where

$$\vartheta_0 = \begin{cases} \arccos\,(4g\,T')^{\frac{1}{4}}/u_0 & \text{if} \quad 4g\,T' \leqq u_0^4 \\ 0 & \text{if} \quad 4g\,T' \geqq u_0^4 \end{cases}$$

and

$$\left.\begin{matrix} k_1(\vartheta) \\ k_2(\vartheta) \end{matrix}\right\} = \frac{u_0^2 \cos^2\vartheta \pm \sqrt{u_0^4 \cos^4\vartheta - 4g\,T'}}{2\,T'}.$$

One may easily show that

$$\nu \sec^2\vartheta < k_1(\vartheta) \leqq \sqrt{g/T'} \leqq k_2(\vartheta) \leqq u_0^2 \cos^2\vartheta/T'.$$

As $T' \to 0$ it is then evident that the integral involving k_2 vanishes and that (24.31) reduces to (13.36).

One may carry out an asymptotic investigation of (24.31), or of φ_x, along the lines of (13.38) and following. However, the analysis is considerably more complicated. The behavior of the wave pattern is roughly as follows. For $u_0^4/4g\,T' \leqq 1$, $\varphi_x(R, \alpha, y)$ is $O(R^{-1})$ for all α, and the disturbance is chiefly local. There is a constant $c > 1$ such that when $1 < u_0^4/4g\,T' < c$ the wave pattern is a superposition of two sets of waves corresponding to the two roots k_1 and k_2. Those corresponding to k_2 are capillary waves which precede the source and bend around it so that their crests eventually make an angle $\frac{1}{2}\pi + \vartheta_0$ with the x-axis. The gravity waves corresponding to k_1 behave similarly except that they follow the source and are longer. If $u_0^4/4g\,T' > c$, a second angle, say ϑ_1, appears, where $\vartheta_1 < \vartheta_0$. There are now three sets of waves. Those associated with k_2 behave as described above. The gravity waves, however, consist of both transverse waves spanning the angle between $\pm(\frac{1}{2}\pi + \vartheta_1)$ and diverging waves which now lie in the wedge bounded by $\frac{1}{2}\pi + \vartheta_1$ and $\frac{1}{2}\pi + \vartheta_0$ and its reflection. One will find a sketch in LAMB's *Hydrodynamics* (1932, p. 470) which was computed for the similar problem of a moving pressure point, the so-called "fishline problem". A precise value for the constant c does not seem to be known. The free surface η may be computed from

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$$\eta(x, z) = \frac{u_0}{g}\left[\varphi_x(x, 0, z) + \frac{T}{\varrho\, u_0^2}\,\varphi_{xy}(x, 0, z)\right].$$

In spite of the general complexity of the asymptotic analysis of (24.31), it is relatively easy to find the asymptotic form of η directly ahead ($\alpha = 0°$) and directly behind ($\alpha = 180°$):

$$\left.\begin{aligned} &\alpha = 0: \\ &\eta(x, z) = -\,8m\,\frac{u_0}{g}\,k_2^2\left[1 + \frac{T'}{u_0^2}\,k_2\right]\sqrt{\frac{\pi}{2k_2 R}}\;\frac{T'\,k_2^2 + g}{[(T'\,k_2^2 - g)\,(3\,T'\,k_2^2 + g)]^{\frac{1}{2}}}\times \\ &\qquad \times e^{k_2 b}\cos(k_2 R - \tfrac{1}{4}\pi) + O(R^{-1}); \end{aligned}\right\} \tag{24.32}$$

$$\left.\begin{aligned} &\alpha = 180°: \\ &\eta(x, z) = 8m\,\frac{u_0}{g}\,k_1^2\left[1 + \frac{T'}{u_0^2}\,k_1\right]\sqrt{\frac{\pi}{2k_1 R}}\;\frac{T'\,k_1^2 + g}{[(g - T'\,k_1^2)\,(3\,T'\,k_1^2 + g)]^{\frac{1}{2}}}\times \\ &\qquad \times e^{k_1 b}\cos(k_1 R - \tfrac{3}{4}\pi) + O(R^{-1}); \end{aligned}\right\} \tag{24.33}$$

here

$$k_1 = k_1(0) = \frac{u_0^2}{2T'}\left[1 - \sqrt{1 - 4\,T' g/u_0^4}\right],$$

$$k_2 = k_2(0) = \frac{u_0^2}{2T'}\left[1 + \sqrt{1 - 4\,T' g/u_0^4}\right]$$

and we assume $u_0^2 > 4T'g$. One may see rather clearly the effect upon k_1 and k_2 of varying T' and u_0 by finding them as the intersection of the graphs of $T'k^2+g$ and $u_0^2 k$.

There is no special difficulty in finding source solutions for two-dimensional motion, and the asymptotic behavior is of course easier to determine. The related problem of a moving concentrated pressure is treated in LAMB (1932, §§ 270,1). For this problem a paper by DEPRIMA and WU (1957) is particularly instructive, for they obtain the solution by first formulating the initial-value problem and then finding the limit as $t \to \infty$. In addition, they analyze the form of the surface for large but finite values of t.

25. Waves in a viscous fluid. If one abandons the assumption of a perfect fluid with irrotational motion, one loses at the same time many convenient and powerful mathematical tools from potential theory and the theory of functions of a complex variable. However, the simplifications introduced by the infinitesimal-wave approximation are sufficient to allow obtaining a number of solutions of interest, most of which have been known for many years. However, discovery of errors in early work has resulted in several recent papers. Furthermore, in connection with the theory of stability of interfaces the subject has again attracted attention; this work will be summarized in Sect. 26. One will find general expositions of many of the fundamental results in LAMB (1932, §§ 348 to 351), and LEVICH (1952, pp. 467—497). LONGUET-HIGGINS (1953b) gives a valuable discussion of the perturbation procedure and carries through certain second-order computations.

Subject to the limitations of the approximation one can find solutions for periodic standing waves in fluid of both infinite and finite depth with a free surface, at the interface of two different fluids in which either may have a fixed horizontal plane as its other boundary, and at the interface and free surface when two different fluids are superposed, the upper one having a free surface. In all cases the presence of surface tension may be admitted. By making use of such solutions together with Fourier analysis one can find the solution to the Cauchy-Poisson initial-value problem [cf. SRETENSKII (1941)].

In general, in the investigation of standing waves one is particularly interested in two things, the effect of viscosity upon the relation between wave-length and frequency, and the rate of decay of amplitude. As an alternative to examining the rate of decay, one may instead assume that a space-and time-periodic pressure has been applied to the free surface and determine the rate of transfer of energy necessary to maintain a steady oscillation.

One may still, as for perfect fluids, combine standing-wave solutions which are out of phase in order to form progressive waves. In a coordinate system moving with the waves the wave system will be stationary but the motion will not be steady for, as a result of viscosity, it will decay unless a periodic pressure distribution is moving with the waves and doing work upon the fluid. Fourier analysis may be used to obtain the fluid motion resulting from an arbitrary moving pressure distribution. Indeed, one need not restrict oneself to a pressure distribution but may also include a distribution of shearing stress at the free surface. If a pressure and shearing distribution of localized extent is moving over the fluid the dissipation of wave energy in viscosity will show up in a diminution of amplitude, as one moves away from the pressure area, which is more rapid than for a perfect fluid. Such problems have been investigated by SRETENSKII (1941, 1957) and by WU and MESSICK (1958). The latter include the effect of surface tension and make a particularly thorough study of the behavior of the solution; they restrict themselves to two-dimensional motion. One should keep in mind that if the fluid is of finite depth it is no longer equivalent to formulate a problem in which the pressure distribution is fixed and the fluid moves with a constant mean velocity.

Instead of attempting to construct a steady progressive-wave solution by means of a moving pressure distribution, one may instead assume that the progressive waves have been somehow initiated and then study their rate of decay with distance from the wave-maker. (This is, of course, closely related to finding the decay with time of an initially given progressive wave.) Studies of this nature have been made by BIESEL (1949) and CARRY (1956), who investigated especially the effect of the bottom, by URSELL (1952), who investigated the effect of side walls for infinite depth, and by HUNT (1952), who combined the two. Dissipation with distance when no walls are present has been treated by DMITRIEV (1953) in connection with the theory of the wave-maker. A point of physical interest in these studies is the relative contribution to dissipation of shearing motion near the surface, near the bottom, near the walls, and within the fluid. CASE and PARKINSON (1957) have studied the damping of standing waves in a circular cylinder of finite depth, making use of the linearized equations of this section; their experimental data seem to confirm the theoretical predictions when the cylinder walls are sufficiently polished. KEULEGAN (1959) has made further measurements with rectangular basins; he finds a striking difference between fluids which wet the container walls and those which do not, but confirms the theory for large enough containers.

The fluid motion resulting from a submerged stationary source of pulsing strength has been derived by DMITRIEV (1953) for two-dimensional motion and infinite depth. SRETENSKII (1957) has carried through the calculations for steady motion of a source in three dimensions. Unfortunately, the source function is not now as useful a tool for constructing solutions to special boundary-value problems as it is for perfect fluids. In particular, one can no longer satisfy the proper boundary condition on a steadily moving body by means of distributions of sources and sinks, as was possible in Sect. 20β. On the other hand, distributions of pulsating sources may still be used to satisfy the linearized boundary conditions on certain types of stationary oscillating bodies. Thus, if the motion is such that the linearized boundary condition specifies the velocity normal to the surface together with zero tangential velocity, then a source distribution may prove useful. For example, the wave-maker problems formulated in (19.26) and (19.31) may be treated in this fashion; DMITRIEV (1953) has done this.

A fundamental assumption of the preceding remarks is that the motion is laminar. Such an assumption seems to be in harmony with the asumption of small motions which is made in deriving the equations of the present section. However, the possible occurrence of turbulent motion in progressive waves has been reported by DMITRIEV and BONCHKOVSKAYA (1953) who found experimental evidence for it near the surface, where the vorticity was highest. The photographs in Fig. 7 do not seem to show any evidence of it, but this may result from special circumstances of the experiments. BOWDEN (1950) has essayed a theory based on VON KÁRMÁN's similarity hypothesis; further references are given there. In the case of steady free-surface flow in a channel the importance of turbulence in modifying the mean-velocity profile is almost obvious. However, investigations have been confined to the necessary modifications of the shallow-water approximation and will be discussed elsewhere.

Finally, we note that much of the theory given below for a constant surface tension T can, in fact, be extended to a more general surface condition. This is indicated in Lamb (1932, §§ 351) and carried out by Dorrestein (1951) in some detail for infinite depth. He includes compressibility of the surface film, hysteresis and a "surface viscosity". An earlier investigation of the effect of generalized surface conditions is due to Wieghardt (1943).

α) Linearized equations and simple solutions. The linearized equations and boundary conditions have already been derived in Sect. 10. For a stratified fluid with interface at $y=0$ the zeroth-order equations are given in (10.2), the first-order in (10.3). For a single fluid with free surface they are given in (10.4). It is customary and convenient to combine the zeroth- and first-order equations. Thus, if in (10.4) we let $p=p^{(0)}+\varepsilon p^{(1)}$ and $\boldsymbol{v}=\varepsilon\boldsymbol{v}^{(1)}$, then the equations become

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$$\left.\begin{aligned} &u_x+v_y+w_y=0\\ &\boldsymbol{v}_t=-\frac{1}{\varrho}\operatorname{grad}(p+\varrho g y)+\nu\Delta\boldsymbol{v},\\ &u_y+v_x=w_y+v_z=0 && \text{for}\quad y=0,\\ &p-\varrho g\eta-2\mu v_y=-T(\eta_{xx}+\eta_{zz})+\bar{p} && \text{for}\quad y=0,\\ &\eta_t(x,z,t)=v(x,0,z,t). \end{aligned}\right\} \tag{25.1}$$

One may clearly combine (10.2) and (10.3) in the same way. In order to obtain the proper equations in a coordinate system moving to the right with velocity u_0, one need only replace $\partial/\partial t$ by $\partial/\partial t-u_0\,\partial/\partial x$.

The standard procedure for solving the equations is to represent the motion as a potential flow plus a rotational flow and to determine the pressure from the potential part. Thus, let

$$\boldsymbol{v}=\boldsymbol{v}^{(p)}+\boldsymbol{v}^{(r)} \tag{25.2}$$

where

$$\boldsymbol{v}^{(p)}=\operatorname{grad}\Phi \tag{25.3}$$

and let

$$p=-\varrho\Phi_t-\varrho g y. \tag{25.4}$$

It then follows from the second equation in (25.1) that $\boldsymbol{v}^{(r)}$ must satisfy

$$\frac{\partial}{\partial t}\boldsymbol{v}^{(r)}=\nu\Delta\boldsymbol{v}^{(r)}. \tag{25.5}$$

The relation between $\boldsymbol{v}^{(p)}$ and $\boldsymbol{v}^{(r)}$ is established through the boundary conditions. In the several examples treated below the motion is two-dimensional. However, there is no difficulty in principle and not much additional algebraic complexity in solving the analogous three-dimensional problems. The essential simplification in two dimensions is that the components of $\boldsymbol{v}^{(r)}$ may be expressed, as a consequence of the continuity equation, in terms of a single function Ψ:

$$u^{(r)}=\frac{\partial\Psi}{\partial y},\qquad v^{(r)}=-\frac{\partial\Psi}{\partial x}. \tag{25.6}$$

It then follows easily from (25.5) that

$$\frac{\partial\Psi}{\partial t}=\nu\Delta\Psi. \tag{25.7}$$

Standing waves-infinite depth. We shall try to find a solution to the equations which has a profile of the form

$$\eta(x,t)=A(t)\cos(mx+\alpha). \tag{25.8}$$

If such a solution exists, the nature of $A(t)$ will, of course, be of especial interest. We take Φ and Ψ of the form

$$\Phi = F(y,t)\cos(mx+\alpha), \qquad \Psi = G(y,t)\sin(mx+\alpha). \tag{25.9}$$

Eq. (25.7) then implies that

$$\Psi = (c\,e^{ly} + d\,e^{-ly})\,e^{\omega t}\sin(mx+\alpha), \tag{25.10}$$

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$$l^2 = m^2 + \frac{\omega}{\nu}. \tag{25.11}$$

Neither l nor ω need be real. The form of Φ is further determined by $\Delta\Phi = 0$ and its relation to Ψ through the third boundary condition in (25.1). It must be

$$\Phi = (a\,e^{my} + b\,e^{-my})\,e^{\omega t}\cos(mx+\alpha). \tag{25.12}$$

If, as usual, we require the motion to remain bounded as $y \to -\infty$, we must take $b=0$. If l has a non-vanishing real part, which we assume for the present, we may without loss of generality take it to be positive. Hence one must have $d=0$. It follows from the third condition of (25.1) that

$$a = c\,\frac{l^2+m^2}{2m^2}. \tag{25.13}$$

Substitution in the formula for η_t and integration with respect to t yield

$$\eta = c\,\frac{1}{2\nu m}\,e^{\omega t}\cos(mx+\alpha) = A_0\,e^{\omega t}\cos(mx+\alpha). \tag{25.14}$$

Finally, one must substitute into the dynamical boundary condition in (25.1). There p is computed from (25.4) with $y=0$. For future use we retain the external pressure distribution $\bar{p}$, which we take in the form

$$\bar{p} = p_0\,e^{\omega t}\cos(mx+\alpha), \tag{25.15}$$

where p_0 may be complex. The boundary condition yields an equation relating l and m:

$$\nu^2(l^2+m^2)^2 - 4\nu^2 m^3 l + g m + T' m^3 = -m\,\frac{p_0}{\varrho}\,\frac{2m\nu}{c} = -m\,\frac{p_0}{\varrho A_0}, \tag{25.16}$$

or, by making use of (25.11), an equation relating ω and m:

$$(\omega + 2m^2\nu)^2 - 4\nu^2 m^3\sqrt{m^2+\frac{\omega}{\nu}} + g m + T' m^3 = -m\,\frac{p_0}{\varrho A_0}. \tag{25.17}$$

Consider first Eq. (25.16) with $p_0 = 0$ and let

$$z = \frac{l}{m}, \qquad K = \frac{g m + T' m^3}{\nu^2 m^4}. \tag{25.18}$$

Then (25.16) takes the dimensionless form

$$(z^2+1)^2 - 4z + K = 0.$$

An examination of this equation shows that two of its roots are always complex with negative real parts. These roots are discarded since the corresponding motion would not die out as $y \to -\infty$; in fact, we explicitly assumed earlier that l has a positive real part. [Note that if we had made the other possible assumption, i.e., that l had a negative real part, the resulting equation corresponding to (25.16) would have had roots with positive real part, again to be discarded.]

The other two roots have positive real part. Whether or not there is an imaginary part depends upon the value of K. There is a critical value $K_c \approx 0.581$ such that if $K < K_c$ the two allowable solutions are both real. If $K > K_c$, the solutions are complex conjugates. Let the two complex roots of positive real part be denoted by $l_1 \pm i l_2$. Then one may establish that $l_1/m > 0.683$. When the two admissible roots are real, both of them lie between 0 and m.

One may find the values of ω associated with the two admissible roots from (25.11). If they are both real ($K < K_c$), then $\omega = -\nu(m^2 - l^2) < 0$. In this case the motion is critically damped and the initial configuration of the surface gradually subsides. This occurs for a given m if ν is large enough. On the other hand, no matter how small ν is, it also occurs when m is large enough, i.e., for very small wavelength. If the two admissible roots are complex ($K > K_c$), then

$$\left.\begin{aligned} \omega &= -\nu m^2 \left(1 - \frac{l_1^2}{m^2} + \frac{l_2^2}{m^2} \pm 2i \frac{l_1 l_2}{m^2}\right) \\ \text{and}\qquad e^{\omega t} &= 2\,e^{-\nu m^2 \left(1 - \frac{l_1^2}{m^2} + \frac{l_2^2}{m^2}\right) t} \cos 2 \frac{l_1 l_2}{m^2} t. \end{aligned}\right\} \tag{25.19}$$

One may establish that $1 - l_1^2/m^2 + l_2^2/m^2 > 0.534$, so that this motion consists of damped standing-wave oscillations. The larger m is, the more quickly it is damped.

Because of the relative complexity of Eqs. (25.16) and (25.17), it is convenient and leads to more perspicuous results to find the relation between ω and m in the two limiting cases of small and large viscosity. First consider the case of small viscosity. If in (25.17) one lets $\nu \to 0$, one regains the relation $\omega^2 = -gm - T' m^3$ of (24.9); let $\sigma_0^2 = gm + T' m^3$. However, if one retains all terms of the first power in ν, (25.17) becomes

$$\omega^2 + 4\nu m^2 \omega + gm + T' m^3 = 0, \tag{25.20}$$

which has roots

$$-2m^2\nu \pm \sqrt{4m^4\nu^2 - gm - T' m^3} \approx -2m^2\nu \pm i\sigma_0 \tag{25.21}$$

if $4m^4\nu^2 \ll gm + T' m^3$. Hence the surface profile can be described by

$$\eta = A_0\, e^{-2m^2\nu t} \cos(\sigma_0 t + \tau) \cos(mx + \alpha). \tag{25.22}$$

To this order of approximation, the frequency σ_0 is related to m as in a perfect fluid, but the amplitude is gradually damped. To have some idea of the orders of magnitude involved in the damping, one should consult the table on p. 645 where the row τ_0 gives computations relevant to this.

In order to find the behavior for large ν, divide equation (25.17) by $4m^4\nu^2$ and expand the term $[1 + \omega/m^2\nu]^{\frac{1}{2}}$ in a series. If one retains only terms in ν^{-1} and ν^{-2}, the resulting equation leads to

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$$3\omega^2 + 4m^2\nu\omega + 2(gm + T'\omega^3) = 0. \tag{25.23}$$

The two solutions, both real and negative, are approximately, if $4m^4\nu^2 \gg gm + T' m^3$,

$$\omega_1 = -\frac{gm + T' m^3}{2m^2\nu}, \qquad \omega_2 = -\frac{4}{3} m^2 \nu. \tag{25.24}$$

Here $|\omega_1| < |\omega_2|$ and hence ω_1 is the more important root inasmuch as it represents a slower damping of the motion. As is pointed out by LAMB (1932, p. 628), the root ω_1 corresponds to a value of l only slightly less than m, so that the motion is nearly irrotational. It should also be noted that by different methods of analyzing (25.17) for large ν one may obtain somewhat different coefficients for ω_2.

In the preceding analysis it was assumed explicitly that l had a non-vanishing real part. If l is pure imaginary, $l = i l'$, another family of solutions exists. It is now convenient two write Φ and Ψ in the forms

$$\left.\begin{aligned} \Phi &= a\, e^{my}\, e^{\omega t} \cos(mx + \alpha), \\ \Psi &= (c \cos l' y + d \sin l' y)\, e^{\omega t} \sin(mx + \alpha), \end{aligned}\right\} \tag{25.25}$$

where

$$\omega = -\nu(l'^2 + m^2) < 0. \tag{25.26}$$

The motion is thus a purely subsiding one. The boundary conditions determine the following relations between a, c, and d:

$$a = c\,\frac{m^2 - l'^2}{2m^2}, \qquad d = c\,\frac{1}{4\nu^2 m^3 l}\left[\nu^2 (m^2 - l'^2)^2 + g m + T' m^3\right]. \tag{25.27}$$

All real values of l' are now admissible. The surface profile is given by

$$\eta = c\,\frac{1}{2m\nu}\, e^{\omega t} \cos(mx + \alpha). \tag{25.28}$$

The two sets of solutions may now be used to investigate the development of an initial disturbance [cf. SRETENSKII (1941)].

Forced standing waves. We may apply Eq. (25.16) or (25.17) to answer the following question. Suppose that m is given. Can we determine p_0 in such a way that a steady standing wave

$$\eta = A_0\, e^{-i\sigma t} \cos(mx + \alpha) \tag{25.29}$$

of prescribed frequency σ is maintained? From (25.17) p_0 is then determined by

$$-m\,\frac{p_0}{\varrho A_0} = (2m^2\nu - i\sigma)^2 - 4\nu^2 m^3 \sqrt{m^2 - i\frac{\sigma}{\nu}} + g m + T' m^3. \tag{25.30}$$

If, for small viscosity, one discards terms higher than the first in ν, one obtains

$$p_0 = 4 i \sigma \mu m A_0 - \sigma^2 + g m + T' m^3. \tag{25.31}$$

If we take $\sigma^2 = g m + T' m^3$, the frequency obtained from perfect-fluid theory, the necessary pressure distribution becomes

$$\bar{p} = 4\sigma\mu m A_0 i\, e^{-i\sigma t} \cos(mx + \alpha). \tag{25.32}$$

Thus the pressure must lead the surface displacement by a quarter of a period.

Standing waves-finite depth. If the fluid is of depth h, the analysis is similar to that above, but yields expressions of much greater complexity. The functions Φ and Ψ may be shown to have the forms

$$\left.\begin{aligned} \Phi &= \frac{1}{m}\left[d l \cosh m(y+h) + c m \sinh m(y+h)\right] e^{\omega t} \cos(mx + \alpha), \\ \Psi &= \left[c \cosh m(y+h) + d \sinh m(y+h)\right] e^{\omega t} \sin(mx + \alpha), \end{aligned}\right\} \tag{25.33}$$

where again

$$\omega = \nu(l^2 - m^2). \tag{25.34}$$

Let

$$L = \cosh l h, \qquad L' = \sinh l h, \qquad M = \cosh m h, \qquad M' = \sinh m h.$$

Then c and d are related by

$$2m(c m M + d l M') - (l^2 + m^2)(c L + d L') = 0. \tag{25.35}$$

The relation between l and m corresponding to (25.16) becomes

$$\left.\begin{aligned}\nu^2(l^2+m^2)^2\frac{(l^2+m^2)(l\,L\,M-m\,L'\,M')-2m^2 l}{(l^2+m^2)(l\,L\,M'-m\,L'\,M)}-\\-4\nu^2 m^3 l\frac{2m\,(m\,M\,L-l\,M'\,L')-(l^2+m^2)}{2m\,(m\,M\,L'-l\,M'\,L)}+g\,m+T'm^3=-m\frac{p_0}{\varrho A_0}\end{aligned}\right\}\tag{25.36}$$

and the surface profile is

$$\eta=\frac{1}{2\nu\,m}(c\,L+d\,L')\,e^{\omega t}\cos(m\,x+\alpha)=A_0\,e^{\omega t}\cos(m\,x+\alpha).\tag{25.37}$$

The formulas become more perspicuous in the case of small viscosity and no external pressure and exhibit the importance of the presence of the bottom. If in (25.36) one sets $p_0=0$ and retains only terms of order ν^0, $\nu^{\frac{1}{2}}$ and ν, the following equation results:

$$\left.\begin{aligned}\omega^3-m\sqrt{\nu}\tanh m h\,\omega^{\frac{5}{2}}+\tfrac{9}{2}m^2\nu\,\omega^2+(g\,m+T'm^3)\tanh m h\,\omega-\\-(g\,m+T'm^3)\,m\sqrt{\nu}\,\omega^{\frac{1}{2}}+\tfrac{1}{2}(g\,m+T'm^3)\,m^2\nu\tanh m h=0.\end{aligned}\right\}\tag{25.38}$$

One may solve this equation by expanding ω in powers of $\nu^{\frac{1}{2}}$,

$$\omega=\omega_0+\omega_1\sqrt{\nu}+\omega_2\nu+\cdots,$$

substituting in (25.38) and keeping only terms in ν^0, $\nu^{\frac{1}{2}}$ and ν. The term independent of ν yields $\omega_0=\pm i\sigma_0$, where σ_0 is given in (24.10) and is the frequency for an inviscid fluid. To the order of accuracy consistent with (25.38), one finds

$$\omega=\pm i\sigma_0-(1\pm i)\,\frac{1}{2}m\sqrt{2\sigma_0\nu}\,\mathrm{cosech}\,2m h-2m^2\nu\,\frac{\cosh 4m h+\cosh 2m h-1}{\cosh 4m h-1}.\tag{25.39}$$

The first two terms were given by HOUGH (1897). The correct expression (25.39) was first given by BIESEL (1949); HOUGH had given $-2m^2\nu$ for the last term but the apparently made an error in calculation, for (25.39) was derived independently of BIESEL's work and has also been checked by CARRY (1956) [BASSET's analysis (1888, p. 314) overlooks the terms in $\nu^{\frac{1}{2}}$].

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The formula (25.39) should be compared with (25.21), the corresponding formula for infinite depth. There the effect of viscosity enters only with the first power of ν. The dissipation of energy in the body of the fluid is evidently of less importance than in the vicinity of the bottom. When two fluids are superposed, a similar phenomenon occurs in the neighborhood of the interface [cf. (25.44)].

Standing waves-stratified fluids. Consider now the situation in which a fluid typified by ϱ_1 and μ_1 fills the space $y<0$ and another typified by $\varrho_2<\varrho_1$ and μ_2 the space $y>0$. The equations to be satisfied in the two fluids and at their interface are given in (10.3). The method of solution is analogous to that used for a single fluid. However, separate functions Φ_1, Ψ_1, and Φ_2, Ψ_2 are needed for the lower and upper fluids. For a standing-wave solution they may be taken in the form

$$\left.\begin{aligned}\Phi_1&=a_1\,e^{\omega t}\,e^{m y}\cos(m\,x+\alpha), &\Psi_1&=b_1\,e^{\omega t}\,e^{l_1 y}\sin(m\,x+\alpha),\\\Phi_2&=a_2\,e^{\omega t}\,e^{-m y}\cos(m\,x+\alpha), &\Psi_2&=b_2\,e^{\omega t}\,e^{-l_2 y}\sin(m\,x+\alpha),\end{aligned}\right\}\tag{25.40}$$

where we assume both l_1 and l_2 to have positive real parts. ω, l_1, l_2 and m are related by the equation

$$\omega=\nu_1(l_1^2-m^2)=\nu_2(l_2^2-m^2).\tag{25.41}$$

Substitution of (25.40) in the various boundary conditions at $y=0$ gives four homogeneous equations relating a_1, a_2, b_1, and b_2. The determinant of the coefficients set equal to zero yields another relation between ω_1, l_1, l_2 and m:

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$$\left.\begin{aligned}&[(\varrho_1+\varrho_2)\,\omega^2+(\varrho_1-\varrho_2)\,g\,m+T\,m^3]\,[\mu_1\,m+\mu_2\,l_2+\mu_2\,m+\mu_1\,l_1]+\\&\quad+4\omega\,m\,(\mu_1\,m+\mu_2\,l_2)\,(\mu_2\,m+\mu_1\,l_1)=0.\end{aligned}\right\}\tag{25.42}$$

In the limiting case of small viscosity, (25.42) gives

$$\omega^2+\frac{4m}{\varrho_1+\varrho_2}\,\frac{\sqrt{\varrho_1\,\varrho_2\,\mu_1\,\mu_2}}{\sqrt{\mu_1\,\varrho_1}+\sqrt{\mu_2\,\varrho_2}}\,\omega^{\frac{3}{2}}+\frac{(\varrho_1-\varrho_2)\,g\,m+T\,m^3}{\varrho_1+\varrho_2}=0.\tag{25.43}$$

This has the approximate solutions, when the coefficient of $\omega^{\frac{3}{2}}$ is small relative to the last term,

$$\omega=\pm i\,\sigma_0-\frac{1\pm i}{\sqrt{2}}\sqrt{\sigma_0}\,\frac{2m}{\varrho_1+\varrho_2}\cdot\frac{\sqrt{\varrho_1\,\varrho_2\,\mu_1\,\mu_2}}{\sqrt{\varrho_1\,\mu_1}+\sqrt{\varrho_2\,\mu_2}}-\frac{2m^2}{\varrho_1\,\varrho_2}\cdot\frac{\varrho_1\,\mu_1^2+\varrho_2\,\mu_2^2}{\left(\sqrt{\varrho_1\,\mu_1}+\sqrt{\varrho_2\,\mu_2}\right)^2}\tag{25.44}$$

where σ_0 is the perfect-fluid frequency given in Eq. (24.14). This solution was first given by HARRISON (1908). The most significant physical fact about (25.44) when compared with (25.21) is that, to the order of approximation considered, the latter shows a rate of decay proportional to $m^2\nu$ and no influence of viscosity on the frequency, whereas (25.44) shows a rate of decay and an alteration of the frequency proportional to $m\sqrt{\nu}$ (in a dimensional sense). The greater importance of viscosity for stratified fluids may be ascribed to the different boundary condition at the interface. HARRISON computed the wave velocity and modulus of decay (time required for the amplitude to decrease by a factor e^{-1}) for an air-water interface at 17° C ($\varrho_1=1$, $\varrho_2=0.00129$, $\nu_1=0.0109$, $\nu_2=0.139$, $T=74$ in c.g.s. units). In the following table reproduced from HARRISON's paper v_0, v_c

Wavelength (cm)	1	10	100	1000
v_0 (cm/sec)	12.48	39.46	124.79	394.62
v_c	24.90	40.05	124.81	394.62
v	24.89	40.04	124.81	394.62
τ_0	$1^{\mathrm{s}}.162$	$1^{\mathrm{m}}\,56^{\mathrm{s}}.2$	$3^{\mathrm{h}}\,12^{\mathrm{m}}\,39^{\mathrm{s}}.4$	$321^{\mathrm{h}}\,5^{\mathrm{m}}\,40^{\mathrm{s}}$
τ	$1^{\mathrm{s}}.125$	$1^{\mathrm{m}}\,34^{\mathrm{s}}.1$	$1^{\mathrm{h}}\,21^{\mathrm{m}}\,40^{\mathrm{s}}.6$	$36^{\mathrm{h}}\,50^{\mathrm{m}}\,36^{\mathrm{s}}$
τ_c	$1^{\mathrm{s}}.106$	$1^{\mathrm{m}}\,34^{\mathrm{s}}.0$	$1^{\mathrm{h}}\,21^{\mathrm{m}}\,40^{\mathrm{s}}.3$	$36^{\mathrm{h}}\,50^{\mathrm{m}}\,34^{\mathrm{s}}$

and v are the wave velocities neglecting, respectively, both surface tension and viscosity, viscosity, and neither; τ_0, τ, τ_c are the moduli of decay taking account of the water viscosity only, a water-air interface without surface tension and a water-air interface with surface tension. A striking aspect is the apparent importance of the air-water interface in damping long waves and almost total lack of influence on wave velocity [the latter fact is obvious from (25.44)].

For very large viscosities the results are analogous to those for a single fluid. The two values of ω analogous to those in (25.24) are

$$\omega_1=-\frac{(\varrho_1-\varrho_2)\,g\,m+T\,m^3}{\varrho_1+\varrho_2}\,\frac{1}{2m^2}\,\frac{\varrho_1+\varrho_2}{\mu_1+\mu_2},\qquad\omega_2=-m^2\,\frac{\mu_1+\mu_2}{\varrho_1+\varrho_2}.\tag{25.45}$$

The analysis of the roots of (25.42) for general values of ν_1 and ν_2 is difficult. However, it has been carried through by CHANDRASEKHAR (1955, especially pp. 170—173) for the special situation $\nu_1=\nu_2$ and $T=0$. In this case $l_1=l_2$.

The behavior is similar to that described for (25.17) except that the critical value K_c separating a steadily decaying motion from an oscillatory decaying one is now a function of $(\varrho_1 - \varrho_2)/(\varrho_1 + \varrho_2)$. This value (actually a different one since he chooses a different parameter) is tabulated for a variety of density combinations. Further analysis of (25.17) may be found in a paper by HIDE (1955) and TCHEN (1956b).

KUSAKOV (1944) has carried through an analysis similar to HARRISON's when the upper fluid is of depth h_2, the lower of depth h_1. However, the results do not seem to be consistent with HARRISON's (or those above) when h_1 and h_2 become large. This problem has also been considered by HIDE (1955), but with an approximation that neglects the viscous boundary conditions on the walls. HARRISON, in the same paper, has treated also the problem when the upper fluid is of finite depth and with a free surface. We shall not reproduce the results except to remark that his computations show that a thin layer of fluid of slightly different density exerts a very marked influence on the damping. The effect of a variable surface tension upon wave motion is investigated briefly in LAMB (1932, § 351) and at some length in LEVICH (1952, pp. 477—490).

see errata

Pulsing stationary source. DMITRIEV (1953) has derived the form of the functions Φ and Ψ and the surface profile in the presence of a submerged source of pulsating intensity $-Q\cos\sigma t$. We shall give here only his expression for the surface profile and an asymptotic expression for large distances from the source. Let the source be located at $(0, -h_0)$ and let

$$h = h_0\sqrt{\frac{\sigma}{\nu}}, \qquad \bar{x} = x\sqrt{\frac{\sigma}{\nu}}, \qquad \bar{y} = y\sqrt{\frac{\sigma}{\nu}}, \qquad \varepsilon = \frac{\sigma^2}{g}\sqrt{\frac{\nu}{\sigma}}.$$

The surface profile is then represented by

$$\left.\begin{aligned} \eta &= \operatorname{Re}\frac{Q\,e^{i\sigma t}}{\pi}\,\frac{\sigma}{g}\int_0^\infty \frac{1-2i\chi^2}{4\varepsilon\left[i\chi^3\left(\chi-(i+\chi^2)^{\frac{1}{2}}\right)-\chi^2\right]+i(\chi-\varepsilon)}\,e^{-h\chi}\cos\bar{x}\chi\,d\chi \\ &= Q\frac{\sigma}{g}\,(1+100\,\varepsilon^4)^{\frac{1}{2}}\,e^{-h\varepsilon-4\varepsilon^3\xi\bar{x}}\cos\left(\sigma t-\varepsilon\bar{x}+4\varepsilon^3 h-\arctan 10\,\varepsilon^2\right)+\cdots. \end{aligned}\right\} \qquad (25.46)$$

26. Stability of free surfaces and interfaces. In this section we wish to examine the circumstances under which a small disturbance of a free surface or of an interface between two fluids will increase in magnitude with time. The energy for this increase may come either from available potential energy, e.g. if the lower fluid is lighter than the upper one, available kinetic energy in the case of flowing fluids, from forced motion of solid boundaries, or possibly some other source such as a given pressure distribution over a free surface. Surface tension and viscosity may be expected to have a stabilizing effect, so that special interest attaches to the study of their influence. We shall use the nature of the energy source as a convenient one for separating classes of problems, even though not every situation falls clearly into one of them.

Since the boundary conditions and equations which we shall use for the mathematical analysis have been linearized, following the assumption that the disturbances are small, one cannot expect the predictions of the theory to be valid quantitatively much beyond the initiation of an unstable motion. However, a great advantage in the use of linearized theory is that an arbitrary initial disturbance can be analyzed into Fourier components and the behavior of individual components examined separately.

α) *Interface between stationary superposed fluids.* Following our earlier notation, let us identify quantities referring to the lower fluid by the subscript 1 and to the upper fluid by 2. Let a sinusoidal disturbance of wave number m exist at the interface. Consider first the case of perfect fluids with no surface tension. Then, if both fluids are infinitely deep, the relation (14.28) must hold. If $\varrho_1 > \varrho_2$, the standing-wave solution of Sect. 14δ obtains. However, if $\varrho_1 < \varrho_2$, then $\sigma^2 < 0$ and σ is imaginary. Let $\omega^2 = -\sigma^2$, i.e.

$$\omega^2 = \frac{\varrho_2 - \varrho_1}{\varrho_2 + \varrho_1} g m. \tag{26.1}$$

Then one must replace $\cos(\sigma t + \tau)$ in the Φ_i of that section by, say, $\sinh \omega t$. The profile of the free surface is then, according to (10.8), given by

$$\eta = A \sin m x \cosh \omega t. \tag{26.2}$$

The amplitude of the initial corrugations of the surface evidently increases very rapidly with time, and the solution is a valid approximation for only a limited time interval. The nature of the disturbance need not have been restricted to $\sin m x$; any function $\varphi(x, z)$ satisfying $\Delta \varphi + m^2 \varphi = 0$ would have yielded the same behavior. Eq. (26.1) still holds if the two fluids are bounded below and above, respectively, by $y = -h_1$ and $y = h_2$ except that ω is given by

see errata

$$\omega^2 = \frac{\varrho_2 - \varrho_1}{\varrho_2 \coth m h_2 + \varrho_1 \coth m h_1} g m < \frac{\varrho_2 - \varrho_1}{\varrho_2 + \varrho_1} g m. \tag{26.3}$$

The surface is still unstable, but the rate of growth of the amplitude is slower.

Effect of surface tension. Let us now suppose that surface tension acts at the interface. Then the relation between σ and m given in (24.14) or (24.15) must hold, and a standing-wave solution is possible even if $\varrho_2 > \varrho_1$, provided that (24.16) holds, i.e.

$$\varrho_2 < \varrho_1 + \frac{T m^2}{g}. \tag{26.4}$$

Thus the interface is stable under small disturbances of sufficiently small wave length. If the unequality in (26.4) is reversed and we again set $\omega^2 = -\sigma^2$, then (26.2) holds once more and the solution is unstable. However, the value of ω^2 is less than that when $T = 0$, so that the rate of growth of the disturbance is retarded. It is also clear from the form of the relationship between ω^2 and m that there is a wave number for which ω^2, that is the rate of growth of the disturbance, is a maximum. If both fluids are of infinite depth this mode of maximum instability occurs when

$$m^2 = (\varrho_2 - \varrho_1) g/3 T. \tag{26.5}$$

The effect of finite depth of the fluids is to displace the position of the maximum to higher values of m (smaller wavelengths) but a precise calculation requires solving a transcendental equation.

Effect of viscosity. The influence of viscosity in stabilizing interfacial disturbances has been the subject of a number of recent papers, in particular Bellman and Pennington (1954), Chandrasekhar (1955), Hide (1955) and Tchen (1956). The relevant equation relating ω and m is now (25.42). Because of the high degree of this equation it is not easy to give a complete discussion of its admissible roots. However, it is easy to establish that if

$$(\varrho_1 - \varrho_2) g + T m^2 < 0, \tag{26.6}$$

then (25.42) has a positive real root ω_0 satisfying

$$0 < \omega_0 < \sqrt{(\varrho_2 - \varrho_1) g m - T m^3}. \tag{26.7}$$

Thus the presence of viscosity does not alter the conditions for instability, as the presence of surface tension did, but it does have a stabilizing effect in that the rate of growth of a disturbance is slower.

In order to show the existence of a positive root under condition (26.6), one can write (25.42) in the form

$$(\varrho_1+\varrho_2)\,\omega^2+(\varrho_1-\varrho_2)\,g\,m+T\,m^3=-4\omega\,m\,\frac{(\mu_1 m+\mu_2 l_2)\,(\mu_2 m+\mu_1 l_1)}{\mu_1 m+\mu_2 l_2+\mu_2 m+\mu_1 l_1} \tag{26.8}$$

and sketch as functions of ω the curves represented by the two sides of the equation (remembering that l_1 and l_2 are functions of ω). The statement above then follows easily from the fact that both curves are continuous and the one represented by the right-hand function starts at the origin like

$$-2\,m^2\,(\mu_1+\mu_2)\,\omega$$

and goes to $-\infty$ in the fourth quadrant, behaving as $\omega\to\infty$ like

$$-4\,\omega^{\frac{3}{2}}\,m\,\frac{\sqrt{\varrho_1\,\varrho_2\,\mu_1\,\mu_2}}{\sqrt{\varrho_1\,\mu_1}+\sqrt{\varrho_2\,\mu_2}}\,.$$

A more elaborate discussion of the roots is given by BELLMAN and PENNINGTON (1954).

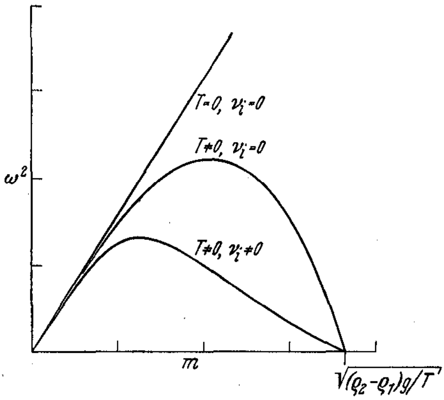

Fig. 33.

The behavior of ω_0 as a function of m in the interval defined by (26.7) and in particular the mode of maximum instability has been investigated by the authors cited earlier. CHANDRASEKHAR has computed the curves $\omega_0(m)$ for $\nu_1=\nu_2$, $T=0$ and a number of values of $(\varrho_2-\varrho_1)/(\varrho_2+\varrho_1)$. HIDE has recomputed these by an approximate method and then applied the method further to a fluid of finite depth with a continuous density variation $\varrho_0\,e^{\beta y}$. TCHEN has devised a different method of approximate computation and includes the effect of surface tension. Fig. 33, which is chiefly qualitative, shows the variation of ω^2 as a function of m in the interval of instability.

Accelerated fluid. If the whole system of fluid is being accelerated in the y-direction by a constant amount $\dot{v}_0=g_1$, then the relative motion in a moving coordinate system is the same as if the system were at rest and g had been replaced by $g+g_1$, as is immediately evident from Eq. (2.15). With this change the reasoning of the preceding paragraphs still applies. This fact was pointed out by G. I. TAYLOR (1950) who, on the basis of it, formulated the following rule (neglecting the influence of surface tension): If the fluids are being accelerated in a direction from the more to the less dense fluid, the interface is stable; in the converse case it is unstable. Experiments carried out by LEWIS (1950) for large accelerations, about $50\,g$, confirm TAYLOR'S observation and the predicted initial rate of growth. TAYLOR'S paper gave rise to a number of others treating various aspects of the instability of accelerated interfaces. In addition to those cited in the last paragraph, we mention INGRAHAM (1954), PLESSET (1954), BIRKHOFF (1954), KELLER and KOLODNER (1954), and LAYZER (1955) but shall not summarize the contents. The effect of an imposed acceleration oscillating in magnitude will be discussed in Sect. 26γ.

β) Interface between moving fluids. Consider the situation in which the fluid occupying the region $y<0$ $(y>0)$ is moving to the left with velocity $-c_1$ $(-c_2)$, and suppose that a small disturbance exists near the interface. If we suppose that the fluid is perfect and the motion in each fluid irrotational, then we may describe it by the velocity potentials

$$\Phi_i(x, y, z, t) = -c_i x + \phi_i(x, y, z, t). \tag{26.9}$$

We shall assume $c_1 \neq c_2$.

The kinematic boundary condition at the interface may be written, after linearization appropriate to the assumption of a small disturbance, in the form:

$$\eta_t(x, y, t) = c_1 \eta_x + \phi_{1y}(x, 0, z, t) = c_2 \eta_x + \phi_{2y}(x, 0, z, t). \tag{26.10}$$

The dynamical boundary condition (3.9) yields the following generalization of (10.8):

$$\varrho_1(\phi_{1t} - c_1\phi_{1x}) - \varrho_2(\phi_{2t} - c_2\phi_{2x}) + (\varrho_1 - \varrho_2) g\eta = T(\eta_{xx} + \eta_{zz}) \quad \text{for} \quad y = 0. \tag{26.11}$$

If η is eliminated between (26.10) and (26.11), one finds

$$\left.\begin{aligned} \varrho_1(\phi_{1tx} - c_1\phi_{1xx}) - \varrho_2(\phi_{2tx} - c_2\phi_{2xx}) + \frac{\varrho_1 - \varrho_2}{c_1 - c_2} g(\phi_{2y} - \phi_{1y}) + \\ + \frac{1}{c_1 - c_2} T(\phi_{2yyy} - \phi_{1yyy}) = 0. \end{aligned}\right\} \tag{26.12}$$

Let us now restrict our attention to two-dimensional motion of fluids bounded above by $y = h_2$ and below by $y = -h_1$, and let the initial displacement be $\eta(x, 0)$. Then from (15.2) we know that the subsequent motion may be resolved into harmonic progressive waves moving to the right and left. It will be sufficient for our purpose to examine a single component of the spectrum. Hence, we look for a solution in the form

$$\left.\begin{aligned} \phi_1 &= a_1 \cosh m(y + h_1)\, e^{i(mx - \sigma t)}, \\ \phi_2 &= a_2 \cosh m(y - h_2)\, e^{i(mx - \sigma t)}. \end{aligned}\right\} \tag{26.13}$$

It follows from (26.10) that $(c_1 - c_2)\eta_x = -\phi_{1y} + \phi_{2y}$. Hence

$$\eta = \frac{-i}{c_1 - c_2}[a_1 \sinh m h_1 + a_2 \sinh m h_2]\, e^{i(mx - \sigma t)}. \tag{26.14}$$

It then follows from (26.10) that

$$\frac{a_1 \sinh m h_1 + a_2 \sinh m h_2}{c_1 - c_2} = \frac{a_1 m}{\sigma + c_1 m} \sinh m h_1 = -\frac{a_2 m}{\sigma + c_2 m} \sinh m h_2. \tag{26.15}$$

Substitution of (26.13) in (26.12) and use of (26.15) yield the following relation between σ and m:

$$\varrho_1(\sigma + c_1 m)^2 \coth m h_1 + \varrho_2(\sigma + c_2 m)^2 \coth m h_2 - (\varrho_1 - \varrho_2) g m - T m^3 = 0. \tag{26.16}$$

The solution may be expressed as follows:

$$\left.\begin{aligned} \frac{\sigma}{m} = -\frac{c_1\varrho_1 \coth m h_1 + c_2\varrho_2 \coth m h_2}{\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2} \pm \\ \pm \sqrt{\frac{(\varrho_1 - \varrho_2)\frac{g}{m} + T m}{\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2} - (c_1 - c_2)^2 \frac{\varrho_1\varrho_2 \coth m h_1 \coth m h_2}{(\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2)^2}}. \end{aligned}\right\} \tag{26.16}$$

It is evident from the form of the term under the radical that σ cannot be real unless

$$(\varrho_1 - \varrho_2)\frac{g}{m} + T m > (c_1 - c_2)^2 \frac{\varrho_1\varrho_2 \coth m h_1 \coth m h_2}{\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2}. \tag{26.17}$$

It is thus evident that there are no real solutions unless the left-hand side is positive and that there may even then exist an interval of wave numbers for which the disturbance is unstable (if both g and T are zero, such a velocity discontinuity is always unstable). If one assumes $\varrho_1 > \varrho_2$, the minimum value of the left-hand side is

$$2\sqrt{(\varrho_1 - \varrho_2)\, g\, T} \tag{26.18}$$

and occurs for $m^2 = (\varrho_1 - \varrho_2)\, g/T$. Since

$$\frac{\varrho_1 \varrho_2 \coth m h_1 \coth m h_2}{\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2} > \frac{\varrho_1 \varrho_2}{\varrho_1 + \varrho_2}, \tag{26.19}$$

the disturbance will be unstable for some wave numbers whenever

$$(c_1 - c_2)^2 > 2\, \frac{\varrho_1 + \varrho_2}{\varrho_1 - \varrho_2} \sqrt{(\varrho_1 - \varrho_2)\, g\, T}. \tag{26.20}$$

One may conclude from (26.19) that the horizontal walls have a destabilizing effect in the sense that wave numbers which are stable for infinitely deep fluids may become unstable modes in the presence of walls. For an air-water interface the right side of (26.20) is about $(646 \text{ cm/sec})^2$. The corresponding wavelength is 1.71 cm; if the water is at rest ($c_1 = 0$), then the wave velocity is 0.84 cm/sec in the direction of the wind.

Let us suppose that c_1 and c_2 are both positive, i.e. that both fluids really do flow to the left. Then it follows from (26.16) that, if the roots are real, one of them is always negative and thus, from (26.13), represents a wave moving along the interface in the direction of the stream. The other will propagate upstream if

$$(\varrho_1 - \varrho_2)\, \frac{g}{m} + T m > \varrho_1 c_1^2 \coth m h_1 + \varrho_2 c_2^2 \coth m h_2, \tag{26.21}$$

otherwise also downstream.

An investigation along the above lines of the stability of an interface between flowing fluids was first given by KELVIN (1871). Similar treatments with additional information may be found in many texts, especially LAMB (1932, §§ 232, 268) and RAYLEIGH'S *Theory of Sound* (Cambridge 1929, § 365). KELVIN'S intention was to try to predict the minimum wind velocity which will cause a small disturbance on smooth water to increase in amplitude, and to find the unstable wave lengths. The predicted minimum velocity, roughly 650 cm/sec, is much higher than the observed minimum which is about 100 cm/sec. An evident objection to the analysis above is that viscosity of both air and water has been neglected. Since this alters in an essential way the behavior of the fluids near the interface, it is not surprising that the prediction is not accurate. One should not expect confirmation except in circumstances in which it is possible to show that the effect of viscosity is confined to a neighborhood of the interface small with respect to the minimum wave lengths considered. The subject of wind generation of waves is still in an unsettled state. One will find summaries of the present status in the article by H.U. ROLL in Vol. XLVIII of this Encyclopedia, especially pp. 703—717, and also in a critical exposition by URSELL (1956). A summary of some of the work in the USSR on wave generation is included in SHULEIKIN (1956).

The inclusion of viscosity in the analysis above leads to a somewhat more difficult development than in the case of standing waves. An exposition of the present achievements in this theory will be omitted; they consist chiefly of papers by WUEST (1949) and LOCK (1951, 1954).

γ) Vertically oscillated basins. Let S denote the wetted surface of a basin and F the water surface when the basin is at rest. We shall suppose that the basin is being oscillated in the y-direction according to some given law, which may be specified by giving $v_0(t)$, the velocity of a point of the basin. It will be most convenient to describe the motion of the fluid in coordinates fixed in the basin; these will be denoted by x, y, z. We shall assume the oscillations and the resulting motion to be of small amplitude so that we may linearize the equations and boundary conditions.

If Φ is the velocity potential for the motion relative to the basin and η the profile of the surface, both in coordinates fixed in the basin, then it follows easily from (2.17) that the only necessary change is to replace g by $g+\dot{v}_0$ in the boundary conditions at the free surface. They become:

$$\eta_t(x, z, t) = \Phi_y(x, 0, z, t), \tag{26.22}$$

$$(g+\dot{v}_0)\,\eta + \Phi_t(x, 0, z, t) = T'(\eta_{xx} + \eta_{zz}), \qquad T' = T/\varrho. \tag{26.23}$$

On the basin walls one must have

$$\Phi_n = 0 \quad \text{on} \quad S. \tag{26.24}$$

We wish, as usual, to investigate the character of the motion of the fluid.

The problem formulated above is clearly related to the problem considered in Sect. 23γ. However, the resulting motions are quite different. RAYLEIGH (1883) appears to have made the first theoretical investigation of this problem. More recently it has been studied by MOISEEV (1953, 1954), BENJAMIN and URSELL (1954), SCHULTZ-GRUNOW (1955) and BOLOTIN (1956). MOISEEV's analysis is the most general in that the only restriction upon the basin shape is that it should allow construction of a GREEN'S function for the Neumann problem; surface tension is not taken into account. BENJAMIN and URSELL restrict themselves to basins in the form of a vertical cylinder with horizontal bottom, but include the effect of surface tension. However, at the intersection with the walls they assume a 90° angle of contact with the free surface. This is in contradiction with the observed behavior of fluids but simplifies the mathematical treatment. In spite of this shortcoming it seems desirable to include the effect of surface tension, and this will be done below. BOLOTIN's paper considers a modification for viscous damping. The treatment below follows closely that of BENJAMIN and URSELL.

Let the basin be of depth h, let C denote the intersection of the walls with the plane $y=0$, and let $\boldsymbol{n}$ be a normal to the wall at a point of C. Then, from (26.22) and (26.24) it follows that $\eta_{tn} = \Phi_{yn} = 0$, or $\eta_n = \text{const}$ at each point of C; we take this constant to be zero, thus assuming a 90° contact angle with the wall. It then follows from (26.23) that $(\eta_{xx}+\eta_{zz})_n = 0$.

Let $\varphi_k(x, y, z)$ be a set of functions harmonic in the region bounded by the basin and the plane $y=0$ and satisfying (26.24), and such that $\varphi_k(x, 0, z)$ form a complete set of orthonormal functions in the area of the (x, z)-plane bounded by C. Then $\Phi(x, 0, z, t)$, $\eta(x, z, t)$ and $\eta_{xx}+\eta_{zz}$ can each be expanded in series in $\varphi_k(x, 0, z)$. The expansion of $\Phi(x, 0, z, t)$ determines $\Phi(x, y, z, t)$ as as eries in $\varphi_k(x, y, z)$. In the case at hand, when the basin is a vertical cylinder, one may separate variables as in Sect. 12α and construct a set φ_k in the form

$$\varphi_k(x, y, z) = \frac{\cosh m_k(y+h)\,\varphi_k(x, z)}{\cosh m_k h}, \tag{26.25}$$

where

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\varphi_k(x,z)+m_k^2\,\varphi_k(x,z)=0. \tag{26.26}$$

The eigenvalues m_k^2 are determined by the boundary condition on the contour C, namely $(\partial/\partial n)\,\varphi_k=0$.

Let the expansion for η be written in the form

$$\eta(x,z,t)=\sum_{n=1}^{\infty}a_k(t)\,\varphi_k(x,z). \tag{26.27}$$

Then, by differentiating (26.27) and using (26.26) one gets

$$\eta_{xx}+\eta_{zz}=-\sum a_k(t)\,m_k^2\,\varphi_k(x,z). \tag{26.28}$$

If

$$\Phi(x,0,z,t)=\sum b_k(t)\,\varphi_k(x,z),$$

then

$$\Phi_y(x,y,z,t)=\sum b_k(t)\,m_k\,\frac{\sinh m_k(y+h)}{\cosh m_k h}\,\varphi_k(x,z)$$

and, from (26.22),

$$b_k(t)\,m_k\tanh m_k h=\dot a_k(t).$$

Hence

$$\Phi(x,y,z,t)=\sum\dot a_k(t)\,\frac{\cosh m_k(y+h)}{m_k\sinh m_k h}\,\varphi_k(x,z). \tag{26.29}$$

Now substitute (26.27) to (26.29) in the remaining boundary condition (26.23):

$$\sum\left[(g+\dot v_0)\,a_k+T'm_k^2a_k+\frac{1}{m_k}\,\ddot a_k\coth m_k h\right]\varphi_k=0.$$

Since the φ_k are orthogonal, we may set each coefficient of φ_k equal to zero. With the special choice

$$\dot v_0=c\cos\sigma t \tag{26.30}$$

the following set of differential equations determine the a_k:

$$\ddot a_k(t)+\left[(g m_k+T'm_k^3)\tanh m_k h+c\,m_k\tanh m_k h\cos\sigma t\right]a_k(t)=0. \tag{26.31}$$

If we set

$$\left.\begin{aligned}&\tau=\tfrac{1}{2}\sigma t,\qquad p_k=\frac{4}{\sigma^2}\,(g m_k+T'm_k^3)\tanh m_k h=4\,\frac{\sigma_k^2}{\sigma^2},\\&q_k=-\frac{2}{\sigma^2}\,c\,m_k\tanh m_k h,\end{aligned}\right\} \tag{26.32}$$

where σ_k is the frequency of free oscillations in the mode m_k when the basin is fixed, then (26.31) takes one of the standard forms for the Mathieu equation:

$$\frac{d^2}{d\tau^2}\,a_k+\left[p_k-2q_k\cos 2\tau\right]a_k=0. \tag{26.33}$$

Of particular interest in the present context is the behavior of the solutions a_k as τ, or t, becomes large. It is known from the theory of differential equations with periodic coefficients that a pair of fundamental solutions can be given in the form

$$e^{\mu\tau}\,Q(\tau),\qquad e^{-\mu\tau}\,Q(-\tau), \tag{26.34}$$

where Q is of period π, unless $i\mu$ is an integer. In the latter case there exists a periodic solution, of period π if $i\mu$ is even and of period 2π if odd, and another independent nonperiodic solution. The coefficient μ will be a function of the parameters p_k, q_k and it is particularly pertinent to the present investigation to know for what regions in the (p, q)-plane μ has a nonzero real part. These regions have been investigated for other purposes and may be found, for example, in N. W. McLachlan's *Theory and application of Mathieu functions* (Oxford, 1947, pp. 40, 41). In Fig. 34, reproduced from Benjamin and Ursell, the shaded regions represent the unstable regions of the (p, q)-plane where μ has a nonzero real part. In the unshaded regions μ is pure imaginary (but not an integer) and the two solutions (26.34) are bounded for all τ. The boundaries between regions correspond to the periodic solutions occurring when $i\mu$ is an integer. In the unstable regions the periodicity behavior of the solutions is of two types. In the second, fourth, ... regions μ is real and the solutions (26.34) are functions of period π multiplied by exponentials. In the first, third, ... regions $\mu = \mu_1 + i$, μ_1 real, and the solutions (26.34) now become functions of period 2π multiplied by exponentials. In terms of t the two sets of regions correspond, respectively, to frequencies σ and $\frac{1}{2}\sigma$.

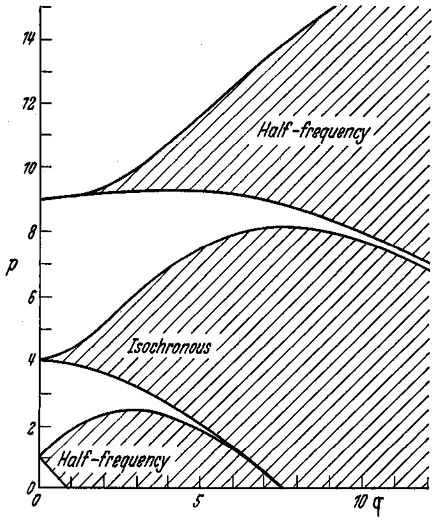

Fig. 34.

For a given mode of oscillation m_k one must compute p_k and q_k and plot (p_k, q_k) on the stability chart in order to find out whether the mode is stable or not. It seems likely, and, in fact, has been proved by Moiseev (1954, p. 44), that for any given values of σ and c some of the possible modes will be unstable. However, the analysis above has neglected the damping effect of viscosity and it may be supposed that the only unstable modes which actually occur are those associated with the smaller values of m_k. In any case, as has been emphasized earlier, the analysis is only suitable for describing the initial stages of the motion.

If the half-frequency of oscillation $\frac{1}{2}\sigma$ is equal, or nearly so, to one of the frequencies σ_k for free oscillation of the fluid, or to a subharmonic of σ_k, i.e. $\frac{1}{2}\sigma = \sigma_k/n$, then $p_k = 1$, or n^2, and it is evident from Fig 34 that (p_k, q_k) will be in an unstable region. If $\frac{1}{2}\sigma = \sigma_k$, (p_k, q_k) will lie in the lowest region and standing waves with half the frequency of the basin will be generated. If $\sigma = \sigma_k$, (p_k, q_k) will lie in the second region and the generated standing waves will have the same frequency as the basin. Thus the mode σ_k can be excited by oscillating the basin with frequency either σ_k or $2\sigma_k$. It is pointed out by Benjamin and Ursell that an apparent discrepancy between experimental observations of Faraday and Rayleigh and of Matthiessen can be explained by the above remarks.

Benjamin and Ursell made an experimental investigation with a circular cylinder in order to determine by experiment the boundaries of the lowest region of instability. The measurements provide a surprisingly good confirmation within certain limitations.

27. Higher-order theory of infinitesimal waves. It is implicit in the theory of infinitesimal waves developed in the preceding sections of this chapter that the approximation given by first-order theory to the solution of a particular problem, assuming that one exists, can be improved by including further terms in the perturbation series. The solution of the resulting boundary-value problems, at least in the simplest cases, can be carried through in a manner similar to that of the first-order theory, although the computations become more and more tedious the higher the order of approximation. Nevertheless, in view of the interest of the results, the computations have been carried through by a number of persons and by a variety of methods.

STOKES (1849) was apparently the first to make the calculation for progressive waves; in fact, the method used below in Sect. 27α is not essentially different from STOKES' first method. Later, in connection with the publication of his collected papers, STOKES (1880) added a supplement describing a different procedure. RAYLEIGH turned to the problem several times (1876, 1911, 1915, 1917) and introduced still another method of approximation. It should be noted, however, that both STOKES' second method and RAYLEIGH's method are limited to two-dimensional irrotational progressive waves. RAYLEIGH (1915) seems to be the first to have given an adequate treatment of the higher-order theory of standing waves. In addition to these classical papers there have been many others extending or improving the earlier theory; some of these will be noted below.

In all such computations, and indeed in the numerous first-order computations carried out in the earlier sections of this chapter, there is the tacit assumption that there exists an "exact solution" which is being approximated and which can be approached more and more closely by pursuing the selected method of approximation. Unfortunately, it is seldom that one is able to prove the existence of an exact solution or of convergence of the method of approximation, and, in fact, BURNSIDE (1916) cast doubt upon the usefulness of the Stokes-Rayleigh type of approximation of periodic progressive waves of permanent type. BURNSIDE's objection was later met by NEKRASOV's (1921, 1922, 1951), LEVI-CIVITA's (1925) and STRUIK's (1926) proofs of the existence of such waves for both infinite and finite depth. However, the existence of a standing wave satisfying the exact boundary conditions has not been demonstrated as yet. The same is true of the more complicated problems considered in earlier sections. However, this mathematical shortcoming is possibly of no more importance than the neglect in many problems of relevant physical parameters such as viscosity.

One should bear in mind that the higher-order infinitesimal waves considered below are not the only higher-order approximations. The solitary and cnoidal waves of the next chapter bear a similar relation to the first-order shallow-water theory. In addition, in the last chapter another method of approximating exact waves, due to HAVELOCK (1919a), will be described.

α) *Periodic progressive waves.* In the following we shall be seeking a wave which moves without change of form, i.e. a progressive wave in the sense of Sect. 7. Hence we shall expect to be able to represent Φ and η in the form

$$\Phi(x, y, z, t) = \varphi(x - ct, y, z), \qquad y = \eta(x - ct, z), \tag{27.1}$$

where c is the velocity of the wave. It will be convenient to represent the motion in a moving coordinate system, say $\bar{x} = x - ct$. However, we shall henceforth drop the bar over the x. The boundary conditions at the free surface are then the following:

see errata

$$\eta_x(x, z)\,\varphi_x(x, \eta(x, z)\,z) - \varphi_y + \eta_z \varphi_z - c\varphi_x = 0, \tag{27.2}$$

$$-c\varphi_x(x, \eta(x, z), z) + \tfrac{1}{2}(\operatorname{grad}\varphi)^2 + g\eta - T'(R_1^{-1} + R_2^{-1}) = 0, \tag{27.3}$$

where $R_1^{-1}+R_2^{-1}$ is given by (3.5′) and, as usual, $T'=T/\varrho$. Surface tension is being taken into account both for the intrinsic interest of the results and because of an interesting phenomenon which occurs in the higher-order approximations. We shall suppose that the wave length $\lambda=2\pi/m$ of the wave system has been given, so that c is still an unknown of the problem.

Let us now, as in Sect. 10α, assume that φ, η and c may all be expanded in a perturbation series in some parameter ε:

$$\left.\begin{aligned} \varphi &= \varepsilon\varphi^{(1)}+\varepsilon^2\varphi^{(2)}+\cdots,\\ \eta &= \varepsilon\eta^{(1)}+\varepsilon^2\eta^{(2)}+\cdots,\\ c &= c_0+\varepsilon c_1+\varepsilon^2 c_2+\cdots. \end{aligned}\right\} \tag{27.4}$$

After substituting in (27.2) and (27.3) and collecting terms in the manner of Sect. 10α, one obtains the following boundary conditions which must be satisfied successively by $\varphi^{(1)}, \eta^{(1)}, c_0$; $\varphi^{(2)}, \eta^{(2)}, c_1$; $\varphi^{(3)}, \eta^{(3)}, c_2$:

$$c_0\eta_x^{(1)}+\varphi_y^{(1)}=0,\qquad g\eta^{(1)}-c_0\varphi_x^{(1)}-T'(\eta_{xx}^{(1)}+\eta_{zz}^{(1)})=0; \tag{27.5}$$

$$\left.\begin{aligned} &c_0\eta_x^{(2)}+\varphi_y^{(2)}=\varphi_x^{(1)}\eta_x^{(1)}+\varphi_z^{(1)}\eta_z^{(1)}-\eta^{(1)}\varphi_{yy}^{(1)}-c_1\eta_x^{(1)},\\ &g\eta^{(2)}-c_0\varphi_x^{(2)}-T'(\eta_{xx}^{(2)}+\eta_{zz}^{(2)})=c_1\varphi_x^{(1)}-\tfrac12(\operatorname{grad}\varphi^{(1)})^2+c_0\eta^{(1)}\varphi_{xy}^{(1)}; \end{aligned}\right\} \tag{27.6}$$

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$$\left.\begin{aligned} c_0\eta_x^{(3)}+\varphi_y^{(3)}=\;&\varphi_x^{(2)}\eta_x^{(1)}+\varphi_z^{(2)}\eta_x^{(1)}-\varphi_{yy}^{(2)}\eta^{(1)}-c_2\eta_x^{(1)}+\varphi_x^{(1)}\eta_x^{(2)}+\varphi_z^{(1)}\eta_z^{(2)}-\\ &-\varphi_{yy}^{(1)}\eta^{(2)}-c_1\eta_x^{(2)}+\eta^{(1)}[\varphi_{xy}^{(1)}\eta_x^{(1)}+\varphi_{zy}^{(1)}\eta_z^{(1)}]-\tfrac12\varphi_{yyy}^{(1)}\eta^{(1)\,2},\\ g\eta^{(3)}-c_0\varphi_x^{(3)}-T'(\eta_{xx}^{(3)}+\eta_{zz}^{(3)})=\;&c_2\varphi_x^{(1)}+c_1\varphi_x^{(2)}+c_1\varphi_{xy}^{(1)}\eta^{(1)}+c_0\varphi_{xy}^{(2)}\eta^{(1)}+\\ &+c_0\varphi_{xy}^{(1)}\eta^{(2)}+\tfrac12 c_0\varphi_{xyy}^{(1)}\eta^{(1)\,2}+\operatorname{grad}\varphi^{(1)}\cdot\operatorname{grad}\varphi^{(2)}+\\ &+\eta^{(1)}\operatorname{grad}\varphi^{(1)}\cdot\operatorname{grad}\varphi_y^{(1)}-T'[\eta_{xx}^{(1)}\eta_z^{(1)\,2}+\eta_{zz}^{(1)}\eta_x^{(1)\,2}-\\ &-2\eta_{xz}^{(1)}\eta_x^{(1)}\eta_z^{(1)}-\tfrac32(\eta_{xx}^{(1)}+\eta_{zz}^{(1)})(\eta_x^{(1)\,2}+\eta_z^{(1)\,2})], \end{aligned}\right\} \tag{27.7}$$

where all conditions are to be satisfied on the plane $y=0$. It is possible, of course, to carry the approximations further, but three steps are ample to illustrate the procedures. The solution will be carried through in outline through the third order for infinite depth and through the second order for finite depth. As an expansion parameter we may take $\varepsilon=Am$, where A is a length determining the amplitude of the waves. The motion will be restricted to be two-dimensional.

Infinite depth. The solutions of (27.5) are already known from (13.5). We take them in the following form

see errata

$$\varphi^{(1)}=\frac{c_0}{m}e^{my}\sin mx,\qquad \eta^{(1)}=\frac{1}{m}\cos mx,\qquad c_0^2 m=g+m^2T'. \tag{27.8}$$

After substitution in (27.9), one finds

$$\left.\begin{aligned} &\varphi_y^{(2)}+c_0\eta_x^{(2)}=c_1\sin mx-c_0\sin 2mx,\\ &c_0\varphi_x^{(2)}-g\eta^{(2)}+T'\eta_{xx}^{(2)}=-c_1c_0\cos mx-\tfrac12 c_0^2\cos 2mx. \end{aligned}\right\} \tag{27.9}$$

Elimination of $\eta^{(2)}$ yields

$$c_0^2\varphi_{xx}^{(2)}+g\varphi_y^{(2)}-T'\varphi_{yxx}^{(2)}=2c_1c_0^2 m\sin mx-3c_0m^2T'\sin 2mx \tag{27.10}$$

as the boundary condition to be satisfied by $\varphi^{(2)}$. If $c_1\neq 0$, one cannot find a periodic potential function satisfying (27.10). Hence we set

$$c_1=0, \tag{27.11}$$

A solution of Laplace's equation satisfying (27.10) with $c_1=0$ and vanishing as $y\to-\infty$ is easily found to be

$$\varphi^{(2)}=\frac{3}{2}\,\frac{c_0}{m}\,\frac{m^2\,T'}{g-2m^2\,T'}\,\mathrm{e}^{2my}\sin 2mx, \tag{27.12}$$

providing $m^2 \neq g/2T'$. The corresponding expression for $\eta^{(2)}$ is

$$\eta^{(2)}=\frac{1}{2}\,\frac{1}{m}\,\frac{g+m^2\,T'}{g-2m^2\,T'}\cos 2mx. \tag{27.13}$$

One could, of course, add terms of the form given in (27.8) but with arbitrary multipliers. However, such solutions are discarded since we wish to allow only first-order terms of this form.

Two striking facts show up in (27.12) and (27.13): First, if surface tension is neglected, $\varphi^{(2)}$ vanishes and $\varphi^{(1)}$ gives the velocity potential correctly to at least the second order. The second fact is the zero in the denominator in both $\varphi^{(2)}$ and $\eta^{(2)}$, which shows that $\varphi^{(2)}$ and $\lambda^{(2)}$ become unbounded as m approaches $\sqrt{g/2T'}$. One may argue, of course, that this simply shows that validity of the perturbation method is limited to smaller and smaller values of Am the closer one comes to $\sqrt{g/2T'}$. However, it seems also to be an indication that near $m=\sqrt{g/2T'}$ the mode represented by $\varphi^{(2)}$ is of the same order of magnitude as that represented by $\varphi^{(1)}$. That this is indeed the case is clear from an examination of the equation determining $\varphi^{(1)}$ and $\varphi^{(2)}$ when $m=\sqrt{g/2T'}$. In fact, $\varphi^{(2)}$ was not determined by (27.10) for this value of m and, furthermore, (27.8) does not give the complete solution of (27.5). The solution with which we must start in this case is

$$\varphi^{(1)}=\frac{c_0}{m}\left[\mathrm{e}^{my}\sin mx+a\,\mathrm{e}^{2my}\sin 2mx+b\,\mathrm{e}^{2my}\cos 2mx\right], \tag{27.14}$$

where a and b are as yet undetermined constants. Thus these two modes of motion are of the same order for $m=\sqrt{g/2T'}$. One may now substitute (27.14) and the corresponding $\eta^{(1)}$ into (27.9). By reasoning similar to that used earlier in setting $c_1=0$, we now find

$$a=\pm\tfrac{1}{2},\qquad b=0,\qquad c_1=\pm\tfrac{1}{4}c_0. \tag{27.15}$$

There are thus two possible first-order modes depending upon the sign of a. $\varphi^{(2)}$ is now a sum of terms with modes $\sin 3mx$ and $\sin 4mx$, but will not be given here. The wave profile, including modes through $\cos 2mx$, may be written as follows:

$$\eta=A\left[\cos mx+\frac{1}{2}\,A\,m\,\frac{g+m^2\,T'}{g-2m^2\,T'}\cos 2mx\right],\qquad m\neq\sqrt{\frac{g}{2T'}}, \tag{27.16}$$

$$\eta=A\left[\cos mx\pm\frac{1}{2}\cos 2mx\right],\qquad m=\sqrt{\frac{g}{2T'}}. \tag{27.17}$$

The two signs in the second solution correspond roughly to the change of sign occurring in the first when k passes through $\sqrt{g/2T'}$. Comparison of the two cases also gives an indication of the limitations upon Am necessary in the first solution, namely,

$$|Am|<\left|\frac{g-2m^2\,T'}{g+m^2\,T'}\right|. \tag{27.18}$$

A reversal of curvature at the center of the wave trough for $m<\sqrt{g/2T'}$, or of the crest for $m>\sqrt{g/2T'}$, will occur when

$$|Am|>\frac{1}{2}\left|\frac{g-2m^2\,T'}{g+m^2\,T'}\right|. \tag{27.19}$$

The existence of the singularity in the expressions for $\eta^{(2)}$ and $\varphi^{(2)}$ was first noticed by HARRISON (1909). WILTON (1915) examined the matter more carefully, found the solutions (27.17) and, in fact, carried all approximations further. Some of WILTON's computed profiles are shown in Fig. 35. Although WILTON casts doubt upon the existence of the solution (27.17) with $+\frac{1}{2}$, such profiles seen to have been observed by KAMESVARA RAV (1920). However, the matter apparently still awaits a thorough experimental investigation, as do also similar higher modes mentioned below.

see errata

Let us now turn to the next order, assuming $m \neq \sqrt{g/2T'}$. Substitution of (27.8) and (27.11) to (27.13) into (27.7) and elimination of $\eta^{(3)}$ yield the following boundary condition to be satisfied by $\varphi^{(3)}$ on $y=0$:

$$\left.\begin{aligned} c_0^2\varphi^{(3)}_{xx}+g\varphi^{3}_{y}-T'\varphi^{(3)}_{yxx} &= c_0^2 m\left[2c_2-\frac{1}{2}c_0\frac{2g-m^2T'}{g-2m^2T'}+\frac{3}{8}c_0\frac{m^2T'}{g+m^2T'}\right]\sin mx+\\ &+\frac{9}{8}c_0^3 m\left[\frac{4m^2T'}{g-2m^2T'}-\frac{m^2T'}{g+m^2T'}\right]\sin 3mx. \end{aligned}\right\} \tag{27.20}$$

Again in order to avoid an unbounded solution we must set the coefficient of $\sin mx$ equal to zero. This yields a value for c_2:

$$c_2=\frac{1}{2}c_0\left[1+\frac{\frac{3}{2}m^2T'}{g-2m^2T'}-\frac{3}{8}\frac{m^2T'}{g+m^2T'}\right]. \tag{27.21}$$

One may now find a potential function satisfying (27.20) and vanishing as $y\to-\infty$. The solutions for $\varphi^{(3)}$ and $\eta^{(3)}$ are as follows:

$$\varphi^{(3)}=-\frac{9}{16}\frac{c_0}{m}\frac{m^2T'(g+2m^2T')}{(g-2m^2T')(g-3m^2T')}e^{3my}\sin 3mx; \tag{27.22}$$

$$\left.\begin{aligned} \eta^{(3)}=\frac{1}{m}&\left[\frac{1}{8}+\frac{3}{16}\frac{m^2T'(5g+2m^2T')}{(g+m^2T')(g-2m^2T')}\right]\cos mx+\\ &+\frac{3}{16}\frac{1}{m}\frac{2g^2-gT'm^2-30(m^2T')^2}{(g-2m^2T')(g-3m^2T')}\cos 3mx, \end{aligned}\right\} \tag{27.23}$$

for $m\neq\sqrt{g/2T'}$, $\sqrt{g/3T'}$. From (27.22) one sees again that $\varphi^{(3)}$ would vanish if surface tension were neglected. Although we shall not carry through the computation, this does not happen for $\varphi^{(4)}$. It is also evident that another singularity has appeared at $m=\sqrt{g/3T'}$. In fact, when one examines the reason for the appearance of the singularities, it is evident that a mode of the form $\cos nmx$ will always show a singularity at $m=\sqrt{g/nT'}$. In each such case the reason is the same as in the situation discussed earlier with $n=2$: for $m=\sqrt{g/nT'}\equiv m_n$ the proper first-order solution is of the form

see errata

$$\varphi^{(1)}=\frac{c_0}{m}\left[e^{my}\sin mx+a_n e^{nmy}\sin mx\right],$$

with a_n to be determined subsequently (according to WILTON only a_2 is not unique). Thus (27.8) should be qualified by $m^2\neq g/nT'$. One should note that, although m_n is getting small (and hence λ_n large) as n increases, the wave number of the second first-order mode is $\sqrt{ng/T'}$. Hence, on the basis of the results in Sect. 25, one will expect this mode to be quickly damped for large values of n. However, one may presume the first few to be observable. We remark that these special associated pairs of first-order waves always straddle the wave number for minimum c_0, namely m_1.

The wave profile, velocity potential and wave velocity are now given by

$$\left.\begin{aligned} \eta &= A m \eta^{(1)} + A^2 m^2 \eta^{(2)} + A^3 m^3 \eta^{(3)} + \cdots, \\ \varphi &= A m \varphi^{(1)} + A^2 m^2 \varphi^{(2)} + A^3 m^3 \varphi^{(3)} + \cdots, \\ c &= c_0 + A m c_1 + A^2 m^2 c_2 + \cdots. \end{aligned}\right\} \tag{27.24}$$

To the third order the profile for pure gravity waves ($T'=0$) is represented by the following function:

see errata

$$\left.\begin{aligned} \eta &= A\left\{\left[1 + \tfrac{1}{8} A^2 m^2\right] \cos m x + \tfrac{1}{2} A m \cos 2 m x + \tfrac{3}{8} A^2 m^2 \cos 3 m x + \cdots\right\} \\ &= A'\left\{\cos m x + \tfrac{1}{2} A' m \cos 2 m x + \tfrac{3}{8} A'^2 m^2 \cos 3 m x + \cdots\right\}, \end{aligned}\right\} \tag{27.25}$$

where $A' = A\left[1 + \frac{1}{8} A^2 m^2\right]$; the velocity becomes

$$c = \sqrt{\frac{g}{m}}\left(1 + \frac{1}{2} A^2 m^2 + \cdots\right). \tag{27.26}$$

The velocity potential to the third order is

$$\varphi = A \sqrt{\frac{g}{m}}\, e^{m y} \sin m x. \tag{27.27}$$

If one sets $g=0$, then the wave profile for pure capillary waves becomes

$$\eta = A\left\{\left[1 - \tfrac{1}{16} A^2 m^2\right] \cos m x - \tfrac{1}{4} A m \cos 2 m x - \tfrac{15}{16} A^2 m^2 \cos 3 m x + \cdots\right\} \tag{27.28}$$

and the velocity

$$c = \sqrt{T' m}\left[1 - \tfrac{1}{16} A^2 m^2 + \cdots\right]. \tag{27.29}$$

For pure gravity waves the approximations were carried to the fifth order by STOKES, RAYLEIGH (1917) and others.

It is of interest to compare the profiles represented in (27.25) and (27.28). The effect of including higher-order terms in pure gravity waves is to sharpen and raise the crests and to broaden and raise the troughs. For pure capillary waves the effect is just the reverse. For combined gravity-capillary waves the increasing importance of the second-order term near $m = \sqrt{g/2T'}$ will first show up as a reversal of curvature at the middle of the flattened part of the wave; formula (27.19) gives the condition for the first occurrence. In Fig. 35 are shown a pure gravity wave as computed by WILTON (1914) for $Am = 0.86$ (here A is the amplitude), and five gravity-capillary waves, the last two corresponding to the solutions (27.17), also computed by WILTON (1915) for a value of $T/\varrho g = 0.075$. It should be remarked that the value of $Am = 0.86$ is much larger than any for which it is possible to prove convergence of the perturbation series and is, in fact, very close to the value of Am for the highest possible irrotational wave of permanent type (see Sect. 33 α), namely 0.891.

Finite depth. When a solid bottom is present at $y=-h$, the only necessary modification of the preceding analysis is substitution of the boundary condition $\varphi_y^{(i)}(x, -h) = 0$ for $\varphi_y^{(i)} \to 0$ as $y \to -\infty$. This increases the computational labor by a substantial amount, but otherwise introduces no difficulties. However, we call attention to the remarks on the definition of wave velocity in Sect. 7; the velocity c below is the one defined there also as c.

The wave profile, velocity potential and wave velocity, including the effect of surface tension, are as follows, to the second order:

see errata

$$\eta = A\left\{\cos m + \frac{1}{2} A m \frac{(2+\cosh 2mh)\operatorname{cosech} 2mh}{\tanh^2 mh - 3T' m^2 (g + T' m^2)^{-1}} \cos 2mx\right\}, \tag{27.30}$$

$$\left.\begin{aligned} \varphi = A c_0 \Bigg\{ & \frac{\cosh m(y+h)}{\sinh mh} \sin mx + \\ & + \frac{3}{4} A m \frac{(g+3T'm^2)\coth mh - (g+T'm^2)\tanh mh}{(g+T'm^2)\tanh^2 mh - 3T'm^2} \frac{\cosh 2m(y+h)}{\sinh 2mh} \sin 2mx \Bigg\}, \end{aligned}\right\} \tag{27.31}$$

$$c^2 = c_0^2 = \left(\frac{g}{m} + T' m\right) \tanh mh. \tag{27.32}$$

The velocity is the same as in the first-order theory; this occurred also for infinite depth. In contrast to the case of infinite depth, the term $\varphi^{(2)}$ does not vanish

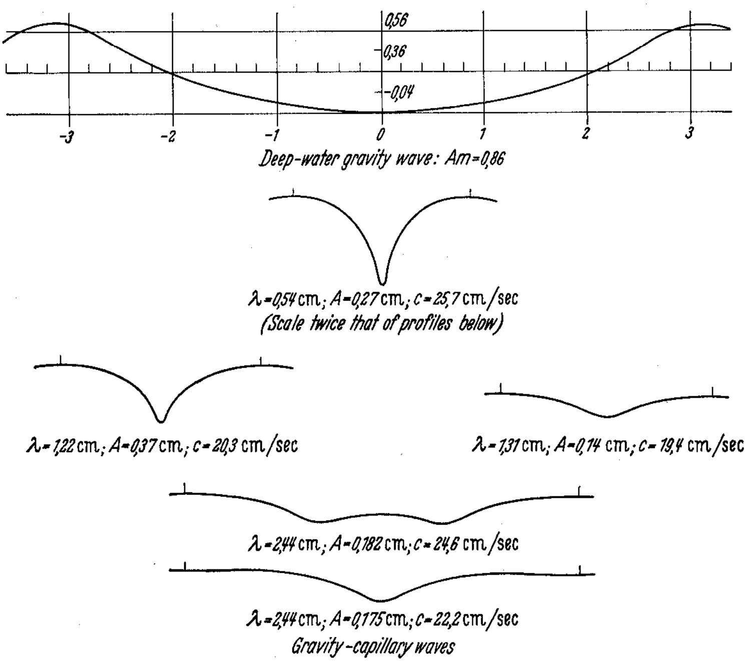

Fig. 35.

when $T' = 0$. The singularity in the coefficient of $\cos 2mx$ still persists provided that $h > \sqrt{3T'/g}$. The earlier discussion of this phenomenon is still relevant, and a detailed one will be omitted here. However, even if surface tension is neglected in (27.30), the second-order term may still become large for small values of mh, as has been emphasized by Miche (1944). If one again takes as an indication of increasing predominance of the second-order term a reversal of curvature at the bottom of the trough, one finds that this occurs for

$$A m > \frac{1}{2} \frac{\tanh^2 mh \sinh 2mh}{2 + \cosh 2mh}, \tag{27.33}$$

or approximately

$$A m > \tfrac{1}{3} \tanh mh \sinh^2 mh$$

as given by MICHE. The occurrence of this secondary crest when mh is small has frequently been observed. It has been investigated experimentally by MORISON and CROOKE (1953) and by HORIKAWA and WIEGEL (1959).

The wave profile and velocity computations were carried by STOKES to the third order, and by DE (1955) to the fifth order, for pure gravity waves in fluid of finite depth. The following expressions are taken from a report by SKJELBREIA (1959):

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$$\left.\begin{aligned} \eta &= A\left\{\cos m\,x + \frac{1}{4}\,A\,m\,\frac{\cosh m\,h\,(2+\cosh 2m\,h)}{\sinh^3 m\,h}\cos 2m\,x + \right.\\ &\qquad\left. + \frac{3}{64}\,A^2 m^2\,\frac{8\cosh^6 m\,h + 1}{\sinh^6 m\,h}\cos 3m\,x + \cdots\right\},\\ c^2 &= \frac{g}{m}\tanh m\,h\left[1 + A^2 m^2\,\frac{8+\cosh 4m\,h}{8\sinh^4 m\,h} + \cdots\right]. \end{aligned}\right\} \tag{27.34}$$

SKJELBREIA has provided comprehensive tables allowing easy computation of η, φ and many other quantities of interest, all to the third order.

Particle orbits. A particularly interesting phenomenon occurs when higher-order approximations are used in the computation of the paths of individual particles. The equations which the coordinates of a particle must satisfy are

$$\frac{dx}{dt} = \varphi_x(x - c\,t,\, y), \qquad \frac{dy}{dt} = \varphi_y(x - c\,t,\, y). \tag{27.35}$$

Since φ depends upon the parameter ε, the solutions x and y also will. We assume then that x and y may be expanded into series of the form

$$x(t) = x_0 + \varepsilon\, x_1(t) + \cdots, \qquad y(t) = y_0 + \varepsilon\, y_1(t) + \cdots, \tag{27.36}$$

substitute them into (27.35) together with the appropriate expansion of φ in powers of ε, and then equate the several powers of ε separately. This results in a sequence of equations of which the first two are as follows

$$\frac{dx_1}{dt} = \varphi_x^{(1)}(x_0 - c_0\,t,\, y_0), \qquad \frac{dy_1}{dt} = \varphi_y^{(1)}(x_0 - c_0\,t,\, y_0); \tag{27.37}$$

$$\left.\begin{aligned} \frac{dx_2}{dt} &= x_1(t)\,\varphi_{xx}^{(1)}(x_0 - c_0\,t,\, y_0) + y_1\,\varphi_{xy}^{(1)} + \varphi_x^{(2)},\\ \frac{dy_2}{dt} &= x_1(t)\,\varphi_{xy}^{(1)}(x_0 - c_0\,t,\, y_0) + y_1\,\varphi_{yy}^{(1)} + \varphi_y^{(2)}. \end{aligned}\right\} \tag{27.38}$$

The first set, (27.37), was already solved in (14.17) and (14.18) and to the first order of approximation gave circular or elliptical orbits. The solution for higher orders is facilitated by neglecting surface tension and assuming $h=\infty$, for then $\varphi^{(2)}$ and $\varphi^{(3)}$ both vanish. From (27.8) one finds easily the orbit to the second order:

$$\left.\begin{aligned} x(t) &= x_0 - A\,e^{m y_0}\sin m(x_0 - c_0\,t) + A^2 m^2 c_0\, e^{2 m y_0}\, t,\\ y(t) &= y_0 + A\,e^{m y_0}\cos m(x_0 - c_0\,t). \end{aligned}\right\} \tag{27.39}$$

The circular orbits of first-order theory are now modified by a general drift in the direction of wave motion. The total amount of fluid transported per unit time (and width) is $\frac{1}{2}A^2 m c_0$. As the formula shows, this additional flow is concentrated chiefly near the surface.

When the depth is finite, or when surface tension is taken into account, the orbits become more complicated. Let

$$K = \frac{(g + 3\,T' m^2) \coth m\,h - (g + T' m^2) \tanh m\,h}{(g + T' m^2) \tanh^2 m\,h - 3 m^2\, T'}\,. \tag{27.40}$$

The particle orbits, accurate to the second order, are as follows:

$$\left.\begin{aligned} x(t) &= x_0 - A\,\frac{\cosh m\,(y_0 + h)}{\sinh m\,h}\sin m\,(x_0 - c_0\,t) + \tfrac{1}{2}\,A^2 m^2\,c_0\,t\,\frac{\cosh 2m\,(y_0 + h)}{\sinh^2 m\,h} + \\ &\quad + \tfrac{1}{4}\,A^2\,m\left[\operatorname{cosech}^2 m\,h - 3\,K\,\frac{\cosh 2m\,(y_0 + h)}{\sinh 2m\,h}\right]\sin 2m\,(x_0 - c_0\,t)\,, \\ y(t) &= y_0 + A\,\frac{\sinh m\,(y_0 + h)}{\sinh m\,h}\cos m\,(x_0 - c_0\,t) + \\ &\quad + \tfrac{3}{4}\,A^2\,m\,K\,\frac{\sinh 2m\,(y_0 + h)}{\sinh 2m\,h}\cos 2m\,(x_0 - c_0\,t)\,. \end{aligned}\right\} \tag{27.41}$$

The mass-transport term in $x(t)$ is still present, and in fact, persists to the very bottom. The elliptical orbits of the first-order theory are now modified not only by the forward drift at all levels, but also by another superposed cyclic motion of twice the frequency. The effect of this is to make the orbits approximately epitrochoidal (neglecting for a moment the drift) with a small hump at the bottom which in extreme cases can become a cusp or a loop. This behavior has, in fact, been observed by MORISON and CROOKE (1953). For capillary waves the situation is reversed and a dimple appears at the top.

The existence of mass transport will be reconsidered in the last chapter, where it will be demonstrated that it is a general consequence of irrotational motion when the exact boundary conditions are satisfied. The theoretically predicted monotonically decreasing forward drift with increasing depth is not confirmed experimentally for small values of $m h$, say $m h < 2$. Instead, with respect to a coordinate system moving with the mean velocity of the fluid, there is an observed forward flow near the bottom and top and a backward flow in the middle portions. It is not surprising that the perfect-fluid model does not give a good prediction for small $m h$, for the high shear rate near the bottom indicates that viscosity should not be neglected. LONGUET-HIGGINS (1953b) has, in fact, devoted a long monograph to development of the higher-order theory of waves in a viscous fluid and finds theoretical drift curves agreeing qualitatively with observed ones. We shall not carry through the details here and refer to LONGUET-HIGGINS' paper.

Wave energy. One of the striking facts about progressive first-order pure gravity waves is that the kinetic and potential energy per wave length are equal (see Sect. 15β). This equal division of energy no longer holds when higher-order terms are taken into account. It is particularly easy to show this for $h = \infty$, for then we may use (27.25) and (27.27). The average potential energy in a wave-length is

$$\mathscr{V}_{\mathrm{av}} = \frac{m}{2\pi}\int_0^{2\pi/m} dx \int_0^{\eta} \varrho\, g\, y\, dy = \frac{m}{2\pi}\int_0^{2\pi/m} \frac{1}{2}\,\varrho\, g\, \eta^2\, dx = \frac{1}{4}\,\varrho\, g\, A^2\left[1 + \frac{1}{2}\,A^2 m^2\right]. \tag{27.42}$$

The average kinetic energy is

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$$\left.\begin{aligned} \mathscr{T}_{\mathrm{av}} &= \frac{m}{2\pi}\int_0^{2\pi/m} dx \int_0^{\infty} \frac{1}{2}\,\varrho\,(\varphi_x^2 + \varphi_y^2)\, dy = \frac{m}{2\pi}\int_0^{2\pi/m} \frac{1}{4}\,\varrho\, A^2\, c_0^2\, m\, e^{2m\eta}\, dx \\ &= \frac{1}{4}\,\varrho\, A^2\, g\,[1 + A^2 m^2]\,. \end{aligned}\right\} \tag{27.43}$$

Composite waves. Previously in this section we have been discussing a wave of permanent type whose prototype is the first-order progressive wave of the form $\eta = A \cos m(x - ct)$. It is natural to inquire into the behavior of higher-order waves whose first-order prototype is composite, say

$$\eta = A_1 \cos m_1 (x - c_1 t) + A_2 \cos m_2 (x - c_2 t). \tag{27.44}$$

To find the corresponding second-order terms one may use Eqs. (10.11) and (10.12); the computations are tedious but not difficult. The third order would introduce modifications of both c_1 and c_2 and lead to a much longer computation. As might be expected in analogy with the theory of sound, the second-order terms introduce waves of wave numbers $m_1 - m_2$ and $m_1 + m_2$, as well as $2m_1$ and $2m_2$. The velocity potential to the second order is given by

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$$\left.\begin{aligned} \Phi = {} & A_1 c_1 e^{m_1 y} \sin m_1 (x - c_1 t) + A_2 c_2 e^{m_2 y} \sin m_2 (x - c_2 t) + \\ & + 2 A_1 A_2 \frac{m_1 m_2 (c_1 - c_2)\, g}{g (m_1 - m_2) - (m_1 c_1 - m_2 c_2)^2}\, e^{(m_1 - m_2) y} \sin [(m_1 - m_2)\, x - (m_1 c_1 - m_2 c_2)\, t]. \end{aligned}\right\} \quad (27.4$$

The profile is then computed from BERNOULLI'S law

$$\eta = -\frac{1}{g} \left[\Phi_t (x, \eta, t) + \frac{1}{2} (\Phi_x^2 + \Phi_y^2)\right]$$

with retention of only terms of first or second order [cf. Eqs. (10.9) and (10.11)]. We omit the rather long expression.

BIESEL (1952) has derived formulas for a composite wave with a finite number of components and for h finite. He computes a number of quantities of interest. However, the formulas are very long and will not be reproduced here.

Three-dimensional waves. By using the full three-dimensional equations as given in (27.5) to (27.7) one may develop a higher-order theory of doubly modulated waves analogous to those considered in Sect. 14γ by first-order theory. This has been done by FUCHS (1952) and SRETENSKII (1954) to whose papers we refer for the resulting motion.

Further references. Development of systematic methods of computation of higher-order approximations has recently attracted the attention of several persons. Among these are SRETENSKII (1952), BORGMAN and CHAPPELEAR (1957), DAUBERT (1957, 1958) in a series of notes, JOLAS (1958) and NORMANDIN (1957). SRETENSKII (1953, 1955) has investigated the higher-order theory of wave motion resulting from a moving pressure distribution and waves in a circular canal.

β) Standing waves. As will be evident below, the formulation of a higher-order theory of standing waves is somewhat clumsier than that for progressive waves of permanent type. Part of the difficulty stems from the fact that one necessarily must deal with one more variable, namely t. The type of motion we are seeking will be represented by a profile $\eta(x, t)$ and a velocity potential $\Phi(x, y, t)$ periodic in both x and t:

$$\eta(x + r\lambda,\ t + s\tau) = \eta(x, t), \qquad \Phi(x + r\lambda, y, t + s\tau) = \Phi(x, y, t). \tag{27.46}$$

If we fix the wave length $\lambda = 2\pi/m$, then the period $\tau = 2\pi/\sigma$ will have to be determined as one of the unknowns of the problem. In addition, we wish to have the first-order standing wave $\eta = A \cos mx \cos \sigma t$ of Sect. 14α serve as a prototype and first-order solution of the more general problem. As a further condition, we shall suppose the motion to be symmetric with respect to a vertical line through a crest.

RAYLEIGH (1915) was apparently the first to consider this problem. It was later attacked in an entirely different way, using Lagrangian coordinates, by SEKERZH-ZENKOVICH (1947, 1951a, b, 1952), who treated both two- and three-dimensional waves for infinite depth, two-dimensional waves for finite depth, and composite waves for infinite depth. PENNEY and PRICE (1952), following approximately RAYLEIGH's method, carried the approximation for two-dimensional motion and $h=\infty$ to the fifth order, and to the second order for h finite and for doubly modulated standing waves. The method used below is a modification of theirs. The two-dimensional problem has recently been studied in a series of notes by CHABERT D'HIÈRES (1957, 1958). CARRY (1953) has carried to the second-order the superposition of two standing waves of the same wave length but 90° out of phase and of differing first-order amplitudes. INGRAHAM (1954) has carried to the second order the stability analysis of superposed two-fluid systems discussed at the beginning of Sect. 26α.

Since η and Φ are periodic in both x and t, we may expand each in a double Fourier series. However, it is also necessary to bring into the form of the series some indications of orders of magnitudes of the components, and in such a way that the first-order term is of the desired sort. We assume the following expansions for an infinitely deep fluid:

$$\left.\begin{aligned}\sigma &= \sigma_0+\varepsilon\,\sigma_1+\varepsilon^2\sigma_2+\cdots,\\ \eta(x,t) &= \sum_{r=1}^{\infty}\varepsilon^r\,\eta^{(r)}=\sum_{r=1}^{\infty}\varepsilon^r\sum_{p,q=0}^{r}\left[a_{pq}^{(r)}\cos q\,\sigma t+b_{pq}^{(r)}\sin q\,\sigma\,t\right]\cos p\,m\,x,\\ \Phi(x,y,t) &= \sum_{r=1}^{\infty}\varepsilon^r\,\Phi^{(r)}=\sum_{r=1}^{\infty}\varepsilon^r\sum_{p,q=1}^{r}\left[c_{pq}^{(r)}\cos q\,\sigma\,t+d_{pq}^{(r)}\sin q\,\sigma t\right]\mathrm{e}^{p\,m\,y}\cos p\,m\,x.\end{aligned}\right\}\quad(27.47)$$

We may immediately set $d_{p0}^{(r)}=0$, $b_{p0}^{(r)}=0$ and with no loss of generality also $c_{00}^{(r)}=0$. Since the mean water level has been fixed at $y=0$, we must also have $a_{00}^{(r)}=0$. We shall again take $\varepsilon=A\,m$, where A is the amplitude of the first-order term.

Substitution of (27.47) into the exact kinematic and dynamic boundary conditions,

$$\left.\begin{aligned}&\Phi_x(x,\eta,t)\,\eta_x-\Phi_y+\eta_t=0\\ &\Phi_t+\tfrac{1}{2}(\Phi_x^2+\Phi_y^2)+g\,\eta-T'(R_1^{-1}+R_2^{-1})=0,\end{aligned}\right\}\quad(27.48)$$

results, as in Sects. 10α and 27α, in a series of equations for successive determination of the coefficients $a_{pq}^{(r)},\ldots,d_{pq}^{(r)}$ and $\sigma_0,\sigma_1,\ldots$. Because of the assumed form of the solution, the equations are now always linear equations between the coefficients. The boundary conditions for $\Phi^{(1)}$ and $\eta^{(1)}$, namely,

$$\left.\begin{aligned}&\Phi_y^{(1)}-\eta_t^{(1)}=0,\\ &\Phi_t^{(1)}+g\,\eta^{(1)}-T'\,\eta_{xx}^{(1)}=0,\end{aligned}\right\}\quad(27.49)$$

yield

$$\left.\begin{aligned}&-\sigma_0\,a_{01}^{(1)}\sin\sigma_0 t+\sigma_0\,b_{01}^{(1)}\cos\sigma_0 t=0,\\ &-\sigma_0\,a_{11}^{(1)}\sin\sigma_0 t+\sigma_0\,b_{11}^{(1)}\cos\sigma_0 t-k\left[c_{10}^{(1)}+c_{11}^{(1)}\cos\sigma_0 t+d_{11}^{(1)}\sin\sigma_0 t\right]=0;\end{aligned}\right\}\quad(27.50)$$

$$\left.\begin{aligned}&g\left[a_{01}^{(1)}\cos\sigma_0 t+b_{01}^{(1)}\sin\sigma_0 t\right]+\left[-\sigma_0\,c_{01}^{(1)}\sin\sigma_0 t+\sigma_0\,d_{01}^{(1)}\cos\sigma_0 t\right]=0,\\ &(g+m^2\,T')\left[a_{10}^{(1)}+a_{11}^{(1)}\cos\sigma_0 t+b_{11}^{(1)}\sin\sigma_0 t\right]+\\ &\qquad+\left[-\sigma_0\,c_{11}^{(1)}\sin\sigma_0 t+\sigma_0\,d_{11}^{(1)}\cos\sigma_0 t\right]=0.\end{aligned}\right\}\quad(27.51)$$

From these follow immediately

$$a_{01}^{(1)} = b_{01}^{(1)} = c_{10}^{(1)} = a_{10}^{(1)} = 0, \qquad d_{11}^{(1)} = -\frac{\sigma_0}{m}\, a_{11}^{(1)}, \qquad c_{11}^{(1)} = -\frac{\sigma_0}{m}\, b_{11}^{(1)}, \tag{27.52}$$

and

$$\sigma_0^2 = g\,m + m^3\,T'. \tag{27.53}$$

We shall in addition fix the phase by making the arbitrary choice

$$a_{11}^{(1)} = \frac{1}{m}, \qquad b_{11}^{(1)} = 0, \tag{27.54}$$

so that

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$$\eta^{(1)} = \frac{1}{m} \cos m\,x \cos \sigma_0\,t, \qquad \Phi^{(1)} = -\frac{c_0}{m^2} \cos m\,x \sin \sigma_0\,t. \tag{27.55}$$

This is a rather clumsy way to derive a first-order solution which was found much more directly earlier in Sect. 14α. However, it provides a caricature of the procedure necessary at each new stage of approximation. Since the higher-order approximations lead to extremely tedious calculations, they will be completely omitted and only the results given.

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The profile and velocity potential through the second order are given by

$$\left.\begin{aligned} \eta &= A \cos \sigma_0\,t \cos m\,x + \frac{1}{4} A^2\,m\,\frac{g + m^2\,T'}{g + 4m^2\,T'} \cos 2m\,x + \\ &\quad + \frac{1}{4} A^2\,m\,\frac{g + m^2\,T'}{g - 2m^2\,T'} \cos 2\sigma_0\,t \cos 2m\,x, \\ \Phi &= -A\,\frac{\sigma_0}{m} \sin \sigma_0\,t\, e^{m y} \cos m\,x + \frac{1}{4} A^2\,\sigma_0 \sin 2\sigma_0\,t - \\ &\quad - \frac{1}{4} A^2\,\sigma_0\,\frac{3m^2\,T'}{g - 2m^2\,T'} \sin 2\sigma_0\,t\, e^{2m y} \cos 2m\,y, \end{aligned}\right\} \tag{27.56}$$

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for $m^2 \neq g/2\,T'$; here $\sigma_1 = 0$. If $m^2 = g/2\,T'$, the situation is similar to that discussed in Sect. 27α following (27.13). For this value of m we must start with a first-order solution of the form:

$$\left.\begin{aligned} \Phi^{(1)} &= -\frac{\sigma_0}{m^2}[\sin \sigma_0\,t\, e^{m y} \cos m\,x + (b_1 \sin 2\sigma_0\,t - b_2 \cos 2\sigma_0\,t)\, e^{2m y} \cos 2m x], \\ \eta^{(1)} &= \frac{1}{m}\,[\cos \sigma_0\,t \cos m\,x + (b_1 \cos 2\sigma_0\,t + b_2 \sin 2\sigma_0\,t) \cos 2m\,x]. \end{aligned}\right\} \tag{27.57}$$

The values of b_1, b_2 and σ_1 are now determined by the second-order equations and are

$$b_1 = \pm\tfrac{1}{2} \qquad b_2 = 0, \qquad \sigma_1 = \pm\tfrac{1}{8}\,\sigma_0. \tag{27.58}$$

Thus the first-order profile for $m^2 = g/2\,T'$ is

$$\eta = A \cos \sigma_0\,t \cos m\,x \pm \tfrac{1}{2} A \cos 2\sigma_0\,t \cos 2m\,x. \tag{27.59}$$

The amplitude relation between the two first-order modes is the same as for progressive waves of this length.

The expression for the third-order standing wave is very clumsy if T' is retained. Also, as might be expected from analogy with the progressive wave, another apparent singularity appears for $m^2 = g/3T'$. If one sets $T' = 0$, the expressions for η and Φ become much simpler and are as follows:

$$\left.\begin{aligned}\eta &= A\cos\sigma t\cos m x + \frac{1}{4}A^2 m\cos 2m x + \frac{1}{4}A^2 m\cos 2\sigma t\cos 2m x + \\ &\quad + \frac{1}{32}A^3 m^2 [-2\cos 3\sigma t\cos m x + 9\cos\sigma t\cos 3m x + \\ &\qquad\qquad + 3\cos 3\sigma t\cos 3m x;\\ \Phi &= -\frac{\sigma A}{m}\sin\sigma t\, e^{my}\cos m x + \frac{1}{4}\sigma A^2\sin 2\sigma t + \\ &\quad + \frac{5}{32}\sigma m A^3\sin 3\sigma t\, e^{my}\cos m x + \frac{3}{16}\sigma m A^3\sin 3\sigma t\, e^{3my}\cos 3m x;\\ \sigma &= \sqrt{g m}\left(1 - \frac{1}{8}A^2 m^2\right).\end{aligned}\right\} \quad (27.60)$$

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As has been mentioned earlier, the approximation has been carried to the fifth order by PENNEY and PRICE (1952). However, it is not necessary to carry the approximation so far in order to see some important features of the motion, namely the sharpening of the crests and flattening of the troughs, the absence of any nodal points and the decrease of frequency with amplitude. One interesting feature does require carrying the approximation to at least the fourth order: this is the absence of any time during a period when the surface is completely flat. In connection with an experimental test of a predicted standing wave of greatest amplitude-length ratio by PENNEY and PRICE, G.I. TAYLOR (1953) has also provided an experimental verification of the correctness of the theory in an extreme case.

Orbits. The method of computation of orbits including higher-order terms is the same as that outlined at the end of Sect. 27α and we omit a detailed exposition. For infinite depth and $T' = 0$ the orbits to the second order are given by

$$\left.\begin{aligned} x &= x_0 - A\, e^{m y_0}\sin m x_0\cos\sigma_0 t,\\ y &= y_0 + A\, e^{m y_0}\cos m x_0\cos\sigma_0 t + \tfrac{1}{4}A^2 m\, e^{2m y_0}\cos 2\sigma_0 t.\end{aligned}\right\} \quad (27.61)$$

The effect of the last term in y is easily seen to be a small wiggle superposed on the first-order straight-line trajectories discussed in Sect. 14α, except directly beneath the crests where the trajectory is still vertical but with the midpoint somewhat above the equilibrium position.

Pressure distribution. A particularly interesting consequence of keeping second-order terms appears in the behavior of the pressure distribution. From (27.56) and BERNOULLI's theorem one finds for the average pressure over a wave length at depth y

$$\left.\begin{aligned}\overline{p - p_0} &= \frac{1}{\lambda}\int_0^\lambda (p - p_0)\,dx = -\varrho g y - \frac{1}{4}\varrho A^2\sigma_0^2 e^{2my} + \\ &\quad + \frac{1}{4}\varrho A^2\sigma_0^2 e^{2my}\cos 2\sigma_0 t - \frac{1}{2}\varrho A^2\sigma_0^2\cos 2\sigma_0 t.\end{aligned}\right\} \quad (27.62)$$

The terms with e^{2my} as a factor drop off quickly. However, the last term is independent of y and at all depths yields a fluctuation about the hydrostatic pressure with double the frequency of the standing waves. The existence of this depth-independent fluctuation, deriving from the term Φ_t in BERNOULLI's theorem and the purely time-dependent term in Φ, was pointed out by MICHE (1944, p. 73). The matter has been investigated more intensively by LONGUET-HIGGINS (1950) who has extended the theory to include a more general wave motion and compressibility of the fluid. He has further applied the theory to give a plausible explanation of recorded microseisms. KIERSTEAD (1952) has extended LONGUET-HIGGINS' analysis to include two-fluid systems. COOPER and LONGUET-HIGGINS (1951) have carried out laboratory experiments showing excellent agreement with the predicted pressure distribution for both progressive and standing waves.

Finite depth. Computations of the surface profile, particle orbits and other quantities for finite depth have been carried to the third order by SEKERZH-see errata ZENKOVICH (1951) and CARRY and CHABERT D'HIÈRES (1957). We reproduce here the results only to the second order (for pure gravity waves):

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$$\left.\begin{aligned}
\eta &= A\cos\sigma t\cos m x + \tfrac{1}{8}A^2 m\tanh m h\,\times\\
&\quad\times\left[1+\coth^2 m h - \coth^2 m h\,(3\coth^2 m h - 1)\cos 2\sigma t\right]\cos 2 m x;\\
\Phi &= -A\frac{\sigma}{m}\,\frac{\cosh m(y+h)}{\sinh m h}\sin\sigma t\cos m x + \frac{1}{16}A^2\sigma(3+\coth^2 m h)\sin 2\sigma t + \\
&\quad+\frac{3}{8}A^2\sigma\,\frac{\coth m h}{\sinh^2 m h}\,\frac{\cosh 2m(y+h)}{\sinh 2 m h}\sin 2\sigma t\cos 2 m x,\\
\sigma^2 &= \sigma_0^2 = g m\tanh m h,\qquad \sigma_1 = 0.
\end{aligned}\right\}\quad(27.63)$$

The pressure averaged over a wave length [cf. (27.62)] is

$$\left.\begin{aligned}
\overline{p-p_0} = -\varrho g y + \frac{1}{8}\,\frac{A^2\sigma^2}{\sinh^2 m h}\,&[1-\cosh 2m(y+h) - \\
&-(2\cosh 2 m h - \cosh 2m(y+h) - 1)\cos 2\sigma t].
\end{aligned}\right\}\quad(27.64)$$

On the bottom, $y=-h$, one finds

$$\overline{p-p_0} = \varrho g h - \tfrac{1}{2}\varrho A^2\sigma^2\cos 2\sigma t. \qquad (27.65)$$

We note that here also, as in the case of progressive waves, the importance of the second-order terms in η and Φ increases as $m h\to 0$.

γ) Waves in a viscous fluid. The Eqs. (10.2) to (10.4), used in Sect. 25 in developing the first-order theory of waves in a viscous fluid, may be considered as the first in a sequence for the determination of higher-order approximations. Although the formulation of the equations appears to be straight forward, if laborious, the higher-order theory does not seem to have attracted many investigators. HARRISON (1909) made a second-order investigation of progressive waves and LONGUET-HIGGINS (1953) has recently made an elaborate study of both progressive and standing waves in an attempt to explain certain observed features of mass transport velocities. We shall not attempt to summarize either paper. However, the following results, taken from HARRISON, may be of interest. For the wave profile to the second order he gives the following expression when ν is small [cf. Eq. (25.22)]:

$$\left.\begin{aligned}
\eta &= A\,e^{-2\nu m^2 t}\cos(m x - \sigma_0 t) + \\
&\quad + A^2 e^{-4\nu m^2 t}\left[\frac{1}{2}m\cos 2(m x-\sigma_0 t) - m^2\left(\frac{\nu^2}{4 g m}\right)^{\frac{1}{4}}\sin 2(m x - \sigma_0 t)\right],
\end{aligned}\right\}\quad(27.66)$$

where $\sigma_0^2 = g m$. The effect of viscosity, besides damping, is to make the leading side of the crest steeper than the trailing side. According to HARRISON the average horizontal velocity of a particle, again for small ν, is

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$$\left.\begin{aligned}
A^2\sigma_0 m\,e^{2 m y - 4\nu m^2 t} - A^2 m^2\sqrt{\tfrac{1}{2}\sigma_0\nu}\,&\times\\
\times[(4\cos l_2 y + \sin l_2 y)\,e^{(m+l_1)y} + \sin 2 m y]\,&e^{-4\nu m^2 t} + \\
+ A^2 m^3\nu\,[4\,e^{(m+l_1)y}\sin l_2 y + 3\,e^{2 l_1 y}]\,&e^{-4\nu m^2 t},
\end{aligned}\right\}\quad(27.67)$$

where, as in (25.19), $l = l_1 + i l_2$ and $\nu(l^2 - m^2) = \omega \approx -2\nu m^2 + i\sigma_0$. This formula should be compared with $A^2 m^2 c_0 e^{2my}$ computed from (27.39), to which it reduces when $\nu = 0$.

E. Shallow-water waves

This chapter will deal with special solutions based on the shallow-water approximation, following the method of FRIEDRICHS (1948) as presented in subsection 10β. The shallow-water approximation for the waves over a rigid bottom yields a set of nonlinear equations [cf. (10.32)] even in the first approximation. If these equations are then linearized, they result in a hyperbolic-type equation which reduces to the simple wave equation for a flat horizontal bottom. Consequently, the solutions resulting from the shallow-water approximation are completely different in character from those resulting from the infinitesimal-wave approximation of subsection 10α and Chap. D, which resulted in linear equations and linear boundary conditions. That is, the shallow-water approximation leads to nonlinear hyperbolic-type equations, whereas the infinitesimal-wave approximation leads to a set of linear equations satisfying the boundary conditions and having each successive approximation to the velocity potential satisfy the simplest elliptic equation, namely the Laplace equation.

After the first-order shallow-water approximation (10.32) has been applied to several problems, the method of FRIEDRICHS (1948) and KELLER (1948) will be extended to obtain the second and third approximations of the shallow-water theory and thereby present, for the first time, the exact second approximation to the cnoidal wave of KORTEWEG and DE VRIES (1895), and the solitary wave of BOUSSINESQ (1871), and RAYLEIGH (1876). These higher-order approximations lead directly to relations predicting the maximum heights of cnoidal waves and solitary waves.

28. The fundamental equations for the first approximation. The shallow-water expansion method introduced by FRIEDRICHS (1948) is discussed in Sect. 10. For this application the expansion parameter ε was selected so that the first approximation would be identical to the nonlinear equations of the classical shallow-water theory, which is based on the assumption of hydrostatic pressure variation throughout and neglect of the variation of the horizontal velocity components with depth, so that the complicated boundary-value problem is greatly simplified to the following nonlinear equations:

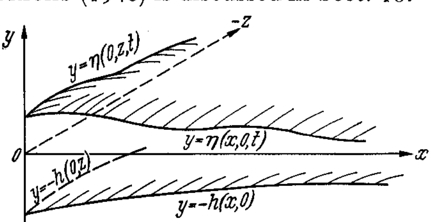

Fig. 36.

$$\left.\begin{aligned} u_t + u\,u_x + w\,u_z &= -g\,\eta_x, \\ w_t + u\,w_x + w\,w_z &= -g\,\eta_z, \\ \eta_t + [u(\eta + h)]_x + [w(\eta + h)]_z &= 0 \end{aligned}\right\} \tag{28.1}$$

[see LAMB (1932, p. 254) or STOKER (1957, p. 23)]. The coordinates and notation are shown in Fig. 36.

The set of nonlinear equations (28.1) is identical to (10.32) and is the first approximation in FRIEDRICHS' (1948) shallow-water expansion method as discussed in Sect. 10; this provides some mathematical justification for these classical equations. It is evident that the higher-order approximations following (10.23) and (10.33) also require that ε be sufficiently small; consequently, as will be shown, this expansion method is applicable if the product of water depth and surface curvature is small. Therefore, in some cases, this shallow-water theory is applicable to extremely large water depths as long as the wave length is sufficiently long, the most common application being to tidal waves, that is, the oceanic tides produced by the gravitational action of the sun and the moon [see, e.g., Lamb (1932) or Defant's article in Vol. XLVIII of this Encyclopedia].

The mathematical justification for this shallow-water expansion method, at least for special cases, lies in the existence proof of Friedrichs and Hyers (1954) for the solitary wave, and the existence proof of Littman (1957) for the more general cnoidal waves. Both of these proofs demonstrate that this expansion method converges to the exact solutions for these particular problems.

see errata

The nonlinear first approximation given by (28.1) is considerably simplified if the rigid bottom surface $h(x, z)$ is flat and horizontal, as may be seen by letting $h=\text{const}$ so that (28.1) may be written as

$$\left.\begin{aligned} &u_t+u u_x+w u_z=-g(\eta+h)_x,\\ &w_t+u w_x+w w_z=-g(\eta+h)_z,\\ &(\eta+h)_t+[u(\eta+h)]_x+[w(\eta+h)]_z=0.\end{aligned}\right\} \qquad (28.2)$$

This is identical to the well known two-dimensional gas-dynamics equation [see, e.g., Lamb (1932)] if we write

$$\left.\begin{aligned} &\varrho(x, z, t)=[\eta(x, z, t)+h],\\ &\frac{\gamma p}{\varrho^2}=\frac{c^2}{\varrho}=\frac{c^2}{\eta+h}=g=\text{const}.\end{aligned}\right\} \qquad (28.3)$$

Since the isentropic gas relationship is $p=\text{const}\times\varrho^\gamma$, the first-order nonlinear shallow-water approximation for a flat horizontal bottom is identical to the isentropic two-dimensional gas flow having a specific heat ratio of $\gamma=2$. This is the basis of the so called hydraulic analogy which has been used for many experimental investigations [see, e.g., Stoker (1957)].

It must be noted, however, that this hydraulic analogy is only valid for a flat horizontal bottom, as may be seen by comparing (28.1) and (28.2), and even more important, it is valid only as a first approximation even for the nonlinear case. It will be shown in Sect. 31 that the second approximation to shallow-water theory yields finite-amplitude waves (the solitary wave or cnoidal waves) which can be propagated without a change in shape or form, a fact which completely invalidates the hydraulic analogy to compressible gas flow since (28.2), or the gas dynamics equation, predicts that any finite disturbance quickly forms a finite discontinuity, e.g. [see, e.g., Lamb (1932), pp. 278, 481)].

In Sect. 29, immediately following, it will be shown that even for the linearized first approximation the hydraulic analogy to compressible gas flow is limited to a flat horizontal bottom.

29. The linearized shallow-water theory. The first approximation to shallow-water theory can now be linearized by two different methods, each suitable for various problems. We shall assume that $u_z=w_x$, so that a velocity potential $\Phi(x, z, t)$ exists. The first method is more appropriate for investigating steady water flow in canals or rivers and consists of the following approximations for carrying out the linearization:

$$u(x, z)=U+\varphi_x\approx U, \qquad w(x, z)=\varphi_z\ll U, \qquad (29.1)$$

$$\eta(x, z)\ll h(x, z), \qquad (29.2)$$