty}\frac{2k^2-1}{k^2}+\cdots\right\}\\
&= \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\frac{a}{h_\infty}} + O\left[\frac{a}{h_\infty}\frac{2k^2-1}{k^2}\right]^{\frac{3}{2}}
\end{aligned}\right\} \tag{31.18}$$

as the exact second-order shallow-water theory solution for the first approximation to the cnoidal waves of Korteweg and de Vries (1895). The remaining terms in (31.6) and (31.7) may similarly be solved to give

$$\left.\begin{aligned}
\frac{p(x,y)}{\varrho g h_\infty} &= \frac{\eta(x)-y}{h_\infty} + O\left[\frac{a}{h_\infty}\frac{2k^2-1}{k^2}\right]^2,\\
\frac{u(x)}{\sqrt{g h_\infty}} &= 1 + \left(1-\frac{1}{2k^2}\right)\frac{a}{h_\infty} - \frac{\eta(x)}{h_\infty} + O\left[\frac{a}{h_\infty}\frac{2k^2-1}{k^2}\right]^2,\\
\frac{v(x,y)}{\sqrt{g h_\infty}} &= -\left(1+\frac{y}{h_\infty}\right)\sqrt{\frac{3}{k^2}\left(\frac{a}{h_\infty}\right)^3}\,\mathrm{cn}(A\,\omega\, x,k)\,\mathrm{sn}(A\,\omega\, x,k)\,\mathrm{dn}(A\,\omega\, x,k) +\\
&\quad + O\left[\frac{a}{h_\infty}\frac{2k^2-1}{k^2}\right]^{\frac{5}{2}}\\
&= +\left(1+\frac{y}{h_\infty}\right)\frac{d\eta(x)}{dx} + O\left[\frac{a}{h_\infty}\frac{2k^2-1}{k^2}\right]^{\frac{5}{2}},
\end{aligned}\right\} \tag{31.19}$$

where cn, sn, dn are the Jacobian elliptic functions with the argument $A\omega x$ defined by (31.18). It must be noted that $0<k\leqq 1$; k can never become identically zero for two reasons. First, because for $k=0$

$$\mathrm{cn}^2(A\,\omega\,x; 0) = \cos^2(A\,\omega\,x)$$

is not a solution of (31.13) or (31.14), and second, because the asymptotic expansions given in (31.18) and (31.19) are only valid as $a^2/k^2 \to 0$.

The limiting case of $k=1$ corresponds to an essentially infinite wavelength since $K(k^2)\to\infty$ as $k^2\to 1$, and the cnoidal-wave solutions reduce to

$$\left.\begin{aligned}
\frac{\eta(x)}{h_\infty} &= \frac{a}{h_\infty}\,\mathrm{sech}^2\left(\frac{x}{h_\infty}\sqrt{\frac{3}{4}\frac{a}{h_\infty}}\right) + O\left(\frac{a}{h_\infty}\right)^2,\\
\frac{p(x,y)}{\varrho\, g\, h_\infty} &= \frac{\eta(x)-y}{h_\infty} + O\left(\frac{a}{h_\infty}\right)^2,\\
\frac{u(x)}{\sqrt{g\,h_\infty}} &= 1 + \frac{1}{2}\frac{a}{h_\infty} - \frac{\eta(x)}{h_\infty} + O\left(\frac{a}{h_\infty}\right)^2,\\
\frac{v(x,y)}{\sqrt{g\,h_\infty}} &= \left(1+\frac{y}{h_\infty}\right)\frac{d\eta(x)}{dx} + O\left(\frac{a}{h_\infty}\right)^{\frac{5}{2}},
\end{aligned}\right\} \qquad (31.20)$$

which provides the exact first approximation to the solitary wave.

All of these solutions for the cnoidal wave and the solitary wave are in exact agreement with the expressions first given by KORTEWEG and DE VRIES (1895, pp. 430—431) if one neglects the terms of $O(a/h_\infty)^2$. It will now be proved that the terms of $O(a/h_\infty)^2$ must be neglected in these first approximations because the second approximations introduce additional terms having this order of magnitude.

We can continue to the next order of approximation by collecting the remaining terms corresponding to ε^3, and adding some of the ε^4 terms that are necessary in order to complete the solution

$$\left.\begin{aligned}
&\pi^{(3)}(Y_0) = Y^{(3)} - Y^{(1)}\pi_\beta^{(2)}(Y_0),\\
&u_\beta^{(3)} = v_\alpha^{(3)}, \qquad u_\alpha^{(3)} = -v_\beta^{(4)},\\
&u_0\,u_\alpha^{(3)} + u^{(1)}u_\alpha^{(2)} + u^{(2)}u_\alpha^{(1)} + \pi_\alpha^{(3)} + v^{(2)}u_\beta^{(2)} = 0,\\
&u_0\,v_\alpha^{(3)} + u^{(1)}v_\alpha^{(2)} + \pi_\beta^{(3)} + v^{(2)}v_\beta^{(2)} = 0,\\
&v^{(4)}(Y_0) = u_0\,Y_\alpha^{(3)} + u^{(1)}\,Y_\alpha^{(2)} + u^{(2)}\,Y_\alpha^{(1)} - v_\beta^{(2)}\,Y^{(2)} - v_\beta^{(3)}\,Y^{(1)},\\
&v^{(4)}(-H) = 0.
\end{aligned}\right\} \qquad (31.21)$$

Now we can combine the expression for $v^{(3)}$ in (31.10) with that in (31.21) to write

$$\left.\begin{aligned}
u^{(3)}(\alpha,\beta) = \int v_\alpha^{(3)}\,d\beta = \tfrac{1}{24}(\beta^4 + 4H\beta^3 - 8H^3\beta)\,f_{\alpha\alpha\alpha\alpha} -\\
- \tfrac{1}{2}(\beta^2 + 2H\beta)\,R_{\alpha\alpha} + S(\alpha).
\end{aligned}\right\} \qquad (31.22)$$

Then the expression for $v^{(4)}$ in (31.21) yields

$$\left.\begin{aligned}
v^{(4)}(\alpha,\beta) = -\int_{-H}^{\beta} u_\alpha^{(3)}(\alpha,\beta)\,d\beta = -\,[(\beta+H)\,S_\alpha +\\
+ \tfrac{1}{120}(\beta^5 + 5H\beta^4 - 20H^3\beta^2 + 16H^5)\,f_{\alpha\alpha\alpha\alpha\alpha} - \tfrac{1}{6}(\beta^3 + 3H\beta^2 - 2H^3)\,R_{\alpha\alpha\alpha}].
\end{aligned}\right\} \qquad (31.23)$$

The boundary condition defined by the expression for $v^{(4)}$ in (31.21) thereby gives one relation for $Y^{(3)}$ that may be written as

$$\left.\begin{aligned} u_0 Y^{(3)}_\alpha(\alpha) &= v^{(4)}(Y_0) - [u^{(1)} Y^{(2)}_\alpha - v^{(2)}_\beta Y^{(2)}] - [u^{(2)} Y^{(1)}_\alpha - v^{(3)}_\beta Y^{(1)}] \\ &= v^4(Y_0) - [u^{(1)} Y^{(2)}]_\alpha - [u^{(2)} Y^{(1)}]_\alpha \end{aligned}\right\} \quad (31.24)$$

which may be directly integrated, upon substituting (31.23) for $v^{(4)}$, as

$$\left.\begin{aligned} u_0 Y^{(3)}_\alpha &= \int v^{(4)}(Y_0)\, d\alpha - u^{(1)}(Y_0)\, Y^{(2)} - u^{(2)}(Y_0)\, Y^{(1)} \\ &= \text{const} - \left\{\frac{1}{120}(Y_0^5 + 5H Y_0^4 - 20H^3 Y_0^2 + 16H^5) f_{\alpha\alpha\alpha\alpha} + \right. \\ &\quad - \frac{1}{6}(Y_0^3 + 3H Y_0^2 - 2H^3) R_{\alpha\alpha} + (Y_0 + H)\, S(\alpha) + \\ &\quad + \frac{f}{u_0}\left[\frac{1}{2} u_0^2 (Y_0^2 + 2H Y_0) f_{\alpha\alpha} - u_0^2 R - \frac{1}{2} u_0 f^2\right] + \\ &\quad \left. + \left[\frac{1}{2}(Y_0^2 + 2H Y_0) f_{\alpha\alpha} - R(\alpha)\right][u_0 f + C]\right\}. \end{aligned}\right\} \quad (31.25)$$

Another relation for $Y^{(3)}$ may also be obtained from the other boundary condition defined by the expression for $\pi^{(3)}$ in (31.21), namely

$$Y^{(3)}(\alpha) = \pi^{(3)}(Y_0) + Y^{(1)} \pi^{(2)}_\beta (Y_0), \quad (31.26)$$

where $\pi^{(3)}$ itself may be obtained by integrating the expressions for $\pi^{(3)}_\alpha$ and $\pi^{(3)}_\beta$ in (31.21) to obtain

$$\pi^{(3)}(\alpha, \beta) = - u_0 u^{(3)} - u^{(1)} u^{(2)} - \tfrac{1}{2}[v^{(2)}]^2 + \text{const}. \quad (31.27)$$

Then substituting $\pi^{(3)}$ from (31.27), $\pi^{(2)}_\beta$ from (31.10), $u^{(1)} = f$, $u^{(2)}$ from (31.10), $u^{(3)}$ from (31.22), $y^{(1)} = -(u_0 f + C)$ and $Y^{(2)}$ from (31.12) into (31.26) we obtain another relation for $Y^{(3)}$, namely,

$$\left.\begin{aligned} u_0 Y^{(3)}(\alpha) &= -\{\tfrac{1}{24} u_0^2 (Y_0^4 + 4H Y_0^3 - 8H^3 Y_0) f_{\alpha\alpha\alpha\alpha} - \tfrac{1}{2} u_0^2 (Y_0^2 + 2H Y_0) R_{\alpha\alpha} + \\ &\quad + u_0^2 S - \tfrac{1}{2} u_0 (Y_0^2 + 2H Y_0) f f_\alpha + u_0 f R + \tfrac{1}{2} u_0^5 (f_\alpha)^2 + \\ &\quad + (u_0 f + C)\, u_0^4 f_{\alpha\alpha} + \text{const}\}. \end{aligned}\right\} \quad (31.28)$$

These two expressions for $Y^{(3)}$, (31.25) and (31.28), must be identically equal; therefore, since u_0 is defined by (31.8), we find that the unknown function R must satisfy the ordinary differential equation

$$\left.\begin{aligned} &\frac{1}{3} u_0^5 R_{\alpha\alpha} - \left(\frac{C}{u_0} + 3f\right) R + \text{const} \\ &\qquad = \frac{1}{30} u_0^5 (u_0^4 - 5H^2) f_{\alpha\alpha\alpha\alpha} - \frac{1}{2}(u_0^4 - 3H^2) f f_{\alpha\alpha} + \\ &\qquad\quad + \frac{C}{2u_0}(u_0^4 + H^2) f_{\alpha\alpha} + \frac{1}{2} u_0^4 (f_\alpha)^2 + \frac{1}{2u_0} f^3, \end{aligned}\right\} \quad (31.29)$$

the other unknown function $S(\alpha)$ having been eliminated since $u_0^2 = Y_0 + H$.

When $f(\alpha)$ is given by (31.15), then the solution of (31.29) is

$$\left.\begin{aligned} R(\alpha) = \frac{C^2}{u_0^3}\left\{\left(\frac{k^2}{2k^2-1}\right)^2\left(1-\frac{9}{4}\frac{H^2}{u_0^4}\right)\mathrm{cn}^4(A\alpha,k) + \right.\\ \left. + \frac{k^2}{2k^2-1}\left(1+\frac{3}{2}\frac{H^2}{u_0^4}\right)\mathrm{cn}^2(A\alpha,k) - \frac{3}{10}\frac{k^2(1-k^2)}{(2k^2-1)^2}\left(1-\frac{5}{2}\frac{H^2}{u_0^4}\right) - \frac{3}{5}\right\}\end{aligned}\right\} \tag{31.30}$$

and (31.11) or (31.12) give the ε^2 term of the wave profile as

$$\left.\begin{aligned} Y^{(2)}(\alpha) = \left(\frac{C}{u_0}\right)^2\left\{\frac{3}{4}\left(\frac{k^2}{2k^2-1}\right)^2\mathrm{cn}^4(A\alpha,k) - \right.\\ \left. - \frac{5}{2}\frac{k^2}{2k^2-1}\mathrm{cn}^2(A\alpha,k) + \frac{12-57k^2+57k^4}{20(2k^2-1)^2}\right\}.\end{aligned}\right\} \tag{31.31}$$

Consequently, the second approximation to the cnoidal-wave profile is obtained from the preceding and (31.5) as

$$\left.\begin{aligned} \eta(x) &= \eta_0 + \omega^2 h^3\, Y^{(1)} + \omega^4 h^5\, Y^{(2)} + O(\varepsilon^3)\\ &= \eta_0 - \eta_1\left[1 - \frac{k^2}{2k^2-1}\mathrm{cn}^2(A\omega x,k)\right] + \\ &\quad + \frac{\eta_1^2}{\eta_0+h_\infty}\left[\frac{3}{4}\left(\frac{k^2}{2k^2-1}\right)^2\mathrm{cn}^4(A\omega x,k) - \frac{5}{2}\frac{k^2}{2k^2-1}\mathrm{cn}^2(A\omega x,k) + \right.\\ &\quad \left. + \frac{12-57k^2+57k^4}{20(2k^2-1)^2}\right],\end{aligned}\right\} \tag{31.32}$$

where

$$\begin{aligned}\eta_1 = C\omega^2 h^3 &= (A\omega)^2\,\tfrac{4}{3}\,(2k^2-1)\,(h\,u_0^2)^3\\ &= (A\omega)^2\,\tfrac{4}{3}\,(2k^2-1)\,(\eta_0+h_\infty)^3.\end{aligned}$$

Then the boundary conditions shown in Fig. 37c yield the relations:

$$\left.\begin{aligned} \eta(0) = a = \eta_0 - \eta_1\frac{k^2-1}{2k^2-1} + \frac{\eta_1^2}{\eta_0+h_\infty}\,\frac{12-7k^2-28k^4}{20(2k^2-1)^2},\\ \eta\left(\frac{K}{A\omega}\right) = 0 = \eta_0 - \eta_1 + \frac{\eta_1^2}{\eta_0+h_\infty}\left[\frac{12-57k^2+57k^4}{20(2k^2-1)^2}\right],\end{aligned}\right\} \tag{31.33}$$

which may be solved to give the second approximation

$$\left.\begin{aligned} \frac{\eta_0}{h_\infty} &= \left(\frac{a}{h_\infty}\right)\frac{2k^2-1}{k^2} + \left(\frac{a}{h_\infty}\right)^2\frac{38-128k^2+113k^4}{20k^4} + O\left(\frac{a}{h_\infty}\right)^3,\\ \frac{\eta_1}{h_\infty} &= \frac{2k^2-1}{k^2}\left(\frac{a}{h_\infty}\right)\left[1+\left(\frac{a}{h_\infty}\right)\left(\frac{85k^2-50}{20k^2}\right)\right] + O\left(\frac{a}{h_\infty}\right)^3,\\ \frac{\eta(x)}{h_\infty} &= \left(\frac{a}{h_\infty}\right)\mathrm{cn}^2(A\omega x,k) - \frac{3}{4}\left(\frac{a}{h_\infty}\right)^2\mathrm{cn}^2(A\omega x,k)\times\\ &\quad\times[1-\mathrm{cn}^2(A\omega x,k)] + O\left(\frac{a}{h_\infty}\right)^3,\end{aligned}\right\} \tag{31.34}$$

where now

$$\left.\begin{aligned} A\omega x &= \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\left(\frac{a}{h_\infty}\right)}\left(1+\frac{\eta_0}{h_\infty}\right)^{-\frac{3}{2}}\left[1+\left(\frac{a}{h_\infty}\right)\frac{85k^2-50}{40k^2}\right]\\ &= \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\left(\frac{a}{h_\infty}\right)}\left[1-\left(\frac{a}{h_\infty}\right)\frac{7k^2-2}{8k^2}\right] + O\left[\left(\frac{a}{h_\infty}\right)\frac{2k^2-1}{k^2}\right]^{\frac{5}{2}}.\end{aligned}\right\} \tag{31.35}$$

The remaining ε^2 terms from (31.10) may then be combined with the ε terms from (31.7), by means of (31.5), to give

$$\left.\begin{aligned}
\frac{p(x,y)}{\varrho g h_\infty} &= \frac{\eta(x)-y}{h_\infty} - \left(\frac{a}{h_\infty}\right)^2 \frac{3}{4k^2}\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right)\left[1-k^2+\right.\\
&\quad \left. +2(2k^2-1)\,\mathrm{cn}^2(A\,\omega\,x,k) - 3k^2\,\mathrm{cn}^4(A\,\omega\,x,k)\right] + O\left[\left(\frac{a}{h_\infty}\right)\frac{2k^2-1}{k^2}\right]^3,\\
\frac{u(x,y)}{\sqrt{g h_\infty}} &= 1+\left(\frac{a}{h_\infty}\right)\left(1-\frac{1}{2k^2}\right) - \left(\frac{a}{h_\infty}\right)^2 \frac{21k^4-6k^2-9}{40k^4} + \\
&\quad -\left(\frac{a}{h_\infty}\right)\left[1-\left(\frac{a}{h_\infty}\right)\frac{7k^2-2}{4k^2} - \left(\frac{a}{h_\infty}\right)\frac{3}{2}\left(2-\frac{1}{k^2}\right)\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right)\right]\mathrm{cn}^2(A\,\omega\,x,k) -\\
&\quad -\left(\frac{a}{h_\infty}\right)^2\left[\frac{5}{4}+\frac{9}{4}\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right)\right]\mathrm{cn}^4(A\,\omega\,x,k) + \\
&\quad +\left(\frac{a}{h_\infty}\right)^2\frac{3}{4}\left(\frac{1}{k^2}-1\right)\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right) + O\left[\left(\frac{a}{h_\infty}\right)\frac{2k^2-1}{k^2}\right]^3,\\
\frac{v(x,y)}{\sqrt{g h_\infty}} &= -\sqrt{\frac{3}{k^2}\left(\frac{a}{h_\infty}\right)^3}\left(1+\frac{y}{h_\infty}\right)\mathrm{cn}(A\,\omega\,x,k)\,\mathrm{sn}(A\,\omega\,x,k)\,\mathrm{dn}(A\,\omega\,x,k)\times\\
&\quad \times\left\{1-\left(\frac{a}{h_\infty}\right)\left(\frac{5k^2+2}{8k^2}\right) - \left(\frac{a}{h_\infty}\right)\left(1-\frac{1}{2k^2}\right)\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right) - \right.\\
&\quad \left. -\frac{1}{2}\left(\frac{a}{h_\infty}\right)\left(1-6\frac{y}{h_\infty}-3\frac{y^2}{h_\infty^2}\right)\mathrm{cn}^2(A\,\omega\,x,k)\right\} + O\left[\left(\frac{a}{h_\infty}\right)\left(\frac{2k^2-1}{k^2}\right)\right]^{\frac{7}{2}}.
\end{aligned}\right\}\quad(31.36)$$

For the solitary wave we have $k=1$ and essentially infinite wavelength, so that (31.36) reduces to

$$\left.\begin{aligned}
\frac{\eta(x)}{h_\infty} &= \frac{a}{h_\infty}\,\mathrm{sech}^2(A\,\omega\,x) - \frac{3}{4}\left(\frac{a}{h_\infty}\right)^2\mathrm{sech}^2(A\,\omega\,x)\times\\
&\quad \times\left[1-\mathrm{sech}^2(A\,\omega\,x)\right] + O\left(\frac{a}{h_\infty}\right)^3,\\
A\,\omega\,x &= \frac{x}{h_\infty}\sqrt{\frac{3}{4}\left(\frac{a}{h_\infty}\right)}\left\{1-\frac{5}{8}\left(\frac{a}{h_\infty}\right)\right\} + O\left(\frac{a}{h_\infty}\right)^{\frac{5}{2}},\\
\frac{p}{\varrho g h_\infty} &= \frac{\eta(x)-y}{h_\infty} - \left(\frac{a}{h_\infty}\right)^2\frac{3}{4}\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right)\times\\
&\quad \times\left[2\,\mathrm{sech}^2(A\,\omega\,x) - 3\,\mathrm{sech}^4(A\,\omega\,x)\right] + O\left(\frac{a}{h_\infty}\right)^3,\\
\frac{u(x,y)}{\sqrt{g h_\infty}} &= 1+\frac{1}{2}\left(\frac{a}{h_\infty}\right) - \frac{3}{20}\left(\frac{a}{h_\infty}\right)^2 - \frac{\eta(x)}{h_\infty} + \\
&\quad +\frac{1}{2}\left(\frac{a}{h_\infty}\right)^2\left[1+6\left(\frac{y}{h_\infty}\right)+3\left(\frac{y}{h_\infty}\right)^2\right]\mathrm{sech}^2(A\,\omega\,x) + \\
&\quad -\frac{1}{2}\left(\frac{a}{h_\infty}\right)^2\left[1+9\left(\frac{y}{h_\infty}\right)+\frac{9}{2}\left(\frac{y}{h_\infty}\right)^2\right]\mathrm{sech}^4(A\,\omega\,x) + O\left(\frac{a}{h_\infty}\right)^3,\\
\frac{v(x,y)}{\sqrt{g h_\infty}} &= -\sqrt{3}\left(\frac{a}{h_\infty}\right)^{\frac{3}{2}}\left(1+\frac{y}{h_\infty}\right)\mathrm{sech}^2(A\,\omega\,x)\tanh(A\,\omega\,x)\times\\
&\quad \times\left\{1-\frac{7}{8}\left(\frac{a}{h_\infty}\right) - \left(\frac{a}{h_\infty}\right)\left(\frac{y}{h_\infty}+\frac{1}{2}\frac{y^2}{h_\infty^2}\right) - \frac{1}{2}\left(\frac{a}{h_\infty}\right)\times\right.\\
&\quad \left.\times\left(1-6\frac{y}{h_\infty}-3\frac{y^2}{h_\infty^2}\right)\mathrm{sech}^2(A\,\omega\,x)\right\} + O\left(\frac{a}{h_\infty}\right)^{\frac{7}{2}}.
\end{aligned}\right\}\quad(31.37)$$

The celerity or propagation velocity c of a solitary wave is defined by (31.37) as the constant uniform motion attained as $x\to\infty$,

$$\frac{c}{\sqrt{g h_\infty}} = \frac{u(\infty)}{\sqrt{g h_\infty}} = 1+\frac{1}{2}\left(\frac{a}{h_\infty}\right) - \frac{3}{20}\left(\frac{a}{h_\infty}\right)^2 + O\left(\frac{a}{h_\infty}\right)^3. \qquad (31.38)$$

In Fig. 48, Eq. (31.38) is shown to be in better agreement with recent experimental data than is the commonly used Boussinesq (1871)-Rayleigh (1876) propagation velocity given by

$$\frac{c}{\sqrt{g h_\infty}} \approx \sqrt{1+\left(\frac{a}{h_\infty}\right)} \approx 1+\frac{1}{2}\left(\frac{a}{h_\infty}\right)-\frac{1}{8}\left(\frac{a}{h_\infty}\right)^2+\cdots.$$

The past success of the Boussinesq-Rayleigh equation, as opposed to the propagation velocities derived by McCowan (1891), as indicated in Fig. 48, is easily explained when one notices the close numerical agreement of the coefficients of the Boussinesq-Rayleigh equation with the exact second approximation given by (31.38).

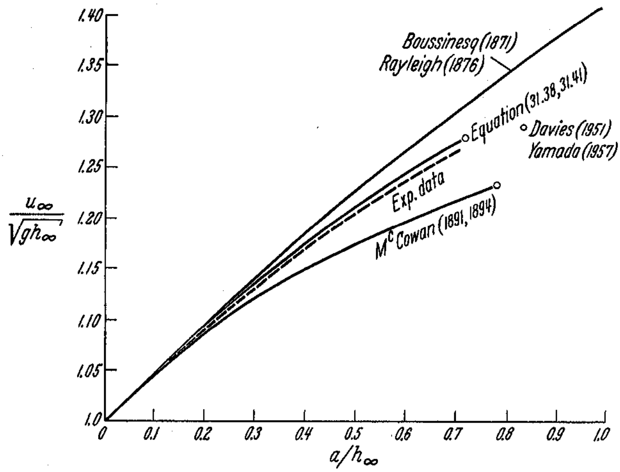

Fig. 48. Propagation velocity of solitary waves.

A comparison of the second approximations with the first approximations to the cnoidal waves proves conclusively that only the proper order of a/h_∞ must be retained for each order of approximation. For example, a comparison of (31.18) with (31.35) shows that a completely erroneous second approximation would be obtained by trying to extend the first approximation to include an additional a/h_∞ term. The reason for this is evident upon comparing the first and second approximations for η_0 in (31.17) and (31.34). Each successive approximation directly affects all the coefficients of the corresponding a/h_∞ terms. Fig. 49 shows the effect of the second approximation on a solitary wave.

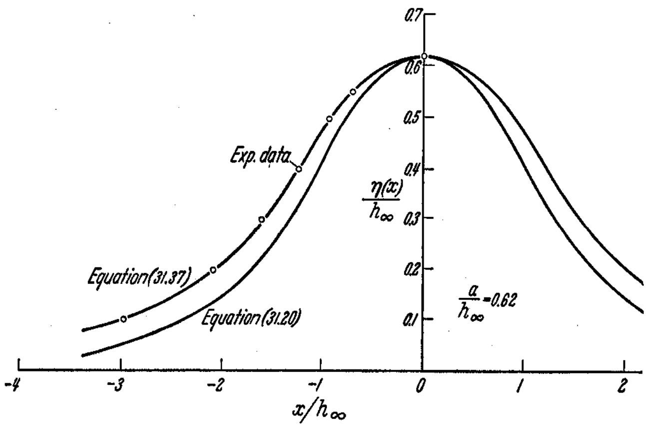

Fig. 49. Comparison of first (31.17) and second (31.37) approximation to the solitary-wave profile, $\eta(x)$.

Of course it must be remembered that the expansion method of Friedrichs (1948), which was used to obtain all the preceding results, is applicable only to shallow water, or long-wavelength wave propagations. However, this is precisely the nature of the solitary wave, especially if the amplitude a/h_∞ is relatively small, since its wavelength, as a member of the family of cnoidal waves, is essentially infinite since $K(k^2)\to\infty$ when $k\to 1$. Also, Friedrichs and Hyers (1954) proved that this expansion method does yield an existence proof for the solitary wave, and thereby demonstrate that it will at least provide asymptotic descriptions of the exact solution for the solitary-wave problem. The corresponding existence proof for cnoidal waves (in the neighborhood of the critical speed defined by a Froude number of unity) was given by LITTMAN (1957). Again this justified the Friedrichs expansion method, at least as an asymptotic type of series development. An additional discussion of these existence proofs is given in Sect. 35.

β) The limiting height and velocity of propagation of cnoidal and solitary waves. It is interesting to note that with the second approximation the pressure is still hydrostatic for $y \approx 0$, but is no longer hydrostatic as the bottom $(y = -h_\infty)$ is approached. Similarly, the variation of the horizontal velocity component with depth below the surface becomes important in the second approximation only upon approaching the flat horizontal bottom. However, the finite vertical velocity component is now seen to be the principal variation from the basic assumptions of first-order shallow-water theory. The first approximation given in (31.19) gives a monotonic variation in $v(y)$ that is obviously necessary from physical considerations in order to satisfy the continuity equation. However, this monotonic variation in $v(y)$ is of the higher order $(a/h_\infty)^{\frac{3}{2}}$ so that it can be neglected in the first-order equations (28.2) as long as the resulting local variations in η are sufficiently small.

The second approximation to the vertical velocity component, as given in (31.36), now shows that the variation of $v(y)$ will no longer be monotonic as a/h_∞ increases. This leads one to suspect that there is a limiting value to a/h_∞ for cnoidal and solitary waves. For example, (31.36) shows that in the neighborhood of the wave crest, where $x \approx 0$ and

$$\mathrm{cn}^2 (A\,\omega\, x) \approx 1 - (A\,\omega\, x)^2 = 1 - \frac{3}{4k^2}\left(\frac{a}{h_\infty}\right)\left(\frac{x}{h_\infty}\right)^2 + O\left(\frac{a}{h_\infty}\right)^2,$$

$v(y)$ actually has a reversal in its direction if a/h_∞ exceeds the value given by

$$\left(\frac{a}{h_\infty}\right)_{\max} = \frac{8k^2}{9k^2+2} \tag{31.39}$$

for any value of $y \geqq 0$.

This limiting value can be substantiated, at least in the limit as $k \to 1$, by noting that (31.33) has a real solution for η_1 only if

$$\frac{2k^2-1}{k^2}\,\frac{a}{h_\infty} < \frac{\eta_0}{h_\infty} \leqq \frac{5(2k^2-1)^2}{7-37k^2(1-k^2)},$$

leading to a limiting value of

$$\left(\frac{a}{h_\infty}\right)_{\max} < \frac{5k^2(2k^2-1)}{7-37k^2(1-k^2)}. \tag{31.40}$$

The most interesting application of these results is to the solitary wave, defined by $k=1$, in which case we find from (31.38), (31.39) and (31.40) that the limiting heights and the corresponding total velocity at infinity are given by

$$\left(\frac{a}{h_\infty}\right)_{\max} = \frac{8}{11} = 0.7273 > \frac{5}{7} = 0.7143, \qquad \left[\frac{u(\infty)}{\sqrt{g\,h}}\right]_{\max} = 1.284 > 1.281. \tag{31.41}$$

Either of these limiting heights would be satisfactory for a solitary wave since recent experimental investigations by IPPEN and KULIN (1955), DAILY and STEPHAN (1952), and PERROUD (1957) have shown that under properly controlled conditions most solitary waves have $a/h_\infty < 0.7$, the maximum recorded value being 0.72. Not only are the limiting values given by (31.39) or (31.40) in excellent agreement with recent experimental data, but they are consistent with the order of approximation involved. The value $\frac{8}{11}$ is derived from the vertical velocity variation given to the order $(a/h_\infty)^{\frac{5}{2}}$ by (31.36), while the value $\frac{5}{7}$ corresponds to the terms governed by ε^2 or $(a/h_\infty)^2$ in (31.32).

Many attempts have been made to determine the limiting height of a solitary wave. However, nearly all of the theoretical calculations have been based on STOKES' (1880, p. 227) relation which assumes that for the limiting heights of any wave the wave crest must form a sharp peak or corner having an enclosed angle of 120° in order to reduce the relative local velocity to zero at the crest itself [see, e.g., Sect. 33 or LAMB (1932, p. 418)]. This 120° enclosed angle at the wave crest was assumed by MCCOWAN (1894), STOKES (1905), GWYTHER (1900), DAVIES (1952), PACKHAM (1952), GOODY and DAVIES (1957) and YAMADA (1957). Several of these values are compared in Fig. 48 with experimental data, and with the theoretical values given by (31.38) and (31.41). It is seen that none of these limiting heights for solitary waves are in as good an agreement with the experimental data as is (31.41). A reasonable explanation of the failure of the 120° sharp crest wave to provide a satisfactory limiting height for a solitary wave may be obtained by noting that KORTEWEG and DE VRIES (1895) proved that any finite-amplitude profile that did not correspond to (31.17) or (31.20) would not be steady with respect to time. Consequently, (31.37) defines the only possible steady-state solitary wave, and when $a/h_\infty > \frac{8}{11}$ the vertical velocity variation reverses its direction near the crest. This probably leads to an unsteady wave crest that breaks unsymmetrically.

Eqs. (31.38) or (32.52) show that the solitary wave occurs only in supercritical flow since the Froude number corresponding to the propagation velocity is always greater than unity. Its velocity of propagation is always less than that of the corresponding hydraulic jump of the same height as may be seen by comparing (30.55) with (31.38), after expanding it in powers of $a/d_1 = a/h_\infty$:

$$F_1 = \frac{u_1}{\sqrt{g\,h_\infty}} = 1 + \frac{3}{4}\left(\frac{a}{h_\infty}\right) - \frac{1}{32}\left(\frac{a}{h_\infty}\right)^2 + O\left(\frac{a}{h_\infty}\right)^3. \tag{30.55'}$$

However, the cnoidal wave can occur in subcritical as well as in supercritical flow, and as shown by BENJAMIN and LIGHTHILL (1954), the undulating flow in the subcritical region behind a hydraulic jump produced at all Froude numbers less than $\sqrt{3}$ may well be represented by these cnoidal waves. The fact that cnoidal waves can form in subcritical flow is easily shown, even in the first approximation, by writing the horizontal velocity component from (31.19) as

$$\left.\begin{aligned} \frac{u(0)}{\sqrt{g\,h_\infty}} &= 1 - \frac{1}{2k^2}\left(\frac{a}{h_\infty}\right) < 1 \qquad &&\text{for all } k \leq 1,\\ \frac{u\left(\frac{K}{A\omega}\right)}{\sqrt{g\,h_\infty}} &= 1 + \left(1 - \frac{1}{2k^2}\right)\left(\frac{a}{h_\infty}\right) < 1 \qquad &&\text{for } k^2 < \frac{1}{2}. \end{aligned}\right\} \tag{31.42}$$

Therefore (31.19) shows that any definition of the wave propagation velocity would be subcritical when $k^2 < \frac{1}{2}$. STOKES (1847) (see Sect. 7) has given two logical definitions of the celerity or propagation velocity of permanent periodic wave forms, and each one would define a critical celerity corresponding to a different value of k, varying as $\frac{1}{2} \leq k^2 < 1$, the solitary wave ($k = 1$) being always supercritical for a finite amplitude. However, the existence proof for cnoidal waves by LITTMAN (1957) is only valid for average velocities (defined as the velocity of the vertical plane that would have zero average flux across it) that are near critical. An interesting physical and mathematical explanation of these flow restrictions is given by BENJAMIN and LIGHTHILL (1954). The main consideration, as shown in Fig. 56 on page 754 and Sect. 35, is that the finite-amplitude periodic waves corresponding to $k^2 < 0.9$ may be better described by using infinitesimal wave theory. This becomes necessary because the wavelength of the cnoidal waves decreases rapidly with k^2 when k is near unity. Fig. 56 indicates that not only must the wavelength be large compared to the water depth, in order to satisfy the shallow-water expansion method, but also the amplitude of the cnoidal wave must become extremely small for values of $k^2 < 0.9$, or for subcritical flow.

KORTEWEG and DE VRIES (1895) have also shown how negative cnoidal or solitary waves can be formed when the water is very shallow and surface tension T is considered. Their correct first approximation may be written as

$$\left.\begin{aligned} \frac{\eta(x)}{h_\infty} &= \pm\left(\frac{a}{h_\infty}\right)\operatorname{cn}^2(A\,\omega\,x, k), \\ A\,\omega\,x &= \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\,\frac{a/h_\infty}{|1 - 3T/\varrho g h_\infty^2|}}, \end{aligned}\right\} \tag{31.43}$$

where the negative algebraic value is assigned to the surface profile whenever

$$h_\infty < \sqrt{\frac{3T}{\varrho g}} \approx \frac{1}{2}\,\text{cm} \qquad \text{for water}. \tag{31.44}$$

These negative waves have a very small amplitude and a very large wavelength, but can create a surprising particle motion. It is interesting to notice that the depth of $h_\infty = \frac{1}{2}$ cm, which, if it could be maintained, would eliminate both solitary and cnoidal waves, is the same depth found from (31.1′) and (31.2) to give nearly the same value of $\sqrt{g h_\infty}$ for both the propagation velocity and the group velocity of infinitesimal waves (also see Sect. 15). Consequently the depth of $\frac{1}{2}$ cm seems to be the optimum for ordinary water ($T = 72.8$ dynes/cm) whenever one uses small models to simulate results appropriate to the first-order shallow-water theory of (28.2), since this particular depth minimizes the effect of group velocity and variation with wavelength for the infinitesimal waves, and minimizes the second-order effect due to the existence of finite-amplitude cnoidal or solitary waves. However, the variation of η must remain sufficiently small since a finite increase in η above $h_\infty = \frac{1}{2}$ cm could still produce cnoidal or solitary waves. Also, the short-wavelength or capillary ripples that will form must be neglected in these model tests.

F. Exact solutions.

The word "exact" in this context is generally understood to mean solutions in which there has been no approximation in the equations or boundary conditions. However, this usage of the word does not exclude neglect of viscosity and, in fact, since positive results have been obtained only for perfect fluids, the treatment below will be restricted to them. Indeed, the present results in the theory of exact solutions are restricted, with few exceptions, to a very special class of motions, namely, those which can be represented as steady two-dimensional flows.

In Sect. 32 some general theorems will be established. In Sect. 33 waves of maximum amplitude-to-length ratio are discussed; because the methods are intimately related, we have also included in this section a discussion of HAVELOCK's method of approximating periodic waves. Sect. 34 treats methods of obtaining explicit exact solutions and of various ones which have been obtained. In Sect. 35, the last, existence theorems are discussed, but only in a descriptive way, for proofs are highly technical and lengthy.

32. Some general theorems. This section will be devoted to several theorems of a rather general nature concerning the motion of a fluid with free surface in a gravitational field. The theorems in subsection 32α are mostly of a kinematical nature and are associated with the phenomenon of mass transport already discussed in subsection 27α. The last part of this section is devoted to several theorems on energy and momentum. In subsection 32β some theorems concerning waves in heterogeneous fluids will be established. In subsection 32γ several different ways of formulating the problem of motion with a free surface will be described.

see errata

α) *Kinematical theorems—mass transport—energy integrals.* The first theorem, due to M. S. LONGUET-HIGGINS (1953), is independent of the presence of a free surface or of the nature of the time dependence. Let $f(z)=\Phi+i\Psi$ describe a space-periodic motion, i.e. $f(z+n\lambda)=f(z)$. The definition of φ will be normalized so that

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$$\int_0^\lambda \Phi(x, y, t)\,dx=0. \tag{32.1}$$

Note that if this condition holds for one value of y, it holds for all since

$$\frac{\partial}{\partial y}\int_0^\lambda \Phi\,dx=\int_0^\lambda \Phi_y\,dx=-\int_0^\lambda \Psi_x\,dx=-\Psi(\lambda, y, t)+\Psi(0, y, t)=0.$$

In Eq. (2.10′) we shall write

$$\frac{p}{\varrho}=\frac{p_0}{\varrho}-g\,y+\frac{p_d}{\varrho}, \qquad \frac{p_d}{\varrho}=A(t)-\Phi_t-\frac{1}{2}(u^2+v^2). \tag{32.2}$$

In the following we define an average by

$$\overline{F}(y, t)=\frac{1}{\lambda}\int_0^\lambda F(x, y, t)\,dx. \tag{32.3}$$

Theorem. In a non-uniform space-periodic motion $\overline{u^2}$, $\overline{v^2}$, $-\overline{p_d}$ each decrease with increasing depth, provided either $\Phi_y(x, -h, t)=0$ or $\lim_{y\to-\infty}\Phi_y=0$.

This may be proved as follows. Consider first $q^2=u^2+v^2$. Then

$$\left.\begin{aligned}\frac{\partial}{\partial y}\overline{q^2}&=\frac{\partial}{\partial y}\frac{1}{\lambda}\int_0^\lambda(\Phi_x^2+\Phi_y^2)\,dx=\frac{2}{\lambda}\int_0^\lambda(\Phi_x\Phi_{xy}+\Phi_y\Phi_{yy})\,dx\\&=\frac{2}{\lambda}\int_0^\lambda[(\Phi_y\Phi_x)_x-2\Phi_y\Phi_{xx}]\,dx\\&=\frac{2}{\lambda}[\Phi_y\Phi_x]_0^\lambda-\frac{4}{\lambda}\int_0^\lambda\Phi_y\Phi_{xx}\,dx\\&=-\frac{4}{\lambda}\int_0^\lambda\Phi_y\Phi_{xx}\,dx.\end{aligned}\right\} \tag{32.4}$$

By a similar computation it follows that

$$\frac{\partial^2}{\partial y^2}\overline{q^2} = \frac{4}{\lambda}\int_0^\lambda (\Phi_{xx}^2 + \Phi_{xy}^2)\,dx > 0, \tag{32.5}$$

since we have assumed that Φ_x is not constant. It is evident from (32.4) that, if the fluid is bounded below by $y = -h$, then

$$\frac{\partial}{\partial y}\overline{q^2}(-h, t) = 0; \tag{32.6}$$

if it is infinitely deep, it is an assumed boundary condition that $\Phi_y \to 0$ as $y \to -\infty$ and hence

$$\frac{\partial}{\partial y}\overline{q^2} \to 0 \quad \text{as} \quad y \to -\infty. \tag{32.7}$$

In either case it then follows from (32.5) that $\partial q^2/\partial y$ is an increasing function of y and hence

$$\frac{\partial}{\partial y}\overline{q^2} \geqq 0, \tag{32.8}$$

with equality occurring only for $y = -h$. In fact, even more can be concluded, for (32.5) is like $\overline{q^2}$ itself with Φ replaced by $2\Phi_x$. Hence, by repeating the above reasoning one may establish that

$$\frac{\partial^{2n}}{\partial y^{2n}}\overline{q^2} > 0, \qquad \frac{\partial^{2n-1}}{\partial y^{2n-1}}\overline{q^2} \geqq 0, \qquad n = 1, 2, \ldots. \tag{32.9}$$

Next consider $\overline{u^2} - \overline{v^2}$. A similar computation shows that

$$\frac{\partial}{\partial y}(\overline{u^2} - \overline{v^2}) = \frac{2}{\lambda}\int_0^\lambda (\Phi_x \Phi_y)_x\,dx = \frac{2}{\lambda}[\Phi_x \Phi_y]_0^\lambda = 0.$$

Hence

$$\overline{u^2} - \overline{v^2} = C(t) = \overline{u^2}|_{y=-h \text{ or } -\infty}. \tag{32.10}$$

It follows from (32.8) that

$$\overline{u^2} = \tfrac{1}{2}[\overline{q^2} + C], \qquad \overline{v^2} = \tfrac{1}{2}[\overline{q^2} - C], \qquad -\overline{p_d} = \tfrac{1}{2}\overline{q^2} - A(t) \tag{32.11}$$

are each increasing functions of y, i.e., they decrease with increasing depth. For infinite depth LONGUET-HIGGINS shows further that, if axes are chosen such that $u = 0$ at $y = -\infty$, then the quantites $|u|$, $|v|$ and $|p_d|$ all decrease exponentially to zero. He had shown earlier (1950) for exact waves (we shall not carry through the proof) that

$$\overline{p_d} = \frac{1}{2}\frac{\partial^2}{\partial t^2}\overline{\eta^2} - \overline{v^2}. \tag{32.12}$$

Hence it follows that

$$p_d|_{y=-h \text{ or } -\infty} = \frac{1}{2}\frac{\partial^2}{\partial t^2}\overline{\eta^2}. \tag{32.13}$$

For purely progressive waves this quantity vanishes, but we recall that for standing waves we found earlier a constant pressure fluctuation of double the wave frequency [see (27.62) and (27.65)] if second-order terms were retained.

Mass transport. In Sect. 27 [see (27.39) and (27.41)] it was shown that a forward drift, called "mass transport", occurred in progressive waves if second-order terms were taken into account. It was shown by RAYLEIGH (1876) in a proof valid only for infinitely deep fluid that mass transport must always occur. The proof is independent of the dynamical free-surface condition. LEVI-CIVITA (1912) and later URSELL (1953) developed methods of analysis to include both finite depth and nonperiodic waves; essentially URSELL'S analysis has also been given by NEKRASOV (1951) for infinite depth. The analysis given below is due to LONGUET-HIGGINS (1953) and is similar to that used in the preceding theorem. We note that STARR (1945) has also given an instructive and perspicuous derivation of RAYLEIGH'S theorem for infinite depth.

Take the wave as moving to the left with velocity c (in the sense of Sect. 7) and impose a uniform velocity c in the opposite direction, so that the motion is reduced to a steady one, generally in the positive x-direction in the sense that $u>\varepsilon>0$. We may then write the complex potential in the form

$$f(z)=\Phi+i\Psi=cz+\varphi+i\psi, \tag{32.14}$$

where $\operatorname{Re} f'>\varepsilon>0$ and

$$\Phi(x+n\lambda,y)=nc\lambda+\Phi(x,y),\qquad \Psi(x+n\lambda,y)=\Psi(x,y). \tag{32.15}$$

We take $\Phi=0$ at a crest and assume $\Psi=0$ as the free-surface streamline and $\Psi=-Q$ as the bottom streamline if the depth is finite. One may invert the relation $f=f(z)$ and obtain $z=z(f)$. Then, since $q^2\neq 0$,

$$z'(f)=\frac{1}{f'(z)}=\frac{\Phi_x+i\Phi_y}{\Phi_x^2+\Phi_y^2}=\frac{1}{q^2}(u+iv)=x_\Phi+iy_\Phi. \tag{32.16}$$

Denote by $T(\Psi)$ the time required for a given particle to progress one wavelength along a streamline $\Psi=$const. In the original wave motion, the time elapsed between the passage of two successive crests over a given point is λ/c. If $T>\lambda/c$, the particle is being transported with the wave and it will be reasonable to call

$$U(\Psi)=c-\frac{\lambda}{T(\Psi)} \tag{32.17}$$

see errata

the mass transport in the direction of wave motion. The following theorem is true.

Theorem. Both T and U decrease with increasing depth, and, with the assumed definition of c, $U>0$.

The theorem may be proved by the following computation:

$$T(\Psi)=\int_0^{s(\lambda)}\frac{1}{q}\,ds=\int_0^{c\lambda}\frac{1}{q}\frac{\partial s}{\partial\Phi}\,d\Phi=\int_0^{c\lambda}(x_\Phi^2+y_\Phi^2)\,d\Phi=\int_0^{c\lambda}(x_\Phi^2+x_\Psi^2)\,d\Phi, \tag{32.18}$$

$$T'(\Psi)=4\int_0^{c\lambda}x_\Phi\,x_{\Phi\Psi}\,d\Phi, \tag{32.19}$$

$$T''(\Psi)=4\int_0^{c\lambda}(x_{\Psi\Phi}^2+x_{\Psi\Psi}^2)\,d\Phi. \tag{32.20}$$

The details of the computation are almost identical with those used in deriving (32.4) and (32.5). Since

$$x_\Psi=-y_\Phi=-\frac{1}{q^2}\Phi_y=0\quad\text{on}\quad\Psi=-Q,$$

it follows from (32.19) that $T'(-Q)=0$. Then, since $T''(\Psi)>0$ unless the flow is uniform, it follows that

$$T'(\Psi)\geqq 0, \tag{32.21}$$

with equality holding only for $\Psi=-Q$. As in the earlier theorem, the computations can be extended to yield

$$T^{(2n)}(\Psi)>0, \qquad T^{(2n-1)}(\Psi)\geqq 0. \tag{32.22}$$

It now follows immediately from (32.17) that

$$U'(\Psi)\geqq 0, \tag{32.23}$$

with the equality holding only for the bottom streamline. If the fluid is infinitely deep, then $U'>0$ for all Ψ. To complete the proof we must show that $U>0$. If the fluid is infinitely deep, it is evident that

$$\lim_{\psi\to-\infty} T(\Psi)=\frac{\lambda}{c}. \tag{32.24}$$

Hence $\lim U=0$ as $\Psi\to-\infty$ and the conclusion follows from $U'>0$. If the depth is finite, we compute

$$T(-Q)=\int_0^{c\lambda} x_\Phi^2\,d\Phi>\frac{1}{c\lambda}\left[\int_0^{c\lambda} x_\Phi\,d\Phi\right]^2=\frac{1}{c\lambda}\lambda^2=\frac{\lambda}{c}; \tag{32.25}$$

here use has been made of the Schwarz-Bunyakovskii inequality. (We have written $>$ rather than $\geqq$, for the equal sign will hold only in the trivial case of a uniform flow.) It now follows that $U(-Q)>0$ and hence that

$$U(\Psi)>0 \tag{32.26}$$

since $U'\geqq 0$. This completes the proof of the theorem.

The method of analysis can be extended to prove an analogous theorem for nonperiodic steady motions which approach uniform flows as $x\to\pm\infty$, in particular, to the solitary wave.

Momentum and energy integrals. We close this section with several momentum and energy integrals, most of which have been found by LEVI-CIVITA (1912, 1921), STARR (1947a, b, 1948) and STARR and PLATZMAN (1948).

Let us again take the wave as moving to the left without change of form and impose an opposite velocity c which brings the profile to rest (or, equivalently, consider the motion relative to a coordinate system moving with the wave). Let the velocity potential be as in (32.14). Consider the area bounded by two streamlines $\Psi=\Psi_1$ and $\Psi=\Psi_2$, say $y=\eta_1(x)$ and $y=\eta_2(x)$ and two vertical lines a wavelength λ apart. To this area apply the theorem

$$\iint(\Phi_x^2+\Phi_y^2)\,d\sigma=\oint\Phi\Phi_n\,ds. \tag{32.27}$$

This yields

$$\iint[(c+u)^2+v^2]\,d\sigma=\int_{\eta_1(x_0+\lambda)}^{\eta_2(x_0+\lambda)}\Phi(x_0+\lambda,y)\,\Phi_x\,dy-\int_{\eta_1(x_0)}^{\eta_2(x_0)}\Phi(x_0,y)\,\Phi_x\,dy \tag{32.28}$$

since $\Phi_n=0$ on the streamlines. Moreover, since $\Phi(x+\lambda,y)=c\lambda+\Phi(x,y)$, $\Phi_x(x+\lambda,y)=\Phi_x(x,y)=c+u$ and $\eta_i(x+\lambda)=\eta_i(x)$, the right-hand side of (32.28) may be written as

$$c^2\lambda[\eta_2(x_0)-\eta_1(x_0)]+c\lambda\int_{\eta_1(x_0)}^{\eta_2(x_0)}\varphi_x(x_0,y)\,dy=c^2\lambda[\eta_2(x)-\eta_1(x)]+c\lambda\int_{\eta_1(x)}^{\eta_2(x)}u\,dy. \tag{32.29}$$

Expanding $(c+u)^2$ and rearranging give

$$\iint(u^2+v^2)\,d\sigma+2c\iint u\,d\sigma+c^2\iint d\sigma=c^2\lambda[\eta_2(x)-\eta_1(x)]+c\lambda\int_{\eta_1(x)}^{\eta_2(x)}u\,dy. \tag{32.30}$$

If one now applies the operator $\lambda^{-1}\int_0^\lambda \ldots dx$ to (32.30), one obtains

$$\iint (u^2+v^2)\,d\sigma + c\iint u\,d\sigma = 0 \tag{32.31}$$

or, after multiplying by $\frac{1}{2}\varrho$ and rearranging,

$$\iint \tfrac{1}{2}\varrho\,(u^2+v^2)\,d\sigma = \tfrac{1}{2}c\iint -\varrho\,u\,d\sigma, \tag{32.32}$$

i.e., the kinetic energy per wavelength between two streamlines equals $\frac{1}{2}c$ times the momentum in the direction of the wave (here to the left).

Next let us write the integral (2.10′) in the form

$$\tfrac{1}{2}\varrho\,[(c+u)^2+v^2] + \varrho\,g\,y + p = \tfrac{1}{2}\varrho\,c_1^2, \tag{32.33}$$

the form of the constant having been chosen for later convenience. Write the terms $p+\varrho g y$ as follows:

$$\left.\begin{aligned} p+\varrho\,g\,y &= \frac{\partial}{\partial y}[y(p+\varrho\,g\,y)] - y\frac{\partial}{\partial y}(p+\varrho\,g\,y) \\ &= \frac{\partial}{\partial y}[y(p+\varrho\,g\,y)] + y\frac{D}{Dt}v \\ &= \frac{\partial}{\partial y}[y(p+\varrho\,g\,y)] - v^2 + \frac{D}{Dt}(y\,v). \end{aligned}\right\} \tag{32.34}$$

Here we have used the second equation of (2.6). We may now write (32.33) as follows

$$\frac{1}{2}\varrho\,(u^2-v^2) + \varrho\,c\,u + \frac{\partial}{\partial y}[y(p+\varrho\,g\,y)] + \frac{D}{Dt}(y\,v) = \frac{1}{2}\varrho\,(c_1^2-c^2). \tag{32.35}$$

Next let us integrate Eq. (32.35) over the same area as is described in the preceding paragraph. First consider $D(y\,v)/Dt$. Since the motion is steady in the selected coordinate system,

$$\frac{D}{Dt}(y\,v) = (u+c)\frac{\partial(yv)}{\partial x} + v\frac{\partial(y\,v)}{\partial y} = \frac{\partial}{\partial x}(u+c)\,y\,v + \frac{\partial}{\partial y}\,y\,v^2,$$

where the last equality follows from the continuity equation. Hence

$$\iint \frac{D}{Dt}(y\,v)\,d\sigma = \oint y\,v\,(u+c,\,v)\cdot\boldsymbol{n}\,ds = \oint y\,v\,\Phi_n\,ds = 0 \tag{32.36}$$

since $\Phi_n = 0$ on the streamline boundaries and the integrals over the vertical boundaries cancel from periodicity. The integrated equation then becomes

$$\left.\begin{aligned} &\iint \tfrac{1}{2}\varrho\,(u^2-v^2)\,d\sigma + c\iint \varrho\,u\,d\sigma + \int_0^\lambda \{\eta_2(x)\,[p(x,\eta_2)+\varrho\,g\,\eta_2] - \\ &\quad - \eta_1(x)\,[p(x,\eta_1)+\varrho\,g\,\eta_1]\}\,dx = \tfrac{1}{2}\varrho\,(c_1^2-c^2)\iint d\sigma. \end{aligned}\right\} \tag{32.37}$$

If one eliminates the second integral by means of (32.32), one obtains

$$\left.\begin{aligned} &\iint \tfrac{1}{2}\varrho\,u^2\,d\sigma + 3\iint \tfrac{1}{2}\varrho\,v^2\,d\sigma - \int_0^\lambda \{\eta_2[p(x,\eta_2)+\varrho\,g\,\eta_2] - \\ &\quad - \eta_1[p(x,\eta_1)+\varrho\,g\,\eta_1]\}\,dx = \tfrac{1}{2}\varrho\,(c^2-c_1^2)\iint d\sigma. \end{aligned}\right\} \tag{32.38}$$

Eq. (32.38) has a simpler aspect if the two streamlines are taken as the free surface $\eta(x)$ and the bottom $y=-h$. Then $p(x,\eta(x))=0$ and the third integral becomes

$$\int_0^\lambda \varrho\,g\,\eta_2^2(x)\,dx + h\int_0^\lambda [p(x,-h) - \varrho\,g\,h]\,dx.$$

Moreover,

$$\int_0^\lambda [p(x, -h) - \varrho g h]\, dx = 0 \tag{32.39}$$

if the x-axis is taken at the mean water level. This follows from the following sequence of equations, similar to those used in (32.36):

$$\left.\begin{aligned}\int_0^\lambda [p(x, -h) - \varrho g h]\, dx &= \iint \frac{\partial}{\partial y}[p(x, y) + \varrho g y]\, d\sigma - \\ -\iint\left[(u+c)\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v\right] d\sigma &= -\iint\left[\frac{\partial}{\partial x}v(u+c) + \frac{\partial}{\partial y}v^2\right] d\sigma \\ = \oint v(u+c, v)\cdot \boldsymbol{n}\, d\sigma &= -\oint v\,\Phi_n\, d\sigma = 0.\end{aligned}\right\} \tag{32.40}$$

Eq. (32.39) now allows us to give a simple physical interpretation of the constant c_1 in (32.33). For if (32.33) is integrated along $y = -h$, and account is taken of (32.39), one finds

$$\frac{1}{\lambda}\int_0^\lambda (u+c)^2\, dx = c_1^2 \geqq c^2, \tag{32.41}$$

i.e., c_1^2 is the mean square velocity of fluid along the bottom. The inequality follows easily from

$$\int_0^\lambda u(x, -h)\, dx = \int_0^\lambda \varphi_x(x, -h)\, dx = \varphi(\lambda, -h) - \varphi(0, -h) = 0. \tag{32.42}$$

If the fluid is infinitely deep, $u \to 0$ as $y \to -\infty$, and (32.41) reduces to

$$c^2 = c_1^2. \tag{32.43}$$

If, following (15.27), we let $\mathcal{T}_{\text{av}}, \mathcal{T}_{x\,\text{av}}, \mathcal{T}_{y\,\text{av}}, \mathcal{V}_{\text{av}}, \mathcal{M}_{\text{av}}$ denote the average kinetic energy, the contributions to this due to the x and y velocity components, the potential energy, and the momentum in the direction of wave motion, respectively, then (32.32) and (32.88) may be expressed as follows:

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$$2\mathcal{T}_{\text{av}} = c\mathcal{M}_{\text{av}}, \qquad \mathcal{T}_{x\,\text{av}} + 3\mathcal{T}_{y\,\text{av}} = 2\mathcal{V}_{\text{av}} - \tfrac{1}{2}\varrho(c_1^2 - c^2)\, h, \tag{32.44}$$

where the last term of the second equation is zero for $h = \infty$. The first equation is essentially due to LEVI-CIVITA (1912, 1921), the second to STARR (1947b).

We note another simple consequence of (32.41), due to LEVI-CIVITA (1924). Let us integrate (32.33) along the free surface for a wavelength and divide by $\frac{1}{2}\varrho\lambda$. Then, remembering our choice of x-axis as the mean water level, we find

$$\frac{1}{\lambda}\int_0^\lambda [(c+u)^2 + v^2]\, dx = c_1^2. \tag{32.45}$$

On the other hand, if we compute the velocity at the intersection of the mean water level and the profile, we also find

$$(c+u)^2 + v^2\big|_{y=0} = c_1^2. \tag{32.46}$$

Hence the absolute value of the velocity at the mean water level equals the root-mean-square velocity along the surface profile or along the bottom, or, indeed, along any streamline, for in the reasoning in (32.40) we could have substituted any streamline $y=\eta_1(x)$ for $y=-h$ and obtained

$$\int_0^\lambda [p(x,\eta_1(x)) - \varrho g \eta_1(x)]\,dx = 0. \tag{32.47}$$

STARR and PLATZMAN (1948) have used the relations above to derive some general relations concerning the flow of energy in a periodic wave. We recall that the average flux of energy in the direction of wave motion is given by [cf. Sect. 8 and Eqs. (15.23) and (15.27)]

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$$\mathcal{F}_{\text{av}} = \int_0^\lambda dx \int_{-h}^{\eta(x)} \varrho c \varphi_x^2(x,y)\,dy = 2\mathcal{T}_{x\,\text{av}}. \tag{32.48}$$

It follows from the second formula in (32.44) that

$$2\mathcal{T}_{x\,\text{av}} = 3\mathcal{T}_{\text{av}} - 2\mathcal{V}_{\text{av}} + \tfrac{1}{2}\varrho(c_1^2 - c^2)\,h. \tag{32.49}$$

Hence, with $\mathcal{E}_{\text{av}} = \mathcal{T}_{\text{av}} + \mathcal{V}_{\text{av}}$, we obtain from (32.48)

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$$\frac{\mathcal{F}_{\text{av}}}{\mathcal{E}_{\text{av}}} = \frac{1}{2} + \frac{5}{2}\,\frac{\mathcal{T}_{\text{av}} - \mathcal{V}_{\text{av}} + \frac{1}{5}\varrho(c_1^2 - c^2)\,h}{\mathcal{E}_{\text{av}}}. \tag{32.50}$$

This should be compared with the result derived in Sect. 15 for infinitesimal waves with neglect of surface tension [cf. (15.25) and (15.26)], namely, $\mathcal{F}_{\text{av}} = \frac{1}{2}\mathcal{E}_{\text{av}}$. Eq. (32.50) is consistent with this, for to the order of approximation involved, $\mathcal{T}_{\text{av}} = \mathcal{V}_{\text{av}}$ and $c_1^2 = c^2$. However, for waves of finite height it was shown in Sect. 27α [cf. Eqs. (27.42), (27.43)] that to the second order of approximation $\mathcal{T}_{\text{av}} > \mathcal{V}_{\text{av}}$. PLATZMAN (1947) has verified that this remains true when 4th-order terms are kept.

Several of the above results have analogues for steady motion of nonperiodic waves, provided that $\eta(x) \to 0$ as $x \to \pm\infty$ in such a way that $\int_{-\infty}^{\infty} \eta\,dx$ is finite. Under such circumstances $c_1^2 = c^2$ and the following results may be established [the notation is that of (15.31) with obvious extensions]:

$$\left.\begin{aligned} \mathcal{M}_{\text{total}} &= c\int_{-\infty}^{\infty} \eta\,dx = c\,\mathcal{A}_{\text{total}},\\ \mathcal{T}_{x\,\text{total}} - \mathcal{T}_{y\,\text{total}} &= \mathcal{V}_{\text{total}},\\ \mathcal{T}_{x\,\text{total}} - \mathcal{T}_{y\,\text{total}} + 2\mathcal{V}_{\text{total}} + (g h - c^2)\,\mathcal{A}_{\text{total}} &= 0. \end{aligned}\right\} \tag{32.51}$$

For details of the proof one may refer to STARR (1947b). From the last two equations follows

$$c^2 = g h + 3\mathcal{V}_{\text{total}}/\mathcal{A}_{\text{total}} > g h. \tag{32.52}$$

We note that the second equation of (32.51) is a special case of a more general one applying to any steady motion:

$$\mathcal{T}_x(x) - \mathcal{T}_y(x) - \mathcal{V}(x) = \text{const} \tag{32.53}$$

where the constant is zero under the conditions of (32.51). The proof is analogous to that of (8.6). Here

$$\mathcal{T}_x(x) = \int_{-h}^{\eta(x)} \tfrac{1}{2}\varrho u^2(x,y)\,dy, \quad \text{etc.}$$

β) Waves in heterogeneous fluids. The first two theorems proved below are also true for homogeneous fluids and were first proved for this case. The last theorems deal specifically with heterogeneous fluids. In the extended form they are all due to DUBREIL-JACOTIN (1932).

A flow will be called barotropic if both the pressure and density are constant along streamlines. We first derive the energy integral for such flows. The Eqs. (2.6) may be written in the following form in two dimensions:

$$-\frac{1}{\varrho}\frac{\partial p}{\partial x} = \frac{\partial E}{\partial x} - v\zeta + \frac{\partial u}{\partial t}, \qquad -\frac{1}{\varrho}\frac{\partial p}{\partial y} = \frac{\partial E}{\partial y} + u\zeta + \frac{\partial v}{\partial t}, \tag{32.54}$$

where

$$E = g y + \frac{1}{2}(u^2 + v^2), \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Since p is assumed constant on a streamline, $u p_x + v p_y = 0$; it follows from (32.27) and the definition of E that

$$0 = u\frac{\partial E}{\partial x} + v\frac{\partial E}{\partial y} + \frac{1}{2}\frac{\partial}{\partial t}q^2 = g v + \frac{1}{2}\frac{D}{Dt}q^2 = \frac{D}{Dt}E. \tag{32.55}$$

In particular, if the flow is steady, E is also constant along a streamline. For steady flow it is a consequence of the incompressibility condition that ϱ is also constant along a stream-line.

The following theorem was proved by BURNSIDE (1915) for a homogeneous fluid. He gives two proofs, of which the second can be carried over to the present more general situation with no change. It will perhaps give more substance to the theorem if we remark that GERSTNER'S wave (see subsection 34β), which is not irrotational, satisfies the other conditions of the theorem.

Theorem. The only steady two-dimensional irrotational motion of a fluid subject to gravity for which all streamlines are also lines of constant pressure is a uniform flow.

Let the streamlines be given by $\psi(x, y) = \text{const}$. Since, from the remark following (32.55), $E = \text{const}$ along a streamline, we may write

$$\tfrac{1}{2}(\psi_x^2 + \psi_y^2) + g y = E(\psi). \tag{32.56}$$

[BURNSIDE shows that one may generalize (32.56) by replacing $g y$ by a function $g(y)$.] Since the motion is irrotational, $\Delta\psi = 0$ and hence also

$$\Delta \log(\psi_x^2 + \psi_y^2) = 0.$$

But then

$$\Delta \log[E(\psi) - g y] = 0,$$

which yields after some computation

$$2y E'(\psi)\psi_y = 2(E - g y)[E'^2 - (E - g y) E''] + g^2. \tag{32.57}$$

We write this in the form

$$\psi_y(x, y) = G(\psi, y). \tag{32.58}$$

It then follows from (32.56) and (32.58) that

$$\psi_{xx} = E' - G G_\psi, \qquad \psi_{yy} = G_\psi \psi_y + G_y$$

or

$$E'(\psi) + G_y(\psi, y) = 0.$$

But then

$$\psi_y = -y E'(\psi) + \text{const}$$

and ψ is a function of y only. Hence, since $\Delta\psi = \psi_{yy} = 0$, ψ_y is a constant and the flow is uniform.

The next theorem was first proved by LEVI-CIVITA (1925) for homogeneous fluids. FENCHEL (1931) showed that his hypotheses could be weakened and DUBREIL-JACOTIN (1932) extended FENCHEL'S proof to heterogeneous fluids. The gist of the theorem is that if the surface profile moves without change of form, then the whole velocity field is steady in a coordinate system moving with the surface. The theorem will be formulated in the moving coordinate system.

Theorem. Let a possibly heterogeneous fluid, bounded below by a horizontal plane $y = -h$, be flowing irrotationally in the x-direction with discharge rate $Q(t)$ and with a fixed surface profile $y = \eta(x)$. If η and u satisfy the conditions

$$-h < b_1 < \eta < b_2, \qquad u > \varepsilon > 0, \tag{32.59}$$

then the velocity potential $f(z)$ is independent of t.

First we derive a boundary condition at the free surface. From the condition of constant pressure and the assumption that the surface profile is an invariant streamline it follows that

$$u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} = 0 \qquad \text{on } \psi = 0.$$

It then follows as in (32.55) that

$$\frac{DE}{Dt} = g v + \frac{Dq^2}{Dt} = 0 \qquad \text{on } \psi = 0. \tag{32.60}$$

However, this conclusion holds now only on this one streamline.

The complex potential $f(z, t) = \varphi + i\psi$ maps the region of the z-plane occupied by fluid onto the strip $-Q(t) \leqq \psi \leqq 0$, where the free surface corresponds to $\psi = 0$, the bottom to $\psi = -Q$ and $x = \pm\infty$ to $\varphi = \pm\infty$. Let $F(z) = \Phi + i\Psi$ be the mapping, unique up to an additive real constant, of the fluid region onto the strip $-1 \leqq \Psi \leqq 0$ with $x = \pm\infty$ corresponding to $\Phi = \pm\infty$. Then

$$f(z, t) = Q(t)\, F(z) \tag{32.61}$$

evidently satisfies the requirements for $f(z, t)$ and, in fact, is determined uniquely, up to the added constant in F, by $Q(t)$ and $\eta(x)$. Now substitute $\varphi(x, y, t) = Q(t)\,\Phi(x, y)$ into (32.60):

$$g\,Q\,\Phi_y + Q\,Q'[\Phi_x^2 + \Phi_y^2] + Q^2[\Phi_{xx}\,\Phi_x^2 + 2\Phi_{xy}\,\Phi_x\,\Phi_y + \Phi_{yy}\,\Phi_y^2] = 0, \tag{32.62}$$

which we may write in the form

$$Q' + AQ + B = 0 \qquad \text{on } \Psi = 0 \tag{32.63}$$

where A and B are independent of t. Division by $\Phi_x^2 + \Phi_y^2$ is possible since (32.59) implies that this does not vanish. Note also that

$$B = g\frac{\Phi_y}{\Phi_x^2 + \Phi_y^2} = g\,y_\Phi\big|_{\Psi=0} = g\frac{d}{d\Phi}\eta, \tag{32.64}$$

and that both A and B may be considered as functions of Φ. Consider two cases: (a) $A = \text{const}$, (b) $A \neq \text{const}$. (a) In this case, since B is independent of t and

$Q' + A\,Q$ is independent of Φ, it follows from (32.63) that both equal constants. It now follows from (32.64) that, unless this constant is zero, the profile $\eta(x)$ will be unbounded and the first part of (32.59) will be contradicted. Hence, in case (a) $\eta = \text{const}$ and the mapping F must be of the form $F = az + b$, a and b real. It then follows that $A = 0$ and hence $Q' = 0$, i.e. the flow is uniform. (b) Let A_1, A_2 be two different values of A, $A_1 \neq A_2$. Write Eq. (32.63) for each value and subtract. This yields

$$Q = -\frac{B_1 - B_2}{A_1 - A_2}. \tag{32.65}$$

But then Q is evidently independent of t. Hence also $f(z, t) = QF(z)$ is also independent of t. This completes the proof.

The next theorem, due to DUBREIL-JACOTIN (1932), specifically requires that the fluid be heterogeneous.

Theorem. Suppose the motion of an incompressible heterogeneous fluid to be irrotational, the free surface to move without change of form, and that, in a coordinate system moving with the surface, conditions (32.59) are satisfied. Then not only is the velocity field steady, but also E, p and ϱ are constant along the streamlines.

It follows from the preceding theorem that the velocity is steady, hence that $E = E(x, y)$. However, we may still conceivably have $p = p(x, y, t)$, $\varrho = \varrho(x, y, t)$. The Eqs. (32.54) may now be written in the form

$$-\frac{1}{\varrho}\frac{\partial p}{\partial x} = \frac{\partial E}{\partial x}, \qquad -\frac{1}{\varrho}\frac{\partial p}{\partial y} = \frac{\partial E}{\partial y}. \tag{32.66}$$

Elimination of first ϱ, then p between these two equations yields

$$\frac{\partial(p, E)}{\partial(x, y)} = 0, \qquad \frac{\partial(\varrho, E)}{\partial(x, y)} = 0.$$

We assume that the corresponding functional relations may be solved and write

$$p = p(E, t), \qquad \varrho = \varrho(E, t), \tag{32.67}$$

where, from (32.66)

$$\varrho = -\frac{\partial p}{\partial E}. \tag{32.68}$$

From the equation expressing incompressibility, namely,

$$\frac{D\varrho}{Dt} = \frac{\partial \varrho}{\partial t} + u\frac{\partial \varrho}{\partial x} + v\frac{\partial \varrho}{\partial y} = 0,$$

follows

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial E}\left(u\frac{\partial E}{\partial x} + v\frac{\partial E}{\partial y}\right) = \frac{\partial \varrho}{\partial t} + \frac{\partial(E, \psi)}{\partial(x, y)}\frac{\partial \varrho}{\partial E} = 0. \tag{32.69}$$

We shall assume $\partial\varrho/\partial E \neq 0$ everywhere, and may thus write

$$\frac{\partial(E, \psi)}{\partial(x, y)} = -\frac{\varrho_t(E, t)}{\varrho_E(E, t)}. \tag{32.70}$$

Since the left-hand side is independent of t, it follows from the form of the right-hand side that we may set both sides equal to $k(E)$, i.e.

$$\frac{\partial \varrho}{\partial t} + k(E)\frac{\partial \varrho}{\partial E} = 0. \tag{32.71}$$

Let us suppose $k(E) \not\equiv 0$, e.g. $k(E_1) \neq 0$. Then ϱ must be a function of the form

$$\varrho = \varrho\left(t - \int_{E_1}^{E} \frac{dE}{k(E)}\right) \tag{32.72}$$

in some neighborhood of E_1. If $k(E)$ vanishes for some values of E, let E_0 be the first zero larger than E_1. Then from (32.71) and (32.72)

$$\varrho_t(E, t) = \varrho'\left(t - \int_{E_1}^{E} \frac{dE}{k(E)}\right) \to 0 \quad \text{as } E \to E_0. \tag{32.73}$$

But (32.73) can hold for all t only if $\varrho' = 0$, i.e. if $\varrho = \text{const}$, which is contrary to the assumed heterogeneity. Moreover, at least one such zero of k exists, for we already know from (32.60) that E is constant along the free surface, so that in steady motion the Jacobian in (32.70) vanishes for $\psi = 0$. Hence $k(E) = 0$ for the corresponding value of E. We must conclude that $k(E) \equiv 0$. This implies, from (32.70) that $E = E(\psi)$ and $\varrho = \varrho(E)$. From (32.68) and the condition $p_t = 0$ on the free surface, it follows that also $p = p(E)$. Hence p, ϱ and E are all constant on streamlines.

The last in this complex of theorems is also due to DUBREIL-JACOTIN (1932).

Theorem. There cannot exist irrotational waves in a heterogeneous fluid such that the profile is propagated without change of form.

This follows immediately from the first and last theorems proved above, and is, of course, subject to condition (32.59). This striking result is all the more so in view of the fact that GERSTNER'S wave (subsection 34β) does provide a steadily propagating wave, even in a heterogeneous fluid. The theorem also casts some doubt upon the significance of the linearized theory of irrotational wave motion in a heterogeneous fluid as developed, for example, in LAMB (1932, § 235). Such a wave evidently cannot be considered as a first approximation to an exact steady solution.

see errata

γ) Some transformations of the boundary-value problem. By means of introduction of new variables or other devices, it is possible to formulate the boundary-value problem for exact solutions in a variety of ways. Several such formulations will be considered in subsection 34α on inverse methods. Here we give a few which seem to be of general interest.

Inversion of $f(z)$. One elementary but important transformation has already been introduced in subsection 32α in the discussion of mass transport. This is the inversion of the velocity potential $f(z)$ when $|f'|$ vanishes nowhere within the fluid, and treatment of f as the independent variable. This has the advantage that under certain circumstances the domain of definition of the independent variable can be given exactly; when z is the independent variable, the domain of definition is one of the unknowns of the problem. For example, if the motion is reducible to a steady flow with discharge rate Q, one may take the surface profile to correspond to $\psi = 0$ and the bottom streamline to correspond to $\psi = -Q$. Hence the domain of definition of $z(f)$ is the strip $0 \geqq \psi \geqq -Q$; if the fluid is infinitely deep, the domain is the half-plane $\psi \leqq 0$. Whenever f can be taken as the independent variable, then one can also express $w = f'$ as a function of f. It has been established independently by GERBER (1951) and LEWY (1952a) that the equation describing the free surface, $z = z(\varphi)$, is an analytic function of φ at all points for which $w \neq 0$.

STOKES' "second method". In the introduction to Sect. 27 it was mentioned that STOKES (1880), in a supplement to an earlier paper in his collected works, developed a method for approximating exact periodic waves which is different from the straightforward generalization of infinitesimal-wave theory expounded in that section. This method is based upon use of f as the independent variable and expansion of z as a Fourier series in f:

$$c z = f + i \frac{c\lambda}{2\pi} \sum_{n=0}^{\infty} a_n \, e^{-i n 2\pi f/c\lambda} \tag{32.74}$$

or

$$c z = f + i \frac{c\lambda}{2\pi} a_0 + \frac{c\lambda}{2\pi} \sum_{n=1}^{\infty} a_n \sin n \frac{2\pi}{c\lambda} (f + i Q) \tag{32.75}$$

for infinite and finite depth respectively; the a_n may be taken to be real. Here ψ is taken as in the preceding paragraph. The coefficients a_n are to be determined from the condition that the pressure be constant on the surface, i.e. from

$$q^2 + 2 g y = C \qquad \text{for } \psi = 0. \tag{32.76}$$

If the mean water level is taken at $y=0$ and the fluid is infinitely deep, then $C = c^2$; we shall consider only this case here. Then Eq. (32.76) may be expressed as

$$(c^2 - 2 g y) \, |z'|^2 = 1. \tag{32.77}$$

Substitution of (32.74) in (32.77) yields

$$\left.\begin{aligned} &\left(1 - \frac{g\lambda}{\pi c^2} \sum_{n=0}^{\infty} a_n \cos \frac{2\pi n \varphi}{c\lambda}\right) \times \\ &\quad \times \left(1 + 2 \sum_{n=1}^{\infty} n \, a_n \cos \frac{2\pi n \varphi}{c\lambda} + \sum_{n,m=1}^{\infty} n \, m \, a_n \, a_m \cos (n-m) \frac{2\pi \varphi}{c\lambda}\right) = 1. \end{aligned}\right\} \tag{32.78}$$

After multiplying the two factors and reducing the cosine products to cosines of sums and differences, the resulting expression may be put into the form

$$\sum_{n=0}^{\infty} \left(\frac{g\lambda}{\pi c^2} b_n + c_n\right) \cos \frac{2\pi n \varphi}{c\lambda} = 0, \tag{32.79}$$

where the b_n's and c_n's are forms of the third degree in the a_n's. The coefficients of the individual cosine terms must then be equated to zero. This results in an infinite sequence of equations, each involving all the a_n's and $g\lambda/\pi c^2$. In order to proceed further, one must devise some method for approximate determination of the a_n's. STOKES' procedure was to assume that each a_n could be expanded in a power series in some parameter, the initial term in the series having the power n. This allows one to carry through a step-by-step improvement in the approximation of the a_n's by including successively higher powers of the parameter. We shall not pursue the matter further, but remark that the most systematic arrangement of such computations seems to have been devised by SRETENSKII (1952).

LEVI-CIVITA's differential-difference equation. The following theorem, due to LEVI-CIVITA (1907), reduces determination of $w(f)$ for steady flow over a horizontal bottom to solution of a differential-difference equation.

Theorem. The complex velocity $w=u-iv$ of an irrotational gravity flow with constant discharge rate Q and with $u\geqq\varepsilon>0$ must satisfy the differential-difference equation

$$\frac{d}{df}\left[w(f+iQ)\,w(f-iQ)\right]-ig\left[\frac{1}{w(f+iQ)}-\frac{1}{w(f-iQ)}\right]=0. \tag{32.80}$$

Conversely, any function $w(f)$ satisfying (32.80) which is regular in the strip $-2Q\leqq \operatorname{Im} f\leqq 0$, finite at ∞, real on $\operatorname{Im} f=-Q$ and has $u>\varepsilon>0$ represents such a flow.

In order to derive (32.80), we note first that the functions $w(f)$ and $z(f)+ih$ both have vanishing real parts for $\psi=-Q$ and consequently can be extended by reflection to the strip $-Q\geqq\psi\geqq-2Q$:

$$w(\bar f-2iQ)=\overline{w(f)},\qquad z(\bar f-2iQ)+ih=\overline{z(f)}-ih. \tag{32.81}$$

The free-surface condition may be expressed by the equation

$$\frac{\partial}{\partial\varphi}w\bar w+2g\frac{\partial}{\partial\varphi}y=0\qquad\text{for }\psi=0, \tag{32.82}$$

or, by making use of the extended definitions of w and z, by

$$\frac{\partial}{\partial\varphi}\{w(\varphi)\,w(\varphi-2iQ)-ig\left[z(\varphi)-z(\varphi-2iQ)\right]\}=0. \tag{32.83}$$

Consider the function

$$H(f)=w(f+iQ)\,w(f-iQ)-ig\left[z(f+iQ)-z(f-iQ)\right]. \tag{32.84}$$

Evidently, H is defined and is regular on the line $\psi=-iQ$ and thus in some neighborhood of this line. From (32.83) it follows that $H'(\varphi-iQ)=0$, hence that $H'(f)\equiv 0$ in its region of definition. Eq. (32.80) follows from the fact that $z'(f\pm iQ)=1/w(f\pm iQ)$. For proof of the converse we refer to LEVI-CIVITA's paper. LEVI-CIVITA also gives a special form of (32.80) appropriate to a space-periodic flow. CISOTTI (1919) generalized the preceding theorem to include a variable discharge rate. The Eq. (32.80) may be considered to contain the Eq. (22.30), when in that equation $f(z,t)=f(z-ct)$, in the sense that linearization of (32.80) by assuming

$$w=c(1+\varepsilon w_1+\cdots)$$

yields (22.30).

RUDZKI's transformation. The following transformation was apparently first introduced by RUDZKI (1898). It has later been used by many others in the investigation of exact water waves. The validity of the reformulated boundary condition is not limited to periodic waves. However, it is assumed that a coordinate system has been selected with respect to which the flow is steady. It is again assumed that $u>\varepsilon>0$. Let ϑ be the angle between the velocity vector $\bar w=u+iv$ and the positive x-axis. Then one may write

$$w=u-iv=q\,e^{-i\vartheta}=c\,e^{-i\omega} \tag{32.85}$$

where

$$\omega=\vartheta+i\tau,\qquad q=c\,e^{\tau}. \tag{32.86}$$

Here c is some typical velocity, say the wave velocity as defined in Sect. 7. We consider ω as a function of f and let $\psi=0$ correspond to the free surface. The free-surface condition may then be expressed by

$$g\frac{\partial y}{\partial\varphi}+q\frac{\partial q}{\partial\varphi}=0\quad\text{for }\psi=0. \tag{32.87}$$

But [see (32.16)]

$$\frac{\partial y}{\partial \varphi} = \frac{1}{q^2}\frac{\partial \varphi}{\partial y} = \frac{1}{q}\sin\vartheta$$

and, from (32.86),

$$\frac{\partial q}{\partial \varphi} = c\,e^{\tau}\frac{\partial \tau}{\partial \varphi} = q\frac{\partial \tau}{\partial \varphi}.$$

Hence (32.87) becomes

$$\frac{\partial \tau}{\partial \varphi} = -g\frac{1}{q^3}\sin\vartheta = -\frac{g}{c^3}e^{-3\tau}\sin\vartheta \quad \text{for} \quad \psi = 0, \tag{32.88}$$

or, since $\partial\tau/\partial\varphi = -\partial\vartheta/\partial\psi$ from the Cauchy-Riemann equations,

$$\frac{\partial \vartheta}{\partial \psi} = \frac{g}{c^3}e^{-3\tau}\sin\vartheta \qquad \text{for } \psi = 0. \tag{32.89}$$

see errata It one can find a function $\omega(f)$ regular in the strip $0 \geqq \psi \geqq -Q$, with $|\vartheta| < \frac{1}{2}\pi - \varepsilon'$, and with its real and imaginary parts satisfying (32.88) or (32.89) on $\psi = 0$, one may then construct from it a free-surface flow with gravity. Of course, further conditions must be imposed at $\psi = -Q$ or as $\psi \to -\infty$.

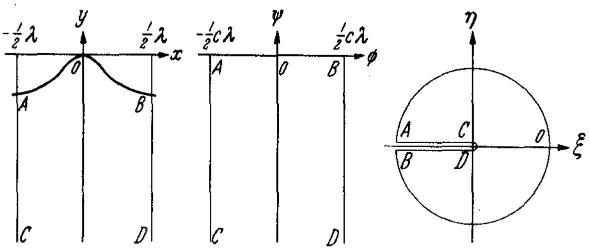

Fig. 50.

NEKRASOV'S transformation. The following transformation is due to NEKRASOV (1921, 1951). It will be assumed that the surface is periodic with period λ, symmetric about a crest and that the fluid is infinitely deep and $\lim_{y\to-\infty} w = c$. Let the origin in the z-plane be taken at a crest, $\psi = 0$ be the free surface, and assume $u > \varepsilon > 0$. In addition to the z- and f-planes, we introduce a ζ-plane,

$$\zeta = \xi + i\eta = \varrho\,e^{i\gamma}, \tag{32.90}$$

related to the f-plane through

$$\zeta = e^{-\frac{2\pi i}{\lambda c}f}. \tag{32.91}$$

With a cut along the negative ξ-axis there is a one-to-one correspondence between the various domains $CAOBD$ shown in Fig. 50.

The relation between the z- and ζ-planes will be determined by

$$\frac{dz}{d\zeta} = -\frac{\lambda}{2\pi i}\frac{h(\zeta)}{\zeta}, \quad h(\zeta) = 1 + a_1\zeta + a_2\zeta^2 + \cdots, \quad a_k \text{ real}, \tag{32.92}$$

where $h(\zeta)$ is regular in the disc and is related to w by

$$w = \frac{df}{d\zeta}\frac{d\zeta}{dz} = \frac{c}{h(\zeta)}. \tag{32.93}$$

The form of h shown in (32.92) follows from the assumed properties of the motion. Since $\varrho = 1$ on the free surface, the condition of constant pressure may be expressed by

$$2g\frac{\partial y}{\partial \gamma} + \frac{\partial q^2}{\partial \gamma} = 0 \quad \text{for} \quad \varrho = 1. \tag{32.94}$$

But

$$\left.\frac{\partial y}{\partial \gamma}\right|_{\varrho=1} = \operatorname{Im} \frac{dz}{d\zeta}\frac{d\zeta}{d\gamma}\bigg|_{\varrho=1} = \operatorname{Im} \frac{-\lambda}{2\pi i}\frac{h(\zeta)}{\zeta} i\zeta\bigg|_{\varrho=1} = \frac{-\lambda}{2\pi} \operatorname{Im} h(e^{i\gamma}). \quad (32.95)$$

It then follows from (32.93) and (32.95) that

$$\frac{d}{d\gamma}\frac{1}{h(e^{i\gamma})\, h(e^{-i\gamma})} = \frac{\lambda g}{\pi c^2} \operatorname{Im} h(e^{i\gamma}). \quad (32.96)$$

In this formulation of the problem one seeks a function $h(\zeta)$, regular in the disc $|\zeta| \leqq 1$, real on the real axis, $h(0) = 1$, and satisfying (32.96). From such a function one can easily construct a periodic gravity flow with free surface.

NEKRASOV's integral equation. NEKRASOV also considers the function ω of (32.92), but as a function of ζ. Let us start from (32.88) and compute

$$\frac{\partial \tau}{\partial \gamma} = \frac{\partial \tau}{\partial \varphi}\frac{\partial \varphi}{\partial \gamma} = -\frac{g}{c^3} e^{-3\tau} \sin\vartheta \cdot \frac{-\lambda c}{2\pi} = \frac{g\lambda}{2\pi c^2} e^{-3\tau} \sin\vartheta \quad \text{for} \quad \varrho = 1. \quad (32.97)$$

One may formally integrate this equation and obtain

$$e^{3\tau} = \frac{3}{2\pi}\frac{g\lambda}{c^2\mu}\left[1 + \mu\int_0^{\gamma} \sin\vartheta(\alpha)\, d\alpha\right], \quad (32.98)$$

where $1/\mu$ is the integration constant; μ is related to the velocity at the crest, $q_0 = \tau(1) = c/h(1)$, by

$$\mu = \frac{3}{2\pi}\frac{g\lambda c}{q_0^3} > 0. \quad (32.99)$$

Substitution of (32.98) into (32.99) yields the following equation for the relation between τ and ϑ on the boundary:

$$\frac{d\tau(\gamma)}{d\gamma} = \frac{1}{3}\,\frac{\mu \sin\vartheta(\gamma)}{1 + \mu\int_0^{\gamma}\sin\vartheta(\alpha)\, d\alpha} \quad (32.100)$$

[it follows from (32.98) that the denominator does not vanish]. It is known from the theory of functions of a complex variable (see, e.g., CARATHÉODORY, Funktionentheorie, Bd. 1, § 147—149, Birkhäuser, Basel, 1950) that, if a function is regular within and on a closed Jordan curve, it is determined up to an additive constant by giving either its real or imaginary part on the boundary. In particular, in the case at hand we may express the value of ϑ on the boundary $|\zeta| = 1$ in terms of τ on the boundary:

$$\vartheta(\gamma) = \text{const} - \frac{1}{2\pi} \operatorname{PV}\int_0^{2\pi} \tau(\beta) \cot\frac{1}{2}(\gamma - \beta)\, d\beta, \quad (32.101)$$

where the constant $= i\vartheta|_{\zeta=0} = 0$. An integration by parts gives

$$\vartheta(\gamma) = -\frac{1}{\pi}\int_0^{2\pi} \frac{d\tau}{d\beta} \log\left|\sin\frac{1}{2}(\gamma - \beta)\right| d\beta. \quad (32.102)$$

From the assumed symmetry about a crest follows $\tau'(-\beta) = -\tau'(\beta)$, so that (32.102) may be expressed as follows:

$$\vartheta(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\tau}{d\beta} \log\left|\frac{\sin\frac{1}{2}(\gamma + \beta)}{\sin\frac{1}{2}(\gamma - \beta)}\right| d\beta. \quad (32.103)$$

Substitution of (32.100) into (32.103) yields NEKRASOV's nonlinear integral equation for $\vartheta(\gamma)$:

see errata

$$\vartheta(\gamma)=\frac{1}{\sigma\pi}\int_0^{2\pi}\frac{\mu\sin\vartheta(\beta)}{1+\mu\int_0^\beta\sin\vartheta(\alpha)\,d\alpha}\log\left|\frac{\sin\frac{1}{2}(\gamma+\beta)}{\sin\frac{1}{2}(\gamma-\beta)}\right|d\beta. \tag{32.104}$$

If one can find ϑ satisfying (32.104), one can then reconstruct $\omega(\zeta)$ and hence the whole flow.

NEKRASOV (1928, 1951) carried through a similar analysis when the depth is finite. We shall only sketch it. In Fig. 50 suppose that $y=-h_1$ represents the bottom (h_1 is not the mean depth) and $\psi=-Q$ the corresponding streamline. In the ζ-plane this maps into a circle of radius $\varrho_0<1$, where

$$\varrho_0=e^{-\frac{2\pi Q}{\lambda c}}. \tag{32.105}$$

In (32.92) $h(\zeta)$ becomes a Laurent series. The integral equation for $\vartheta(\gamma)$ remains the same in form as (32.104), but the kernel function $\log|\ldots|$ is now replaced by

$$\sum_{n=1}^{\infty}\frac{2}{n}\tanh\frac{2\pi Q}{\lambda c}\sin n\gamma\sin n\beta. \tag{32.106}$$

MOISEEV (1957b) has further generalized NEKRASOV's equation so as to allow a wavy bottom.

The solution $\vartheta(\gamma)$ of (32.104) will, of course, depend upon the parameter μ, except for the trivial solution $\vartheta\equiv 0$ corresponding to a uniform flow. It is possible to show that not all μ's are allowable. Let

$$M=\max|\vartheta(\gamma)|. \tag{32.107}$$

It then follows from (32.102) that

$$0\le|\vartheta(\gamma)|<\frac{1}{6\pi}\frac{\mu\sin M}{1-\pi\mu\sin M}\int_0^{2\pi}-\log\left|\sin\frac{1}{2}(\gamma-\beta)\right|d\beta<\frac{1}{3}\frac{\mu\sin M}{1-\pi\mu\sin M}, \tag{32.108}$$

hence that

$$0\le M<\frac{1}{3}\frac{\mu\sin M}{1-\pi\mu\sin M}. \tag{32.109}$$

From this follows

$$\frac{1}{\pi\sin M}>\mu>\frac{1}{\pi\sin M+\frac{1}{3}M^{-1}\sin M}>\frac{3}{1+3\pi}. \tag{32.110}$$

VILLAT's integral equation. Even though we shall not consider its contents in any detail, it would be improper not to mention an important paper of VILLAT (1915). VILLAT wished to find the steady motion of a fluid in a canal of given bottom profile and also with a given top profile over the part of the fluid upstream of some point. Downstream of this point the top profile is one of constant pressure. The boundary condition on the free surface, (32.89), is modified by introduction of new variables, and a pair of integral equations, one of them nonlinear, is derived. The method is also applicable if the upstream "cover" is absent and, in fact, becomes a little simpler. The chief use made of the procedure by VILLAT is as an inverse method in which the free surface is given and the corresponding bottom and cover determined.

33. Waves of maximum amplitude. In the higher-order theory of infinitesimal waves one of the important effects of including higher-order terms was to make the profile more peaked at the crests and flatter in the troughs. The effect was the same for either steady progressive waves or standing waves. Since the peakedness increased with increase of the amplitude-to-wavelength ratio, it seems reasonable to conjecture that there is some bound to this ratio and that, if a wave of maximum amplitude-to-length ratio exists, it will be characterized by a corner or a cusp at the crest, at least if capillarity is neglected. It has never been proved that such waves exist. However, if one assumes their existence, it is possible to prove some necessary properties. This will be done below.

Following an earlier erroneous investigation of RANKINE (1865), STOKES (1880, p. 225) showed that, if a corner occurs in steady motion, the angle included between the tangents must be 120°. MICHELL (1893) assumed that a periodic highest progressive wave exists and showed how to compute the coefficients of an associated series, but without proving convergence. HAVELOCK (1919) made MICHELL'S procedure the basis of a general method of approximation to periodic progressive waves, again with no proof of convergence. MICHELL'S wave was later investigated by a different procedure by NEKRASOV (1920). However, NEKRASOV did not carry his computations to the same degree of accuracy as MICHELL and HAVELOCK, so that the numerical results are discrepant. More recently YAMADA (1957) rediscovered NEKRASOV'S method and carried through the calculations with the necessary accuracy; the results are now in substantial agreement with those of HAVELOCK and MICHELL.

PENNEY and PRICE (1952b), in their work on standing waves of finite amplitude, include an analysis intended to show that, if there exists a standing wave of maximum amplitude with a corner at the crest, then the angle must be 90°. G.I. TAYLOR (1953) has questioned the validity of the proof, and it appears, in fact, to be untenable. On the other hand, in the same paper TAYLOR reports the results of experiments which appear to confirm PENNY and PRICE'S prediction. In view of the present unsatisfactory state of the theory, it will not be further discussed here.

STOKES' theorem. We prove first STOKES' theorem on the angle at a corner in steady flow. Let the corner be at the origin $z=0$, the free surface be the streamline $\psi=0$, and $\varphi=0$ at the corner. Since $z=0$ is assumed to be a corner, it must also be a stagnation point and the constant-pressure condition on the surface may be taken in the form

$$q^2 + 2g\eta(x) = 0. \tag{33.1}$$

In the mapping from the z- to the f-plane the point $z=0$ must be a branch point, so that in the neighborhood of $z=0$ the complex velocity potential will take the form

$$f = A z^n. \tag{33.2}$$

If $\alpha_+ < 0$ is the angle between the right-hand tangent to the corner and OX, then near $z=0$ Eq. (33.1) can be written

$$|A|^2 n^2 r^{2n-2} + 2g r \sin\alpha_+ = 0.$$

This can hold for all small r only if

$$n = \tfrac{3}{2}. \tag{33.3}$$

It also follows that, if α_- is the angle between the left-hand tangent and OX, then $\sin\alpha_- = \sin\alpha_+$ and $\alpha_- = -180° - \alpha_+$ so that the surface is symmetrical about

OX near the corner. If $\psi \leqq 0$ corresponds to the region occupied by fluid and if the branch of $z = r\,e^{i\alpha}$ with $-\frac{3}{2}\pi < \alpha < \frac{1}{2}\pi$ is taken, then the complex velocity potential has the following form near $z = 0$:

$$\left.\begin{aligned} f(z) &= -\tfrac{2}{3}\sqrt{g}\,(-i z)^{\frac{3}{2}} \\ &= -\tfrac{2}{3}\sqrt{g}\,r^{\frac{3}{2}}\left[\cos\tfrac{3}{2}(\alpha - \tfrac{1}{2}\pi) + i\sin\tfrac{3}{2}(\alpha - \tfrac{1}{2}\pi)\right]. \end{aligned}\right\} \tag{33.4}$$

The streamline $\psi = 0$ has a corner at $z = 0$ with included angle 120°. In this case the flow is to the right. The inversion of (33.4) gives

$$z(f) = \left[\frac{3}{2\sqrt{g}}\right]^{\frac{2}{3}} e^{-i\pi/6} f^{\frac{2}{3}} \tag{33.5}$$

for f near 0.

α) *Periodic wave of maximum height.* Let us suppose that a periodic progressive wave of maximum amplitude-length ratio exists. We may take this as a steady flow with complex velocity potential $f(z) = \varphi + i\psi$ and with

$$\lim_{y \to -\infty} f'(z) = c. \tag{33.6}$$

Let the origin of the z-plane be at one of the crests, the surface profile correspond to $\psi = 0$, and the origin of the f-plane to that of the z-plane. Then the free surface condition may be taken in the form (33.1).

MICHELL's method. First we give MICHELL's procedure for finding $f'(z)$. As we have done earlier, we shall write

$$f'(z) = q\,e^{-i\vartheta}, \qquad z'(f) = \frac{1}{q}\,e^{i\vartheta}. \tag{33.7}$$

From the assumed periodicity and symmetry, ϑ is an odd periodic function of φ with period $c\lambda$ for $\psi = 0$. From (33.7) follows

$$\frac{d}{df}\log z'(f) = -\frac{\partial}{\partial\varphi}\log q + i\,\frac{\partial\vartheta}{\partial\varphi}. \tag{33.8}$$

For $\psi = 0$, $\partial\vartheta/\partial\varphi$ is an even periodic function of φ with removable singularities at the crests; we expand it in a Fourier series:

$$\frac{\partial\vartheta}{\partial\varphi} = \frac{\pi}{c\lambda}\left[a_0 + a_1\cos\frac{2\pi\varphi}{c\lambda} + a_2\cos\frac{4\pi\varphi}{c\lambda} + \cdots\right]. \tag{33.9}$$

The a_k are real. Substitute (33.9) into (33.8) and rewrite it in the following way:

$$\left[\frac{d}{df}\log z'(f) - i\,\frac{\pi}{c\lambda}\sum_{n=0}^{\infty} a_n\,e^{-i\,2n\pi f/c\lambda}\right]_{\psi=0} = -\frac{\partial}{\partial\varphi}\log q - \frac{\pi}{c\lambda}\sum_{n=1}^{\infty} a_n \sin\frac{2n\pi\varphi}{c\lambda}. \tag{33.10}$$

Now consider the function

$$Z(f) = \frac{d}{df}\log z'(f) - i\,\frac{\pi}{c\lambda}\sum_{n=0}^{\infty} a_n\,e^{-i\,2n\pi f/c\lambda}. \tag{33.11}$$

$Z(f)$ is defined in the whole lower half-plane, is regular for $\psi < 0$, and, as $\psi \to -\infty$, $Z(f) \to -i\pi a_0/c\lambda$. Moreover, from (33.10) Z is also real on the real axis and hence may be extended by reflection to the upper half-plane. Z is then a function with only singularities on the real axis at the points $\varphi = n c\lambda$ associated with the crests. The form of the singularity may be determined from (33.5). In fact, near $f = 0$

$$\frac{d}{df}\log z' = -\frac{1}{3}\,\frac{1}{f}. \tag{33.12}$$