S' vanishes since both Φ_n and G_n are zero on S. We shall assume that the behavior of G and Φ as $R\to\infty$ is such that the integral over Ω' vanishes as $\varrho\to\infty$. If the fluid is of bounded extent, the situation considered by VOLTERRA (1934), this presents, of course, no difficulty. In the two cases for which G has been given above, it has been shown by FINKELSTEIN that this is true. For finite depth the proof presents no difficulty once the estimates for G are obtained; for infinite depth the analysis is more troublesome and we refer to his paper or to STOKER (1957, pp. 193/194) for proof. After letting $\varrho\to\infty$, one then has

$$\Phi_t(x,y,z,t)=\frac{1}{4\pi}\iint\limits_F [G\Phi_{t\eta}-G_\eta\Phi_t]\,d\sigma+\frac{1}{4\pi}\iint\limits_{S_m}[G\Phi_{t\nu}-G_\nu\Phi_t]\,d\sigma. \tag{22.6}$$

In the integral over F we may replace G_η by $-g^{-1}G_{tt}$ because of the boundary condition at F. Now interchange t and τ and integrate with respect to τ between limits 0 and t. This gives, following an integration by parts,

$$\left.\begin{aligned}\Phi(x,y,z,t)-\Phi(x,y,z,0)&=\frac{1}{4\pi}\iint\limits_F\left\{\left[G\Phi_\eta+\frac{1}{g}\Phi_tG_t\right]\Big|_0^t-\int_0^t\left[G_t\Phi_\eta+\frac{1}{g}\Phi_{tt}G_t\right]d\tau\right\}+\\&\quad+\frac{1}{4\pi}\int_0^t d\tau\iint\limits_{S_m}[G\Phi_{tn}-G_n\Phi_t]\,d\sigma\\&=\frac{1}{4\pi}\iint\limits_F\Big\{G(\xi,0,\zeta,x,y,z;t,t)\,\Phi_y(\xi,0,\zeta,t)+\frac{1}{g}\Phi_t(\ldots,t)\,G_t(\ldots;t,t)-\\&\quad-G(\ldots;0,t)\,\Phi_y(\ldots,0)-\frac{1}{g}\Phi_t(\ldots,0)\,G_t(\ldots;0,t)+\\&\quad+\frac{1}{\varrho g}\int_0^t G_t(\ldots;\tau,t)\,p_t(\xi,\zeta,\tau)\,d\tau\Big\}\,d\xi\,d\zeta+I\end{aligned}\right\} \tag{22.7}$$

where I stands for the last integral. (G_t always represents the derivative with respect to the seventh variable.) Recalling the properties of G in (22.4), one finds

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$$\left.\begin{aligned}\Phi(x,y,z,t)&=\Phi(x,y,z,0)+\frac{1}{4\pi}\iint\limits_F\Big\{-G(\xi,0,\zeta;x,y,z;0,t)\,\eta_t(\xi,\zeta,0)+\\&\quad+G_t(\xi,0,\zeta,x,y,z;0,t)\left[\eta(\xi,\zeta,0)+\frac{1}{\varrho g}p(\xi,\zeta,0)\right]+\\&\quad+\frac{1}{\varrho g}\int_0 G(\xi,0,\zeta;x,y,z,\tau,t)\,p_t(\xi,\zeta,\tau)\,d\tau\Big\}\,d\xi\,d\zeta+I\\&=\Phi(x,y,z,0)+\frac{1}{4\pi}\iint\limits_F\Big\{-G_t(\xi,0,\zeta,x,y,z;0,t)\,\eta_t(\xi,\zeta,0)+\\&\quad+G_t(\xi,0,\zeta,x,y,z;0,t)\,\eta(\xi,\zeta,0)-\\&\quad-\frac{1}{\varrho g}\int_0 G_{tt}(\xi,0,\zeta,x,y,z;\tau,t)\,p(\xi,\zeta,\tau)\,d\tau\Big\}\,d\xi\,d\zeta+I,\end{aligned}\right\} \tag{22.8}$$

where $\Phi(x,y,z,0)$ is determined up to an additive constant as the solution to a Neumann problem, since $\Phi_n(x,y,z,0)$ is given on all boundaries and bounded at infinity. In the integral I we note that $\Phi_{tn}=V_n'(t)$ is known on S_m, but Φ_t is not.

If there are no moving bodies in the fluid, then the integral I is not present and Φ is determined by the initial displacement and velocity of the free surface and the given pressure distribution over it. This is VOLTERRA's result as extended

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to unbounded fluids by FINKELSTEIN. If surfaces S_m are present, one may still use (22.8) to derive an integral equation in the same way that (16.13) was derived. For as (x, y, z) is made to approach a point (x_0, y_0, z_0) of S_m,

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$$\frac{1}{4\pi}\iint\limits_{S_m} G_\nu(\xi,\eta,\zeta,x,y,z;\tau,t)\,\Phi_t(\xi,\eta,\zeta,\tau)\,d\sigma \to \frac{1}{2}\Phi_t(x_0,y_0,z_0,t) + \frac{1}{4\pi}\iint\limits_{S_m} G_\nu(\xi,\eta,\zeta,x_0,y_0,z_0;\tau,t)\,\Phi_t\,d\sigma$$ [1].

Thus, after carrying out the integration with respect to τ, one has an integral equation for $\Phi(x, y, z, t)$ for each value of $t>0$. This may be used to find the value of Φ, and hence Φ_t, on the surface S_m, providing that the integral equation can be solved. One may then use (22.8) to determine $\Phi(x, y, z, t)$ for all values of (x, y, z) in the fluid. The integral equation has the same appearance as (22.8) except that the first two terms have coefficients $\frac{1}{2}$ and (x, y, z) is understood to be a point of S_m. This further extension of VOLTERRA's analysis is also due to FINKELSTEIN.

Uniqueness of $\Phi(x, y, z, t)$, at least up to an additive constant, may be proved a follows. Let Φ_1 and Φ_2 be two solutions satisfying the boundary conditions. Then $\Phi=\Phi_1-\Phi_2$ satisfies (22.8) with f, F, p and V_n all identically zero, i.e.

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$$\Phi(x,y,z,t) = \text{const} - \frac{1}{4\pi}\int_0^t d\tau \iint\limits_{S_m} G_n\,\Phi_t\,d\sigma.$$

If we assume that G_n is $O(R^{-1-\varepsilon})$ as $R\to\infty$, then Φ_t and grad Φ will have the same behavior and the integrals we shall write below may be shown to exist. As has been mentioned above, G_n vanishes much quicker than is required in the cases when the fluid is infinitely deep and when the fixed surface consists of a horizontal bottom; if the fluid is bounded in extent, no such condition is necessary to make the integrals converge.

Consider then, following VOLTERRA,

$$\begin{aligned}\Omega &= \frac{1}{2}\frac{\partial}{\partial t}\iint\limits_F \frac{1}{g}\Phi_t^2\,d\sigma \\ &= \iint\limits_F \frac{1}{g}\Phi_t\Phi_{tt}\,d\sigma = -\iint\limits_F \Phi_t\Phi_y\,d\sigma \\ &= -\iint\limits_{F+S+S_m} \Phi_t\Phi_n\,d\sigma\end{aligned}$$

since Φ_n vanishes on S and S_m. Now apply GREEN's theorem and denote the volume occupied by fluid by T:

$$\Omega = -\iiint\limits_T \text{grad}\,\Phi_t\cdot\text{grad}\,\Phi\,d\tau = -\frac{1}{2}\frac{\partial}{\partial t}\iiint\limits_T (\text{grad}\,\Phi^2)\,d\tau.$$

Hence

$$\frac{\partial}{\partial t}\left\{\iint\limits_F \frac{1}{g}\Phi_t^2\,d\sigma + \iiint\limits_T (\text{grad}\,\Phi)^2\,d\tau\right\} = 0$$

and

$$\iint\limits_F \frac{1}{g}\Phi_t^2\,d\sigma + \iiint\limits_T (\text{grad}\,\Phi)^2\,d\tau = \text{const}. \tag{22.9}$$

[1] Cf. O.D. KELLOGG: Foundations of potential theory, p. 167. Berlin: Springer 1929.

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Since $\Phi_n = 0$ on F, S and S_m for $t=0$, $\Phi(x,y,z,0) = C$, a constant; hence grad $\Phi = 0$ for $t=0$. Also $\Phi_t(x,y,z,0) = 0$. Hence the constant in (22.9) is zero and Φ_t and grad Φ vanish for all t. Thus $\Phi(x,y,z,t) = \text{const}$ and the solution of the initial-value problem is determined up to a constant.

β) The Cauchy-Poisson problem. In this classical problem of water-wave theory, the pressure over the free surface is constant, say zero, the fluid is infinitely deep or bounded below by a horizontal bottom, no obstructions are present and the initial displacement and velocity of the free surface are given. The two- and three-dimensional cases will be separated in order to illustrate different methods of approach.

Three dimensions. The velocity potential may be obtained directly from (22.8) after setting $p(x,z,t)$ and I equal to zero. However, the explicit expressions for G and G_t are needed. As was noted in Sect. 22α, these can be written down immediately from (13.49) for infinite depth and (13.53) for depth h. The resulting expressions, after setting $\eta = 0$, are as follows:

infinite depth:

$$\left.\begin{aligned} G(x,y,z;\xi,0,\zeta;0,t) &= 2\int_0^\infty \left[1-\cos\left(\sqrt{gk}\,t\right)\right] e^{ky} J_0(kR)\,dk, \\ G_t(x,y,z;\xi,0,\zeta;0,t) &= -2\int_0^\infty \sin\left(\sqrt{gk}\,t\right) e^{ky} J_0(kR)\sqrt{gk}\,dk; \end{aligned}\right\} \quad (22.10)$$

depth h:

$$\left.\begin{aligned} G(x,y,z;\xi,0,\zeta;0,t) &= 2\int_0^\infty \left[1-\cos\left(\sqrt{gk\tanh kh}\,t\right)\right] \frac{\cosh k(y+h)}{\sinh kh} J_0(kR), \\ G_t(x,y,z;\xi,0,\zeta;0,t) &= -2\int_0^\infty \sqrt{gk\tanh kh}\,\sin\left(\sqrt{gk\tanh kh}\,t\right) \frac{\cosh k(y+h)}{\sinh kh} J_0(kR), \end{aligned}\right\} \quad (22.11)$$

where $R^2 = (x-\xi)^2 + (z-\zeta)^2$.

There still remains to find $\Phi(x,y,z,0)$ where

$$\Phi_y(x,0,z,0) = \eta_t(x,z,0)$$

and

$$\lim_{y\to-\infty} \Phi_y(x,y,z.0) = 0 \quad \text{or} \quad \Phi_y(x_1-h,z,0) = 0.$$

The solution of these two problems is well known:

infinite depth:

$$\left.\begin{aligned} \Phi(x,y,z,0) &= \frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{\eta_t(\xi,\zeta,0)}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{\frac{1}{2}}}\,d\xi\,d\zeta \\ &= \frac{1}{2\pi}\iint_{-\infty}^{\infty} \eta(\xi,\zeta,0)\,d\xi\,d\zeta \int_0^\infty e^{ky} J_0(kR)\,dk; \end{aligned}\right\} \quad (22.12)$$

depth h:

$$\Phi(x,y,z,0) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \eta_t(\xi,\zeta,0)\,d\xi\,d\zeta \int_0^\infty \frac{\cosh k(y+h)}{\sinh kh} J_0(kR)\,dk, \quad (22.13)$$

where R is defined as above.

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Substituting the several expressions in (22.8), one obtains the expressions for the velocity potential:

infinite depth:

$$\left.\begin{aligned}\Phi(x,y,z,t) &= \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}e^{ky}\cos\sigma t\,J_0(kR)\,dk - \\ &\quad -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma\,e^{ky}\sin\sigma t\,J_0(kR)\,dk, \\ \sigma^2 &= gk;\end{aligned}\right\}\qquad(22.14)$$

depth h:

$$\left.\begin{aligned}\Phi(x,y,z,t) &= \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\frac{\cosh k(y+h)}{\sinh kh}\cos\sigma t\,J_0(kR)\,dk - \\ &\quad -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma\frac{\cosh k(y+h)}{\sinh kh}\sin\sigma t\,J_0(kR)\,dk, \\ \sigma^2 &= gk\tanh kh.\end{aligned}\right\}\qquad(22.15)$$

The equations describing the free surface are as follows:

infinite-depth:

$$\left.\begin{aligned}\eta(x,z,t) &= \frac{1}{2\pi g}\iint\limits_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma\sin\sigma t\,J_0(kR)\,dk + \\ &\quad +\frac{1}{2\pi g}\iint\limits_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma^2\cos\sigma t\,J_0(kR)\,dk, \\ \sigma^2 &= gk;\end{aligned}\right\}\qquad(22.16)$$

depth h:

$$\left.\begin{aligned}\eta(x,z,t) &= \frac{1}{2\pi g}\iint\limits_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma\sin\sigma t\coth kh\,J_0(kR)\,dk + \\ &\quad +\frac{1}{2\pi g}\iint\limits_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma^2\cos\sigma t\coth kh\,J_0(kR)\,dk, \\ \sigma^2 &= gk\tanh kh.\end{aligned}\right\}\qquad(22.17)$$

It has been shown by Kochin (1935) that the integrals with respect to k in (22.16) can be evaluated. Consider the integral

$$K = \int\limits_0^{\infty}\sigma^{-1}\sin\sigma t\,J_0(kR)\,dk,\qquad \sigma^2 = gk. \qquad(22.18)$$

Then the first integral with respect to k in (22.16) is $-K_{tt}$ and the second one is $-K_{ttt}$ To evaluate K make first the following change of variables:

$$\varkappa^2 = kR,\qquad \omega^2 = gt^2/4R. \qquad(22.19)$$

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Then

$$K = \frac{2}{\sqrt{gR}} \int_0^\infty \sin 2\omega\varkappa\, J_0(\varkappa^2)\, d\varkappa$$

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$$= \frac{2}{\sqrt{gR}} \int_0^\infty d\varkappa \int_0^1 \frac{\sin 2\omega\varkappa \cos v\varkappa^2}{\sqrt{1-v^2}}\, dv$$

$$= \frac{2}{\sqrt{gR}} \int_0^\infty d\varkappa \int_0^1 [\sin(2\omega\varkappa + v\varkappa^2) + \sin(2\omega\varkappa - v\varkappa^2] \frac{dv}{\sqrt{1-v^2}}.$$

In the first integral let $u = v\varkappa + \omega$, in the second let $u = v\varkappa - \omega$. Then

$$K = \frac{2}{\sqrt{gR}} \int_{+\omega}^\infty du \int_0^1 \sin\left(\frac{u^2-\omega^2}{v}\right) \frac{dv}{v\sqrt{1-v^2}} - \frac{2}{\sqrt{gR}} \int_{-\omega}^\infty du \int_0^1 \sin\left(\frac{u^2-\omega^2}{v}\right) \frac{dv}{v\sqrt{1-v^2}}$$

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$$= + \frac{2}{\sqrt{gR}} \int_{-\omega}^{\omega} du \int_1^\infty \frac{\sin(\omega^2-u^2)\, v'}{\sqrt{v'^2-1}}\, dv' \qquad (v' = 1/v),$$

$$= \frac{4}{\sqrt{gR}} \frac{\pi}{2} \int_0^\omega J_0(\omega^2 - u^2)\, du,$$

and, after setting $u = \sqrt{2}\omega \sin \frac{1}{2}\vartheta$,

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$$K = \frac{\sqrt{2}\pi}{\sqrt{gR}}\, \omega \int_0^{\frac{1}{2}\pi} J_0\left(2\frac{\omega^2}{2}\cos\vartheta\right) \cos\frac{1}{2}\vartheta\, d\vartheta.$$

Finally, from an identity in WATSON'S *Bessel functions* [§ 5.43, Eq. (1)] one finds

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$$K(\omega) = \frac{\pi^2}{\sqrt{2gR}}\, \omega\, J_{\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) J_{-\frac{1}{4}}\left(\frac{1}{2}\omega^2\right). \qquad (22.20)$$

In order to use the results in (22.16) one needs the first three derivatives with respect to t. Since

$$\frac{\partial}{\partial t} = \frac{1}{2}\sqrt{\frac{g}{R}}\, \frac{\partial}{\partial\omega},$$

the derivatives can be computed by taking derivatives with respect to ω and multiplying by an appropriate factor. After some rather tedious computation one finds

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$$\left.\begin{aligned}
\frac{\partial}{\partial t} K(\omega) &= -\frac{1}{2\sqrt{2}}\pi^2 \frac{\omega^2}{R}\left[J_{\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) J_{\frac{3}{4}}\left(\frac{1}{2}\omega^2\right) - J_{-\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) J_{-\frac{3}{4}}\left(\frac{1}{2}\omega^2\right)\right],\\
\frac{\partial^2}{\partial t^2} K(\omega) &= -\frac{1}{2}\pi^2\omega^3 \sqrt{\frac{g}{2R^3}}\left[J_{\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) J_{-\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) + J_{\frac{3}{4}}\left(\frac{1}{2}\omega^2\right) J_{-\frac{3}{4}}\left(\frac{1}{2}\omega^2\right)\right],\\
\frac{\partial^3}{\partial t^3} K(\omega) &= -\frac{\pi^2 g}{2R^2}\omega^2 \left[J_{\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) J_{-\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) - \right.\\
&\quad \left. - \omega^2\left\{J_{\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) J_{\frac{3}{4}}\left(\frac{1}{2}\omega^2\right) - J_{-\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) J_{-\frac{3}{4}}\left(\frac{1}{2}\omega^2\right)\right\}\right].
\end{aligned}\right\} \qquad (22.21)$$

These are KOCHIN'S formulas, but derived somewhat differently from his original paper; still another derivation may be found in KOCHIN, KIBEL and ROZE (1948, Chap. 8, § 21). Similar formulas for (22.17) do not seem to have been discovered.

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It should be noted that in the final form of (22.16) the dependence upon t is through the dimensionless variable $\omega^2 = g t^2/4R$. Hence, if one examines the contribution to the surface profile from a given locality, say the neighborhood of (ξ, ζ), then a given phase of this contribution, say a maximum, will be described by $g t^2/4R = \text{const}$; i.e., the phase is moving away from (ξ, ζ) with constant acceleration proportional to g. The amplitude of the contribution is modulated by either $R^{-\frac{3}{2}}$ or R^{-2} according as one is considering the first or second summand in (22.16). KOCHIN'S 1935 paper is of some methodological interest inasmuch as he started his analysis with dimensional considerations. This method will be introduced for the two-dimensional case.

One may obtain without great difficulty series expansions for the k-integrals in (22.14) and (22.16), as was first done by CAUCHY and POISSON. We refer to LAMB'S *Hydrodynamics* (1932, § 255) for the derivation and exact expressions. They can also be derived from the the known expansions for $J_{\frac{1}{4}}$, etc., as can asymptotic expressions for large ω. One may also carry out an analysis of the changing shape of the surface profile following the methods of Sect. 15.

It is evident that one can solve explicitly other similar initial-value problems for which the GREEN'S function can be given. For example, the method of images allows one to give an explicit solution for various cases when vertical walls are present as boundaries. Such cases have been considered by RISSER (1925). The Cauchy-Poisson problem in the presence of a vertical half-plane, $z=0$, $x>0$, has been treated by BOIKO (1938), but by more complex methods.

Two dimensions. Rather than repeat the methods used for three-dimensional motion, we shall introduce a method making use of the complex potential and thus special to two-dimensional motion. It is analogous to the method used in deriving (13.28).

Let $f(z, t) = \Phi(x, y, t) + i\Psi(x, y, t)$ be the complex velocity potential. The initial conditions will be taken in the form

$$-\frac{1}{g}\operatorname{Re} f_t(x - i0, 0+) = \eta(x, 0), \quad -\operatorname{Im} f'(x - i0, 0+) = \eta_t(x, 0). \tag{22.22}$$

Let us consider infinite depth first. For $t>0$ we assume that $f(z, t)$ is regular and $|f'| < M(t)$, $|f_{tt}| < M(t)$ for $y<0$ and that both f' and f_{tt} approach zero as $y \to -\infty$. Consider now the function

$$G(z, t) = f_{tt}(z, t) + i g f'(z, t). \tag{22.23}$$

From the assumptions about f it follows that, for $t>0$, $G(z, t)$ is regular for $y<0$, that $|G| < B(t)$ for $y<0$ and that $G \to 0$ as $y \to -\infty$. Moreover, it follows from the condition at the free surface, (11.5), that $\operatorname{Re} G(x - i0, t) = 0$. Hence, the definition of G may be extended into the upper half-plane by defining

$$G(x + iy) = -\overline{G(x - iy)}. \tag{22.24}$$

But then since G is regular and bounded in the whole finite z-plane, it follows from LIOUVILLE'S theorem that $G = \text{const}$; the constant must equal zero from the assumed behavior as $y \to -\infty$. Hence the fundamental differential equation for the Cauchy-Poisson problem in two dimensions is

$$f_{tt}(z, t) + i g f'(z, t) = 0, \quad t > 0, \tag{22.25}$$

an observation usually credited to LEVI-CIVITA (cf. TONOLO, 1913).

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Let us now find the analogous equation for finite depth. The function f will be assumed regular in the strip $0>y>-h$. The boundary condition on the bottom is

$$\operatorname{Im} f'(x-ih,t)=0. \tag{22.26}$$

Hence f may be extended analytically into the strip $-2h<y<-h$ by defining

$$f\big(x-i(y+2h)\big)=\overline{f(x+iy)},\qquad 0>y\geqq -h. \tag{22.27}$$

We may also, as before, extend the function $f_{tt}+igf'$ into the strip $h\geqq y\geqq 0$. The condition $\operatorname{Re}\{f_{tt}+igf'\}=0$ for $y=0$ implies $\operatorname{Re}\{f_{tt}-igf'\}=0$ for $y=-2h$. Hence the function $f_{tt}-igf'$ can be extended by reflection into the strip $-3h\leqq y\leqq -2h$. Now consider the function

$$\left.\begin{aligned} H(z,t)&=[f_{tt}(z+ih,t)+f_{tt}(z-ih,t)]+ig[f'(z+ih,t)-f'(z-ih,t)]\\ &=\{f_{tt}(z+ih,t)+igf'(z+ih,t)\}+\{f_{tt}(z-ih,t)-igf'(z-ih,t)\}.\end{aligned}\right\} \tag{22.28}$$

As a result of the various extended regions of definition, one may verify easily that H is defined for all z in the strip $-2h<y<0$ and is regular there. Moreover, it follows that

$$H(x-ih,t)=0; \tag{22.29}$$

for from (22.27) it follows that the two pairs of summands in the first form of (22.28) are real for $z=x-ih$, whereas from the boundary conditions at $y=0$ and $y=-2h$ it follows that the terms in curly brackets in the second form of (22.28) have zero real parts. Since $H(z,t)$ is regular in the strip $0>y>-2h$ and vanishes on $y=-h$, it must vanish identically in the strip. Hence we have the following differential-difference equation of CISOTTI (1918):

$$f_{tt}(z+ih,t)+f_{tt}(z-ih,t)+ig[f'(z+ih,t)-f'(z-ih,t)]=0. \tag{22.30}$$

Let us now turn to the solution of (22.25) with initial conditions (22.22). We shall follow closely an exposition of SEDOV's (1948, 1957). However, the idea of the derivation is KOCHIN's (1935) and, in fact, really goes back to TONOLO (1913). The use of dimensional analysis can be extended to the three-dimensional problem; this was also done by KOCHIN.

We first remark that the initial-value problem can be solved by solving it for two special cases of (22.22), namely, first with $\eta(x,0)\equiv 0$ and then with $\eta_t(x,0)\equiv 0$. The sum of these two solutions will satisfy (22.22). Next we note that $\eta(x,0)$ has the dimension "length" and $\eta_t(x,0)$ the dimension "velocity", and that the solution f in each of the two initial-value problems will be proportional to some typical parameters associated with $\eta(x,0)$ or $\eta_t(x,0)$, respectively. Let us suppose that a is such a parameter with dimension $L^p T^q$ and that f is proportional to a. Since f has dimension L^2T^{-1} and g has dimension LT^{-2}, the Π theorem of dimensional analysis then states that f can be expressed as follows:

$$f(z,t)=a z^{\alpha} g^{\beta}\chi\left(\frac{igt^2}{4z}\right), \tag{22.31}$$

where

$$\alpha=\tfrac{3}{2}-p-\tfrac{1}{2}q,\qquad \beta=\tfrac{1}{2}(q+1). \tag{22.32}$$

(The factor $i/4$ in the argument of χ is chosen for later convenience.) Now substitute (22.31) into (22.25). One finds after some computation that

$$f_{tt}+igf'=iaz^{\alpha-1}g^{\beta+1}[\zeta\chi''(\zeta)+(\tfrac{1}{2}-\zeta)\chi'(\zeta)+\alpha\zeta]=0, \tag{22.33}$$

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where $\zeta = igt^2/4z$. The differential equation obtained by setting the expression in square brackets equal to zero determines χ in terms of confluent hypergeometric functions:

$$\chi(\zeta) = A\,{}_1F_1(-\alpha, \tfrac{1}{2}; \zeta) + B\zeta^{\frac{1}{2}}\,{}_1F_1(\tfrac{1}{2}-\alpha, \tfrac{3}{2}; \zeta). \tag{22.34}$$

From this it follows that

$$\left.\begin{aligned} f(z,t;\alpha) &= a z^{\alpha} g^{\beta}\left[A\,{}_1F_1\left(-\alpha, \frac{1}{2}; \frac{igt^2}{4z}\right) + B\left(\frac{igt^2}{4z}\right)^{\frac{1}{2}}{}_1F_1\left(\frac{1}{2}, -\alpha, \frac{3}{2}; \frac{igt^2}{4z}\right)\right] \\ &= A f_1(z,t;\alpha) + B f_2(z,t;\alpha). \end{aligned}\right\} \tag{22.35}$$

Remembering that

$${}_1F_1(\gamma, \delta; 0) = 1, \qquad {}_1F_1'(\gamma, \delta; 0) = \gamma/\delta,$$

one may easily derive the following:

$$\left.\begin{aligned} f(z,0) &= A f_1(z,0) = A a g^{\beta} z^{\alpha}, \\ f'(z,0) &= A f_1'(z,0) = A a\alpha g^{\beta} z^{\alpha-1}, \\ f_t(z,0) &= B f_{2t}(z,0) = \tfrac{1}{2} B a i^{\frac{1}{2}} g^{\beta+\frac{1}{2}} z^{\alpha-\frac{1}{2}}. \end{aligned}\right\} \tag{22.36}$$

The solution (22.35) may be further generalized by replacing t by $t-t_0$ and z by $z-x_0$ (i.e., by a different choice of the dimensionless variable ζ). One may then further superimpose these solutions. For the purpose at hand it will be sufficient to retain $t_0=0$. Then we may form the solution

$$f(z,t) = \int_{-\infty}^{\infty} A(x_0)\, f_1(z-x_0, t; \alpha_1)\, dx_0 + \int_{-\infty}^{\infty} B(x_0)\, f_2(z-x_0, t; \alpha_2)\, dx_0. \tag{22.37}$$

One finds from (22.36) that

$$\left.\begin{aligned} f(z,0) &= a_1 g^{\beta_1} \int_{-\infty}^{\infty} A(x_0)\,(z-x_0)^{\alpha_1}\, dx_0, \\ f'(z,0) &= a_1 \alpha_1 g^{\beta_1} \int_{-\infty}^{\infty} A(x_0)\,(z-x_0)^{\alpha_1-1}\, dx_0, \\ f_t(z,0) &= \tfrac{1}{2} a_2 i^{\frac{1}{2}} g^{\beta_2+\frac{1}{2}} \int_{-\infty}^{\infty} B(x_0)\,(z-x_0)^{\alpha_2-\frac{1}{2}}\, dx_0. \end{aligned}\right\} \tag{22.38}$$

Let us now make some special choices of a, and hence of α and β. As a parameter describing the initial profile of the surface take

$$a_2 = \int_{-\infty}^{\infty} \eta(x,0)\, dx; \tag{22.39}$$

as a parameter describing the initial velocity distribution take

$$a_1 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} \eta_t(\xi,0)\, d\xi. \tag{22.40}$$

Then a_1 has the dimension L^3T^{-1}, corresponding to $\alpha_1=-1$, $\beta_1=0$, and a_2 the dimension L^2, corresponding to $\alpha_2=-\tfrac{1}{2}$, $\beta_2=\tfrac{1}{2}$. With these choices of α_1 and α_2 in (22.37) we take

$$\left.\begin{aligned} A(x_0) &= \frac{-1}{a_1\pi} \int_{-\infty}^{x_0} \eta_t(\xi,0)\, d\xi, \\ B(x_0) &= \frac{-2}{a_2 \pi i^{\frac{3}{2}}}\, \eta(x_0,0). \end{aligned}\right\} \tag{22.41}$$

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Then the last two equations of (22.38) become (after an integration by parts in the first one)

$$f'(z,0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta_t(x_0,0)}{x_0-z}\,dx_0,$$

$$f_t(z,0)=\frac{g}{\pi i}\int_{-\infty}^{\infty}\frac{\eta(x_0,0)}{x_0-z}\,dx_0.$$

From the Plemelj-Sokhotskii theorem we have

$$\left.\begin{aligned}f'(x-i0,0)&=-i\eta_t(x,0)+\frac{1}{\pi}\,\mathrm{PV}\int_{-\infty}^{\infty}\frac{\eta_t(x_0,0)}{x_0-x}\,dx_0,\\ f_t(x-i0,0)&=-g\eta(x,0)+\frac{g}{\pi i}\,\mathrm{PV}\int_{-\infty}^{\infty}\frac{\eta(x_0,0)}{x_0-x}\,dx_0.\end{aligned}\right\}\tag{22.42}$$

Thus the initial conditions (22.22) are satisfied.

see errata There remains to point out that for the special choices of $\alpha_1=-1$ and $\alpha_2=-\frac{1}{2}$ the corresponding confluent hypergeometric functions in (22.35) may be expressed in terms of Fresnel integrals or integrals of these. In fact, if we write (22.37) see errata the form

$$f(z,t)=\int_{-\infty}^{\infty}\Omega_1(z-x_0,t)\int_{-\infty}^{x_0}\eta_t(\xi,0)\,d\xi\,dx_0+\int_{-\infty}^{\infty}\Omega_2(z-x_0,t)\,\eta(x_0,0)\,dx_0,\tag{22.43}$$

then

$$\left.\begin{aligned}\Omega_1(z,t)&=-\frac{2i}{z}e^{-i\frac{\pi}{2}\omega^2}\omega-\frac{2i}{z}\int_0^{\omega}du\int_0^{u}e^{-i\frac{\pi}{2}(v^2-\omega^2)}\,dv,\\ \Omega_2(z,t)&=2i\sqrt{\frac{2g}{\pi z}}\int_0^{\omega}e^{-i\frac{\pi}{2}(u^2-\omega^2)}\,du,\end{aligned}\right\}\tag{22.44}$$

where

$$\omega^2=\frac{g\,t^2}{2\pi z}.$$

One should also consult the discussion in LAMB's Hydrodynamics (1932, § 238, 239), where graphs are given which display the behavior of the surface profile corresponding to an initial elevation concentrated in the neighborhood of one point, i.e., essentially $-g^{-1}\Omega_{2t}(x-i0,t)$, and to a concentrated impulse, i.e., essentially $-g^{-1}\Omega_{1t}(x-i0,t)$. However, general aspects of the development of the surface profile have already been discussed in Sect. 15α.

It should be noted that the velocity potential (22.37) represents a much wider class of time-dependent gravity-wave motions than does (22.43). The initial-value problems corresponding to other values of α have been determined by SEDOV (1948) but the discussion will not be repeated here.

A class of solutions of (22.30) analogous to that found by SEDOV for (22.25) does not seem to have been given in the published literature. CISOTTI (1920) expands $f(z,t)$ in a power series in t, thus replacing (22.30) by a recursive set of difference equations. We refer to the original paper for his discussion of this set of equations.

In (15.22) we have already given the velocity potential and surface profile corresponding to a given initial profile; the derivation was based upon a Fourier

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analysis of the initial profile and the result was valid for either finite or infinite depth. The same procedure may be used for an initial velocity distribution. The combined result for the complex velocity potential and surface profile is given below in such a way as to include the possible presence of surface tension:

infinite depth:

$$\left.\begin{aligned} f(z,t) &= \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty d\xi \, \frac{1}{k}\left[-\sigma(k)\,\eta(\xi,0)\sin\sigma t + \eta_t(\xi,0)\cos\sigma t\right] e^{ik(z-\xi)}, \\ \eta(x,t) &= \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty d\xi \left[\eta(\xi,0)\cos\sigma t + \frac{\sigma}{g k}\eta_t(\xi,0)\sin\sigma t\right]\cos k(x-\xi), \end{aligned}\right\} \tag{22.45}$$

where

$$\sigma^2 = g k + T k^3/\varrho;$$

depth h:

$$\left.\begin{aligned} f(z,t) &= \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty d\xi \, \frac{1}{k \sinh k h} \times \\ &\quad \times\left[-\sigma(k)\,\eta(\xi,0)\sin\sigma t + \eta_t(\xi,0)\cos\sigma t\right]\cos k(z-\xi+i h), \\ \eta(x,t) &= \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty d\xi \times \\ &\quad \times\left[\eta(\xi,0)\cos\sigma t + \frac{\sigma(k)}{g k \tanh k h}\eta_t(\xi,0)\sin\sigma t\right]\cos k(x-\xi), \end{aligned}\right\} \tag{22.46}$$

where

$$\sigma^2 = (g k + T k^3/\varrho)\tanh k h.$$

If $T=0$, the coefficients of $\eta_t(\xi,0)$ in the formulas for $\eta(x,t)$ reduce to σ^{-1}. When $T=0$, (22.45) is, of course, another form of (22.43).

It has already been indicated in Sect. 15 that the Cauchy-Poisson problem can also be solved for superposed fluids. Sretenskii (1955) has investigated a further generalization in which the two fluids are each flowing with constant velocities for $t<0$ and then when $t=0$ a disturbance is suddenly created at the interface.

γ) Some other time-dependent problems. It is possible to solve a number of initial-value problems either by using Eq. (22.8) or by using the time-dependent Green functions (13.49) or (13.53) directly. The special situations treated, below fall roughly into the following four categories: wave motions resulting from a pressure distribution suddenly brought into existence at time $t=0$; waves resulting from a body set into motion at time $t=0$; waves resulting from an underwater explosion or a sudden movement of the bottom (tsunamis); and waves resulting from an initially displaced freely floating body.

Time-dependent pressure distributions. Suppose that the fluid is undisturbed for $t<0$ and that starting with $t=0$ the pressure over the free surface is a given function $p(x,z,t)$. The consequent motion of the fluid may be easily obtained, for this is just the problem formulated in (22.2) if we put $\eta(x,z,0) = \eta_t(x,z,0)=0$. Formula (22.8) then gives the velocity potential in the form

$$\Phi(x,y,z,t) = \frac{-1}{4\pi\varrho g}\iint_F d\xi\,d\zeta \int_0^t G_{tt}(\xi,0,\zeta,x,y,z;\tau,t)\,p(\xi,\zeta,\tau)\,d\tau + I. \tag{22.47}$$

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see errata

In the two situations for which explicit GREEN's functions have been given, Eqs. (22.10) and (22.11), we may give explicit solutions for Φ:

infinite depth:

$$\Phi(x, y, z, t) = \frac{-1}{2\pi\varrho} \iint_{-\infty}^{\infty} d\xi\, d\zeta \int_0^t p(\xi, \zeta, \tau)\, d\tau \int_0^{\infty} \cos\left(\sqrt{g k}(\tau - t)\right) e^{k y} J_0(k R)\, k\, dk; \quad (22.48)$$

depth h:

see errata

$$\left.\begin{aligned} \Phi(x, y, z, t) = \frac{-1}{2\pi\varrho} \iint_{-\infty}^{\infty} d\xi\, d\zeta \int_0^t p(\xi, \zeta, \tau)\, d\tau \int_0^{\infty} \left(\cos\sqrt{g k \tanh k h}\,(\tau - t)\right) \times \\ \times \frac{\cosh k(y+h)}{\cosh k h} J_0(k R)\, k\, dk, \end{aligned}\right\} \quad (22.49)$$

where, as usual, $R^2 = (x-\xi)^2 + (z-\zeta)^2$.

The velocity potential for a moving pressure distribution is obtained from these expressions simply by letting

$$p(\xi, \zeta, \tau) = p_0(\xi - c\tau, \zeta, \tau).$$

If $p_0(\xi - c\tau, \zeta, \tau) = p_0(\xi - c\tau, \zeta) \cos\sigma\tau$ the resulting Φ is the velocity potential for a steadily moving pressure distribution of oscillating strength. LUNDE (1951b) has investigated the special case when $p(\xi, \zeta, \tau) = p\left(\sqrt{(\xi - c\tau)^2 + \zeta^2}\right)$ and has shown that as $t \to \infty$ the expressions (22.48) and (22.49), after a change to moving coordinates, approach asymptotically to the expressions (21.26) or (21.31) properly modified for circular symmetry (the assumed symmetry is not essential). The computation is interesting but will not be carried out here. This procedure for obtaining (21.26) or (21.31) yields the velocity potential without necessitating the extra boundary condition requiring the motion to vanish as $x \to +\infty$.

see errata

As was mentioned in connection with the solution of the Cauchy-Poisson problem, the GREEN's function for some other simple configurations can be found by the method of reflection.

The complex velocity potentials for two-dimensional motion which correspond to (22.48) and (22.49) are as follows:

infinite depth:

$$f(z, t) = \frac{-1}{\pi\varrho} \int_{-\infty}^{\infty} d\xi \int_0^t p(\xi, \tau)\, d\tau \int_0^{\infty} \cos\left(\sqrt{g k}(\tau - t)\right) e^{-i k (z - \xi)}\, dk; \quad (22.50)$$

depth h:

$$f(z, t) = \frac{-1}{\pi\varrho} \int_{-\infty}^{\infty} d\xi \int_0^t p(\xi, \tau)\, d\tau \int_0^{\infty} \cos\left(\sqrt{g k \tanh k h}\,(\tau - t)\right) \frac{\cos k(z - \xi + i h)}{\cosh k h}\, dk. \quad (22.51)$$

Certain special cases have been investigated in more detail. STOKER (1953) [see also WURTELE (1955)] has treated the motion resulting when a pressure distribution constant in time for $t > 0$ is suddenly applied to a uniformly moving stream of depth h. The velocity potential may be obtained from (22.51) by taking $p(\xi, \tau) = p_0(\xi - c\tau)$ and transferring to moving coordinates. His aim, as was that of LUNDE in the computations described above, was to show that the potential (21.40) can be derived without a special assumption about its behavior as $x \to +\infty$. The same can be carried through with (22.50) to derive (21.38). If one assumes $p(\xi, \tau) = p_1(\xi) \cos\sigma\tau + p_2(\xi) \sin\sigma\tau$, then one may also derive (21.21) or (21.23) from (22.49) or (22.50), respectively, as asymptotic expressions

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for large t without having to impose a radiation condition. VOIT (1957b) has investigated the surface profile for large t when $p(\xi,\tau)=p(\tau)$ for $\xi<c\tau<cT$, $p(\xi,\tau)=0$ for $\xi\geqq c\tau$ or for $\tau>T$.

Waves resulting when a body is set into motion. Many of the problems solved in Sects. 17 to 20 by means of source distributions can be formulated as initial-value problems and solved by the same procedure if one uses the appro- see errata priate time-dependent GREEN's function. We shall consider briefly several examples, omitting details.

In (19.28) the velocity potential was given for the motion resulting from an oscillator in a wall, described by (19.26). It was assumed there that a steady situation had been reached in which the motion was purely harmonic in the time. Suppose instead that the motion of the oscillator described by (19.26) is to start at $t=0$ and that for $t<0$ the oscillator and fluid are at rest. It is easy to verify that the time-dependent velocity $\Phi(x,y,z,t)$ potential is still given by (19.28) if for the GREEN's function G one uses (13.50) with $m=1$. The last term in (13.50) will give the transient aspects of the motion. For two-dimensional motion the time-dependent wave-maker has been considered by KENNARD (1949), who also gives an estimate of time necessary for the transient terms to die out.

In (20.65) the velocity potential has been given for a "thin" ship moving with constant velocity c; it is assumed there that a steady state has been reached. Let us now suppose the same ship to move with velocity $c(t)$, $t>0$, but that it and the fluid have been at rest for $t<0$. As in (20.64) we take a coordinate system moving with the ship. Then from (20.26) it follows that the velocity potential $\Phi(x,y,z,t)$ must satisfy the boundary condition

$$\Phi_z(x,y,\pm 0,t) = \mp c(t)\,F_x(x,y).$$

A GREEN's function enabling us to construct Φ can be easily obtained from either (13.49) or (13.53). However, let us take $c(t)=C$, a constant, for $t>0$, i.e. we suppose the ship to attain instantaneously its final velocity. The GREEN's function for this situation has already been written out explicitly in (13.51). Setting there $u_0=c$, $a_0=\xi$, $b_0=\eta$, $c_0=\zeta$ and calling the resulting function $G(x,y,z,\xi,\eta,\zeta,t)$, the velocity potential for the problem at hand is

$$\Phi(x,y,z,t)=\frac{c}{2\pi}\iint\limits_{S_0} G(x,y,z,\xi,\eta,0,t)\,F_x(\xi,\eta)\,d\xi\,d\eta. \tag{22.52}$$

see errata Having found Φ, one may then compute the force upon the ship and obtain formulas analogous to (20.67) or (20.69). The computations for infinite depth was originally made by SRETENSKII (1937); LUNDE (1951a) gives an exposition of this result and extends it to include thin ships moving in an infinite expanse of fluid of depth h and down the center of a canal of width b and depth h. In these computations c is allowed to be an arbitrary function of t. We refer to LUNDE's paper for the results.

HAVELOCK (1948, 1949) has considered the accelerated motion of a submerged horizontal circular cylinder in fluid of infinite depth. The complex velocity potential is expanded in a Laurent series about the center, starting with a dipole. In order to satisfy the other boundary conditions, one makes use of (13.54) to obtain singularities of the proper sort. The boundary condition on the circle then yields an infinite set of equations for determining the coefficients in the Laurent series. After finding as many terms as seems necessary for a suitable approximation, one may compute the force on the cylinder. HAVELOCK has

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carried this out for the first two singularities [a slight inconsistency in the approximation is corrected in MARUO (1957)] and has made numerical computations for an impulsive start and for a constant acceleration. Consider an impulsive start with instantaneous acceleration to constant speed c, and let the cylinder have radius a and its center be submerged to depth h. Then the two leading terms in the resistance are given by R_0, the steady-state resistance given in Eq. (20.52), and by the transient term

$$R_1 = \frac{1}{2}\pi g \varrho a^4 \nu^2 \left(\frac{\pi}{\nu c t}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\nu h} \sin\left(\frac{1}{4}\nu c t - \frac{\pi}{4}\right), \quad \nu = \frac{g}{c^2}. \tag{22.53}$$

Fig. 28, taken from MARUO (1957), shows $(R_0+R_1)/R_0$ plotted against ct/h for $c/\sqrt{gh}=1$.

An exposition of the theory of accelerated motion of submerged bodies is given by MARUO (1957, Chap. 3). Both two- and three-dimensional problems in fluid of infinite or finite depth are considered. We note that the use of KOCHIN's H-function may be extended with no difficulty to time-dependent motion; this has been done by MARUO and earlier by HASKIND (1946b).

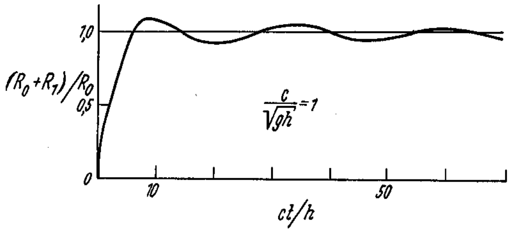

Fig. 28.

An investigation of PALM (1953) also fits into the category of problems under consideration. In considering flow over an uneven bottom in Sect. 20α, it was necessary to impose an upstream boundary condition in order to obtain uniqueness if the velocity is subcritical. In order to avoid this extra condition he formulated an initial-value problem in which the fluid is at rest and the bottom suddenly starts to move. The asymptotic expression for large t in a coordinate system moving with the bottom agrees with the results in Sect. 20α.

Tsunamis and submarine explosions. A tsunami is an ocean wave originating from a sudden upheaval or recession of the ocean floor. If one assumes an ocean of uniform depth h and if the disturbance occurs in a region S of the bottom, one may approximate this situation by the boundary-value problem in which

$$\Phi_y(x, -h, z, t) = \begin{cases} V(x, z, t), & 0 < t < T, \ (x, z) \text{ in } S, \\ 0, & \text{otherwise}. \end{cases} \tag{22.54}$$

If the time-interval of the disturbance is short (i.e., if gT^2/h is small), the solution for Φ is given approximately by distributing over S sources of a form easily derived from (13.53). In fact, in (13.53) let $a=\xi$, $b=-h$, $c=\zeta$, and let $2m(t) = 2m(\xi,\zeta,t) = -\Phi_y(\xi,-h,\zeta,t)/2\pi$; denote the resulting function by $\Phi_s(x, y, z, \xi, -h, \zeta, t)$. Then

$$\Phi(x, y, z, t) = \iint_S \Phi_s(x, y, z, \xi, -h, \zeta, t)\, d\xi\, d\zeta \tag{22.55}$$

is the approximate solution. If one assumes $V(x,z,t) = V(x,z)$ for $0<t<T$, then Φ_s takes the following simple form for $t>T$:

$$\left.\begin{aligned} \Phi_s(x, y, z, \xi, -h, \zeta, t) = -\frac{1}{2\pi} V(\xi,\zeta) \int_0^\infty \frac{\cosh k(y+h)\, J_0(kR)}{\sinh kh \cosh kh} \times \\ \times [\cos\sigma(t-T) - \cos\sigma t]\, dk, \end{aligned}\right\} \tag{22.56}$$

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where $\sigma^2 = gk\tanh kh$. If the deformation is assumed to take place so quickly that one may let $T\to 0$ while keeping $VT = L(\xi,\zeta)$ constant (i.e., keeping the same total deformation), (22.56) becomes

$$\Phi_s(x,y,z;\xi,-h,\zeta,t) = \frac{-1}{2\pi} L(\xi,\zeta)\int_0^\infty \frac{\cosh k(y+h)\,J_0(kR)}{\sinh kh\cosh kh}\,\sigma(k)\sin\sigma(k)\,t\,dk, \quad (22.57)$$

and the solution (22.55) is no longer approximate for the formulated problem.

A further approximation may be obtained by assuming the area of disturbance to be so localized that one may assume the whole disturbance to originate at one point, say $(0,-h,0)$. Then (22.55) becomes simply (22.57) with L replaced by $Q = \iint L\,d\xi\,d\zeta$ and $R^2 = x^2 + z^2$. Although this may be a reasonable approximation to the explosion of a mine on the ocean floor, it is not in general suitable for a tsunami since the diameter of the region of disturbance in the latter may be many times the depth of fluid.

A comparison of (22.55), with (22.57) for Φ_s, with (22.15) shows that one may expect the same qualitative behavior for tsunamis as for waves resulting from an initial deformation of the free surface. In fact, if one makes the substitution (22.19) in the expressions for the surface profiles, they become the following, respectively, for the initially displaced surface and the tsunami:

$$\left.\begin{aligned}
\eta(x,z,t) &= \frac{1}{\pi}\iint_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,\frac{1}{R^2}\,d\xi\,d\zeta\int_0^\infty \varkappa^3 J_0(\varkappa^2)\cos\left(2\omega\varkappa\sqrt{\tanh\varkappa^2\frac{h}{R}}\right)d\varkappa,\\
\eta(x,z,t) &= \frac{1}{\pi}\iint_{-\infty}^{\infty}L(\xi,\zeta)\,\frac{1}{R^2}\,d\xi\,d\zeta\int_0^\infty \varkappa^3\frac{J_0(\varkappa^2)}{\cosh^2\left(\varkappa^2\frac{h}{R}\right)}\cos\left(2\omega\varkappa\sqrt{\tanh\varkappa^2\frac{h}{R}}\right)d\varkappa.
\end{aligned}\right\}\quad (22.58)$$

One may study the development of η along the lines worked out in Sect. 15 for two dimensions.

Many of the investigations of tsunamis have been devoted to an examination of the profile for a given type of initial bottom disturbance. The classical papers on tsunamis are by SANO and HASEGAWA (1915) and SYONO (1936). They have recently been investigated by TAKAHASHI (1942, 1945, 1947), ICHIYE (1950), GAZARYAN (1955) and others. Since the shape of the bottom and the configuration of the shore are of obvious importance in a geophysical application of the theory, much recent attention has been given to this aspect of the propagation of tsunamis. GRIGORASH (1957a) has given a brief survey of the literature together with a substantial bibliography.

The waves resulting from an exploding submerged mine may be represented approximately by using the velocity potential for a source whose strength $m(t)$ has the form of a square pulse of duration T. One may then determine Φ from either (13.49) or (13.53). If one assumes T very small and forms the limit as $T\to 0$ while keeping $mT = Q$ constant, one finds easily the following expressions for Φ:

infinite depth:

$$\Phi = 2Q\int_0^\infty e^{k(y+b)}\,J_0(kR)\,\sigma(k)\sin\sigma t\,dk, \qquad \sigma^2 = gk; \quad (22.59)$$

depth h:

see errata

$$\Phi = 2Q\int_0^\infty \frac{\cosh k(h+b)\cosh k(y+h)}{\sinh kh\cosh kh}\,\sigma(k)\sin\sigma t\,dk, \qquad \sigma^2 = gk\tanh kh. \quad (22.60)$$

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Again one may examine the development of the surface profile by the methods developed in Sect. 15.

Investigations of waves generated by a sudden pulse of the above or similar sort have been made by LAMB (1913, 1922) and TERAZAWA (1915); both took the fluid to be infinitely deep. SRETENSKII (1950, 1949) has made a similar study when the source (two-dimensional) is situated on the bottom of a rectangular basin and within a fluid layer covering a solid sphere. SEZAWA (1929a, b) has included the effect of compressibility of the fluid.

One should recognize that such studies can elucidate only a small part of the phenomena associated with underwater explosions. An investigation of the migration and oscillation of the explosion bubble requires different analytical methods. Furthermore, if the explosion is too violent the linearized boundary condition on the free surface may not be a useful approximation.

Freely floating bodies. The motion of a freely floating body following an initial displacement is of considerable interest and practical importance, but also leads to a difficult mathematical problem. Uniqueness of solution follows from the argument in Sect. 22α. For the sake of perspicuity let us restrict ourselves to motion constrained to be vertical, i.e., heaving motion. Then from (19.59) and (19.62) the boundary conditions to be satisfied on the surface of the body in its equilibrium position, S_0, are

$$\Phi_n(x, y, z, t) = \dot{y}_1(t)\, n_y(x, y, z), \quad (x, y, z) \text{ on } S_0, \tag{22.61}$$

$$M\ddot{y}_1(t) + \varrho g I^A y_1(t) = -\varrho \iint_{S_0} \Phi_t(\xi, \eta, \zeta, t)\, n_y(\xi, \eta, \zeta)\, d\sigma. \tag{22.62}$$

(The notation is explained in Sect. 19β.) In addition Φ must satisfy the free-surface condition

$$\Phi_{tt}(x, 0, z, t) + g\,\Phi_y(x, 0, z, t) = 0 \tag{22.63}$$

and initial conditions, say

$$\Phi_t(x, 0, z, 0) = \Phi_y(x, 0, z, 0) = 0, \tag{22.64}$$

$$\dot{y}_1(0) = \dot{y}_{10}, \qquad y_1(0) = y_{10}. \tag{22.65}$$

As in many previous cases one may reduce the problem to the solution of an integral equation by use of a GREEN's function. In (13.49) replace (a, b, c) by (ξ, η, ζ) and $m(t)$ by $\gamma(\xi, \eta, \zeta, t)$; denote the resulting function by Φ_s:

see errata

$$\left.\begin{aligned} &\Phi_s(x, y, z, \xi, \eta, \zeta, t) = \gamma(\xi, \eta, \zeta, t)\left[\frac{1}{r} - \frac{1}{r_1}\right] + \\ &\quad + 2\int_0^\infty (g k)^{\frac{1}{2}}\, e^{k(y+\eta)} J_0(k R)\, dk \int_0^t \gamma(\xi, \eta, \zeta, \tau) \sin\left[(g k)^{\frac{1}{2}}(t-\tau)\right] d\tau. \end{aligned}\right\} \tag{22.66}$$

We now attempt to express Φ by the integral

$$\Phi(x, y, z, t) = \iint_{S_0} \Phi_s(x, y, z, \xi, \eta, \zeta, t)\, d\sigma, \tag{22.67}$$

for then (22.63) and (22.64) will be satisfied. One should note especially that the relation of Φ to γ is more complicated here than in problems typified by (16.12), for the past history of γ is involved in Φ_s. The conditions (22.61) and (22.62) now become

$$-2\pi\gamma(x, y, z, t) + \iint_{S_0} \Phi_{sn}(x, y, z, \xi, \eta, \zeta, t)\, d\sigma = \dot{y}_1(t)\, n_y(x, y, z), \tag{22.68}$$

$$M\ddot{y}_1(t) + \varrho g I^A y_1(t) = -\varrho \iint_{S_0} d\sigma \iint_{S_0} dS\, \Phi_{st}(x, y, z, \xi, \eta, \zeta, t)\, n_y(\xi, \eta, \zeta), \tag{22.69}$$

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see errata

where γ also enters into the equations through Φ_s. The two equations form a pair of coupled integro-differential equations for γ and y_1. It is evident that one can probably not hope for an analytic solution even for simple configurations.

SRETENSKII (1937b), for two dimensions, and later HASKIND (1946b) for three dimensions simplified the problem further by assuming the body to be "thin", i.e., if the surface is given by $z = \pm F(x, y)$, by replacing the boundary condition (22.61) by

$$\Phi_z(x, y, \pm 0, t) = \mp \dot{y}_1(t)\, F_y(x, y) \tag{22.70}$$

and S_0 by the projection of S_0 on the plane $z=0$ [cf. (20.26) and (20.64)]. With this further assumption one can immediately satisfy (22.68) by taking

$$\gamma(x, y, t) = -\frac{1}{2\pi}\dot{y}_1(t)\, F_y(x, y). \tag{22.71}$$

Eq. (22.69) then becomes an integro-differential equation for $y_1(t)$.

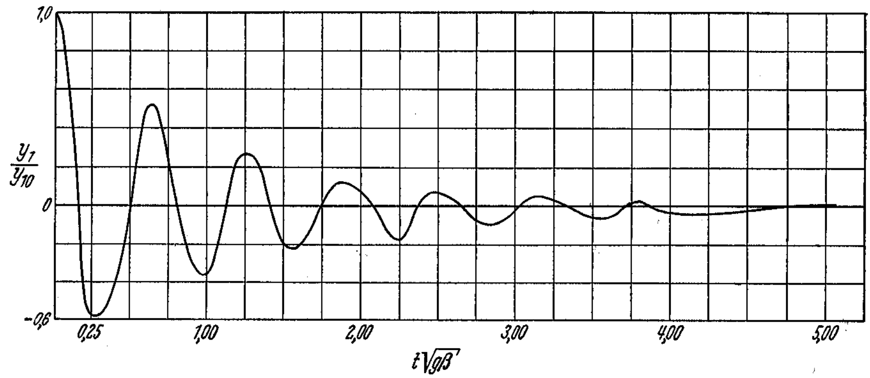

Fig. 29.

The procedure is open to some objection in that the substituted boundary condition (22.70) does not seem to fit into the general perturbation scheme as developed in Sects. 10α, 19α and 20β. It is thus not clear what physical problem really corresponds to the mathematical problem. However, this seems to be the closest anyone has come to reducing the equations to a manageable form. SRETENSKII solved his resulting integro-differential equation numerically for a surface described by $F(y) = l\, e^{\beta|y|}$, where

$$l = \frac{\pi g}{800} = 3.85 \text{ cm}, \qquad \beta = \frac{100}{g} = 0.104 \text{ cm}^{-1}.$$

The resulting graph of y_1/y_{10} is shown in Fig. 29 with a dimensionless abscissa $t\sqrt{g\beta}$. In spite of the questionableness of the formulation of the problem, the graph is instructive in showing the difference between a damped harmonic oscillation and the solution of SRETENSKII's integro-differential equation. Approximate methods of solution to the problem which assume that the fluid motion at any instant is independent of its past history lead to damped harmonic oscillations.

23. Waves in basins of bounded extent. The study of wave motion in a basin presents no special difficulties not already encountered earlier, and has a particular interest because of the many opportunities of observing such waves. Certain general aspects of the problem may be considered as being contained in earlier sections. For example, the general discussion of initial-value problems in Sect. 22α

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applies to motion in a basin. However, in order to make use of the results, in particular of Eq. (22.8), in constructing a solution, one must have prior knowledge of the time-dependent GREEN'S function for the geometric boundary. Although the method of images can be used together with (13.49) or (13.53) to construct the GREEN'S function for certain simple configurations, an explicit analytic solution is generally not available.

The time-dependent problem has also been approached in another manner by HADAMARD (1910, 1916), who derived an integro-differental equation for the function $\eta(x, y, t)$ describing the free surface. HADAMARD'S short notes have been worked out by BOULIGAND (1912, 1926, 1927) and developed further. Certain of BOULIGAND'S investigations indicate that singularities which may occur at the intersection of the plane $y=0$ with the basin walls are a result of linearizing the free-surface boundary condition. For an exact statement one should consult the original papers. There is a brief treatment of HADAMARD'S equation in VERGNE (1928, § 10, 14). MOISEEV (1953) has developed a treatment of the time-dependent problem which generalizes somewhat the method used in Sect. 23 α.

In Sect. 23 α we give some general theorems concerning motions periodic in time, and another solution of the initial-value problem. In Sect. 23 β wave motions for several special configurations of the boundary are given. In Sect. 23 γ the theory of wave motion in movable basins is considered.

α) *Periodic waves in basins: general theorems.* If the motion is periodic in time, the velocity potential may be found by solving a Fredholm integral equation, obtained after introduction of an appropriate GREEN'S function. Assume $\Phi(x, y, z, t) = \varphi(x, y, z)\cos(\sigma t + \tau)$; then φ must satisfy the boundary conditions

$$\left.\begin{aligned} &\varphi_y(x, 0, z) - \nu\varphi(x, 0, z) = 0, \quad (x, z) \text{ in } F, \quad \nu = \sigma^2/g, \\ &\varphi_n = 0, \quad (x, y, z) \text{ on } S, \end{aligned}\right\} \tag{23.1}$$

where F is the part of the plane $y=0$ occupied by the free surface at rest and S is the surface of the basin. Let $G(x, y, z, \xi, \eta, \zeta)$ be the GREEN'S function for NEUMANN'S problem, i.e.,

$$G = \frac{1}{r} + G_0,$$

where G_0 is regular in the region occupied by fluid and G satisfies the conditions

$$G_n = c \text{ on } S, \quad G_y(x, 0, z, \xi, \eta, \zeta) = c \text{ on } F, \tag{23.2}$$

where c is an arbitrary nonzero constant. In addition, in order to make the definition of φ unique we require

$$\iint\limits_{S+F} \varphi\, d\sigma = 0.$$

It then follows from GREEN'S theorem that

$$\varphi(x, y, z) = \frac{1}{4\pi}\iint\limits_F G\varphi_y\, d\xi\, d\zeta = \frac{\nu}{4\pi}\iint\limits_F G\varphi(\xi, 0, \zeta)\, d\xi\, d\zeta. \tag{23.3}$$

If one now lets $y \to 0$, one obtains

$$\varphi(x, 0, z) = \frac{\nu}{4\pi}\iint\limits_F G(x, 0, z, \xi, 0, \zeta)\,\varphi(\xi, 0, \zeta)\, d\xi\, d\zeta, \tag{23.4}$$

a homogeneous Fredholm integral equation for $\varphi(x, 0, z)$. If $\varphi(x, 0, z)$ can be found, then $\varphi(x, y, z)$ is determined by (23.2). From the theory of such integral

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equations there will exist a sequence $\nu_1, \nu_2, \ldots$ of eigenvalues for which (23.4) will yield solutions $\varphi_1, \varphi_2, \ldots$. The functions φ_i corresponding to different ν_i-s are orthogonal on F, as shown in (16.10). If several ν_i-s have the same value, the corresponding φ_i-s can be orthogonalized. The φ_i also form a complete set on F. Each solution φ_i yields a standing wave in the basin.

It is possible to use these solutions to solve the initial-value problem formulated in (22.2), but with $p=0$. Let $\eta(x, z, 0)$ and $\eta_t(x, z, 0)$ be given. We try to express $\Phi(x, 0, z, t)$ in the following form:

$$\Phi(x, 0, z, t) = \sum_{i=1}^{\infty} a_i \varphi_i(x, 0, z) \cos \sigma_i t + b \varphi_i(x, 0, z) \sin \sigma_i t. \tag{23.5}$$

Then

$$\left.\begin{aligned} -g\,\eta(x, z, 0) &= \Phi_t(x, 0, z, 0) = \sum \sigma_i b_i \varphi_i(x, 0, z), \\ -g\,\eta_t(x, z, 0) &= \Phi_{tt}(x, 0, z, 0) = -\sum \sigma_i^2 a_i \varphi_i(x, 0, z). \end{aligned}\right\} \tag{23.6}$$

Since the φ_i form a complete set of orthogonal functions over F, the coefficients a_i and b_i can be determined in the usual manner. $\Phi(x, y, z, t)$ is then determined by (23.5) and (23.3).

In order to use the integral equation (23.4) one must first find G, the GREEN's function to a Neumann problem for a region having a corner along the curve of intersection of the plane $y=0$ and the basin walls. The difficulty with the corner can be overcome in certain cases. If the basin wall intersects the plane perpendicularly, then the basin plus its reflection in the plane $y=0$ has a boundary without this corner. If $\gamma(x, y, z, \xi, \eta, \zeta)$ is a GREEN's function for the Neumann problem for the extended region, then

$$G(x, y, z, \xi, \eta, \zeta) = \tfrac{1}{2}\left[\gamma(x, y, z, \xi, \eta, \zeta) + \gamma(x, -y, z, \xi, \eta, \zeta)\right] \tag{23.7}$$

is a GREEN's function for the original region. For some other special regions one may construct a GREEN's function by the method of images, even though the intersection with the plane $y=0$ is not perpendicular.

As mentioned above, each φ_i represents a standing wave of frequency σ_i. It may happen, as we shall see presently, that two or more σ_i-s are equal. Let σ be such an eigenvalue and $\varphi^{(1)}$ and $\varphi^{(2)}$ two of the corresponding potential functions. By forming the standing-wave solution

$$[\lambda_1 \varphi^{(1)} + \lambda_2 \varphi^{(2)}] \cos \sigma t, \qquad \lambda_1 + \lambda_2 = 1, \tag{23.8}$$

one may vary continuously the position of the nodal curves, say. If n independent φ_i correspond to σ, then the possible nodal curves form an $(n-1)$-parameter family of curves in F. With the two solutions $\varphi^{(1)}$ and $\varphi^{(2)}$ one may also form the solution

$$\Phi(x, y, z, t) = \varphi^{(1)}(x, y, z) \cos \sigma t + \varphi^{(2)}(x, y, z) \sin \sigma t. \tag{23.9}$$

The nodal curves will now migrate from those of $\varphi^{(1)}$ to those of $\varphi^{(2)}$, and then on again to those of $\varphi^{(1)}$. If $\varphi^{(1)}$ and $\varphi^{(2)}$ have a common zero at, say, (x_0, z_0), then a nodal curve for Φ will always pass through (x_0, z_0). Near (x_0, z_0) the waves will appear to progress about (x_0, z_0) like spokes moving about a wheel. There may, of course, be several such centers.

β) Some special boundaries. It is possible to give explicit solutions for standing waves for several particular configurations of the basin. The variety of such configurations, however, is rather small. As a preliminary we note that if the

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basin has a flat bottom at depth h and if the side walls form a vertical cylinder making a section C with $y=0$ then, from Sect. 13α, we have

$$\left.\begin{aligned} \Phi(x, y, z, t) &= \varphi(x, z) \cosh m_0 (y+h) \cos(\sigma t + \tau), \\ m_0 \tanh m_0 h - \frac{\sigma^2}{g} &= 0, \end{aligned}\right\} \tag{23.10}$$

where $\varphi(x, z)$ is a solution of

$$\varphi_{xx} + \varphi_{zz} + m_0^2 \varphi = 0 \tag{23.11}$$

satisfying

$$\varphi_n = 0 \text{ on } C. \tag{23.12}$$

The boundary condition (23.12) will limit m_0, and hence σ, to a discrete sequence of eigenvalues

$$m_0^{(1)}, m_0^{(2)}, \ldots; \sigma_1, \sigma_2, \ldots. \tag{23.13}$$

In a coordinate system in which (23.11) can be separated it is usually possible to find the standing waves in basins whose side walls are constant-coordinate surfaces. These statements will be illustrated below for rectangular and cylindrical coordinates.

In connection with the special cases treated below we call attention to papers by HONDA and MATSUSHITA (1913) and SASAKI (1914). The authors investigated experimentally in a systematic way the various modes of motion in rectangular, triangular, circular, ring-shaped, circular-sectorial and ring-sectorial basins and compared measured with calculated periods. In most cases the agreement is with 2%. Photographs showing the various modes were obtained by sprinkling the surface with aluminum powder and exposing a photographic plate for about one period. The nodes then show up as dots, the rest as streaks. In connection see errata with a study of the excitations of waves in a port, MCKNOWN (1953) has also investigated experimentally the standing waves in circular and square basins; some striking photographs are included. APTÉ (1957) has studied further the theory of the excitation of standing waves in a square basin and has also given experimental results. Perhaps the first theoretical investigation was by RAYLEIGH (1876, pp. 272—279); he compared his predicted frequencies with observations of his own and of GUTHRIE (1875).

Rectangular basin. Let the basin walls be given by

$$x=0, \quad x=l, \quad z=0, \quad z=b, \quad y=-h.$$

Then from (13.6) one may write down immediately the solution

$$\left.\begin{aligned} \Phi &= A \cosh m_0 (y+h) \cos \frac{2\pi}{l} x \cos \frac{p\pi}{b} z \cos(\sigma t + \tau), \\ m_0^2 &= \pi^2 \left(\frac{q^2}{l^2} + \frac{p^2}{b^2}\right), \quad \frac{\sigma^2}{g} = m_0 \tanh m_0 h, \quad p, q = 0, 1, 2, \ldots \end{aligned}\right\} \tag{23.14}$$

Thus the choice of the integers p and q determines m_0 and then σ. If the basin is square, i.e., $l=b$, then the same values of m_0 and σ may correspond to two different solutions obtained by interchanging p and q, assuming $p \neq q$. However, this may also occur for other rectangular basins if b and l are commensurate.

Circular-cylinder basin. Let the basin have radius a. Then from (13.8) we find the solutions

$$\left.\begin{aligned} \Phi &= A \cosh m_0 (y+h) J_n(m_0 R) \cos(n\alpha + \delta) \cos(\sigma t + \tau), \quad n = 0, 1, 2, \ldots, \\ J_n'(m_0 a) &= 0, \quad \frac{\sigma^2}{g} = m_0 \tanh m_0 h. \end{aligned}\right\} \tag{23.15}$$

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Thus m_0 must be selected so that $m_0 a$ is one of the zeros of J_n'; this then determines σ. For $n=0$ the wave crests and nodes lie on concentric circles, the number of such nodal circles depending upon which zero of J_n' is used to determine m_0. If $n \geqq 1$, then to the same σ there correspond two independent solutions ($\delta=0$, $\delta=\frac{1}{2}\pi$), and the remarks made in connection with (23.8) and (23.9) apply.

The standing waves in a basin shaped like a sector of a circle may be obtained from (13.8). If α_0 is the angle of the sector, then

$$\Phi = A \cosh m_0(y+h)\, J_{\frac{n\pi}{\alpha_0}}(m_0 R) \cos\frac{n\pi}{\alpha_0}\alpha \cos(\sigma t+\tau), \qquad n=0,1,\ldots,$$

$$J'_{\frac{n\pi}{\alpha_0}}(m_0 a)=0, \qquad \frac{\sigma^2}{g}=m_0 h \tanh m_0 h.$$

If the basin is ring-shaped, with inner radius b and outer radius a, then from (13.8) one finds [cf. SANO (1913), CAMPBELL (1953)]:

$$\left.\begin{aligned} \Phi &= A\cosh m_0(y+h)\,[Y_n'(m_0 b)\, J_n(m_0 R) - \\ &\quad - J_n'(m_0 b)\, Y_n(m_0 R)] \cos(n\alpha+\delta)\cos(\sigma t+\tau), \\ & Y_n'(m_0 b)\, J_n'(m_0 a) - J_n'(m_0 b)\, Y_n'(m_0 a) = 0, \\ & \frac{\sigma^2}{g} = m_0 h \tanh m_0 h, \qquad n=0,1,\ldots. \end{aligned}\right\} \tag{23.16}$$

Formulas for sectors of a ring may be obtained and are similar to (23.15).

Basins with sloping side-walls. There are very few explicit solutions known when the sides are not vertical. If the basin is a horizontal cylinder bounded at either end by vertical walls at, say, $z=0$ and $z=l$, the theory of progressive waves in canals, developed in Sect. 18γ, can be carried over with only small changes, namely replacement of $\cos(kz-\sigma t)$ by $\cos kz \cos(\sigma t+\tau)$ where now k is restricted to the values $n\pi/l$ and σ correspondingly. Thus (18.39) and (18.43) give the velocity potentials, after the indicated modifications, for various modes of oscillation of a fluid in a basin of triangular section whose sides form an angle of 45° with the horizontal. However, even though these formulas may be used also for the two-dimensional modes, when $k=0$, they do not give the gravest two-dimensional mode except by a limiting process [described, e.g., in LAMB (1932, p. 443)].

The two-dimensional modes of motion in triangular basins whose sides form an angle $\gamma=m\pi/n$ with the horizontal may also be studied by use of the methods introduced in Sect. 17β for standing waves on beaches. Indeed, it is apparent that KIRCHHOFF (1879) considered his investigation of waves on beaches as a preliminary to the problem at hand. Because his approach is systematic we shall describe it.

In order to use the results of 17γ we take one side as $y=-x\tan\gamma$, i.e., $z=r\,e^{i\gamma}$; let the other side be given by

$$z = 2a - r\,e^{i\gamma}. \tag{23.17}$$

Then the complex potential $f(z)$ must satisfy not only (17.31) and (17.32), but also

$$f(2a-r\,e^{i\gamma}) = \bar f(2a - r\,e^{-i\gamma}), \tag{23.18}$$

which, taken with (17.31), implies that

$$f(z) = f\big(z\,e^{-i4\gamma} + 2a\,e^{-i2\gamma}(1-e^{-i2\gamma})\big). \tag{23.19}$$

In order to satisfy (17.31), (17.32) and (23.19) KIRCHHOFF first takes

$$f(z) = B_h z^h + \cdots + B_{h+k} z^{h+k}. \tag{23.20}$$

Substitution in (17.32) yields (with $\beta = e^{-2i\gamma}$ as before)

$$1 - \beta^h = 0, \quad B_{n+1} = \frac{-i\nu}{n+1} \frac{1+\beta^n}{1-\beta^{n+1}} B_n, \quad 1 + \beta^{h+k} = 0. \tag{23.21}$$

Thus, since $\gamma = m\pi/n$, one must have $h = pn$, $p = 0, 1, \ldots$, and $k = \frac{1}{2}n = q$, an integer. If one takes $p = 0$, then (23.20) becomes

$$f(z) = B_0 \left\{1 + \sum_{s=1}^{q} (-1)^s z^s \frac{\nu^s}{s!} \frac{e^{is\gamma}}{\cos s\gamma} \cot\gamma \ldots \cot s\gamma\right\}. \tag{23.22}$$

Conditions (17.31) requires B_0 to be real. Condition (23.18) or (23.19) remains to be satisfied. The function $f(z)$ in its assumed form, is apparently overdetermined, and it is possible to show that for $q > 3$ not all conditions can be satisfied. For $q = 2$, $m = 1$ and $q = 3$, $m = 1$, (23.18) can be satisfied. The potential functions are as follows:

$$\left.\begin{aligned}
&\gamma = \pi/4: \\
&f(z) = B_0\left[1 - (1+i)\nu z + \tfrac{1}{2} i\nu^2 z^2\right] = \tfrac{1}{2} i B_0 (\nu z - 1 + i)^2 \\
&\qquad = B_0\left[(1-\nu x)(1+\nu y) - i\nu(x+y)(1-\nu(x-y))\right], \quad a = 1/\nu,\ h = 1/\nu;
\end{aligned}\right\} \tag{23.23}$$

$$\left.\begin{aligned}
&\gamma = \pi/6: \\
&f(z) = B_0\left[1 - (\sqrt{3}+i)\nu z + \tfrac{1}{2}(1+i\sqrt{3})\nu^2 z^2 - \tfrac{1}{6} i\nu^3 z^3\right] \\
&\qquad = -\tfrac{1}{6} i B_0\left[2 + i(\nu z - \sqrt{3} + i)^3\right] \\
&\qquad = -\tfrac{1}{6} B_0\left[2 + (\nu y + 1)\left[(\nu y+1)^2 - 3(\nu x - \sqrt{3})^2\right] + \right. \\
&\qquad\quad \left. + i(\nu x + \nu y\sqrt{3})(\nu x - \nu y\sqrt{3} - 2\sqrt{3})(\nu x - \sqrt{3})\right], \quad a = \sqrt{3}/\nu,\ h = 1/\nu.
\end{aligned}\right\} \tag{23.24}$$

Here h is the depth of fluid at the deepest point. The surface profile for $\gamma = 45°$ is a straight line, for $\gamma = 30°$ a parabola.

In order to find the higher modes of oscillation KIRCHHOFF returns to the form (17.33) for $f(z)$. It then follows as before that (17.34) must hold and that n must be even, say $2q$. Now, however, instead of taking $\lambda = 1$ it is left to be determined by (23.19). Substitution of (17.33) into (23.19) gives

$$A_{k+2} = A_k \exp\left[i\, 2\lambda\nu a \beta^{k+1}(1-\beta)\right], \quad k = 0, 1, \ldots, n-3. \tag{23.25}$$

Altogether there are then $n - 1 + n - 2 = 2n - 3$ independent equations to determine $A_1, \ldots, A_{n-1}$ and also λ and νa. Again the conditions can be satisfied for $\gamma = \pi/4$ and $\gamma = \pi/6$.

The solutions for $\gamma = \pi/4$ are as follows, where C is an arbitrary real constant:

$$\left.\begin{aligned}
f(z) &= C\left[\cos\lambda\nu(z - a(1-i)) + \cosh\nu\lambda(z - a(1-i))\right], \\
\lambda &= \coth\lambda\nu a = -\cot\lambda\nu a; \\
f(z) &= C\left[\cos\lambda\nu(z - a(1-i)) - \cosh\lambda\nu(z - a(1-i))\right], \\
\lambda &= \tanh\lambda\nu a = \tan\lambda\nu a.
\end{aligned}\right\} \tag{23.26}$$

The values of λ and ν can easily be determined graphically. For the first set of solutions the values of $\lambda\nu a$ will be slightly more than $3\pi/4, 7\pi/4, \ldots$, for the second set slightly less than $7\pi/4, 11\pi/4, \ldots$ These two sets of solutions correspond,

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respectively to (18.39) and (18.43) with $k=0$; the eigenvalues λ may be identified with m_i/v and n_i/v, respectively. KIRCHHOFF and HANSEMANN (1880) carried out an experimental investigation of the first three antisymmetric modes [Eq. (23.23) gives the first one]; they compare frequencies and positions of maxima and minima. The agreement seems satisfactory, although corrections for surface tension were necessary for the two higher modes.

The solution for $\gamma=30°$ is the following:

$$\left.\begin{aligned} f(z) = C\Big[&\frac{1}{\lambda+1}\,e^{i\lambda v[z-a]}+\frac{1}{\lambda-1}\,e^{-i\lambda v[z-a]}+\\ &+\frac{1}{\lambda+1}\,e^{i\beta^2\lambda v[z-a-\beta^2 a]}+\frac{1}{\lambda-1}\,e^{-i\beta^2\lambda v[z-a-\beta^2 a]}+\\ &+\frac{1}{\lambda+1}\,e^{-i\beta\lambda v[z-a-\beta a]}+\frac{1}{\lambda-1}\,e^{i\beta\lambda v[z-a-\beta a]}\Big], \end{aligned}\right\} \tag{23.27}$$

where C is an arbitrary real constant, $\beta=\frac{1}{2}(1-i\sqrt{3})$, $\beta^2=-\bar{\beta}=-\frac{1}{2}(1+i\sqrt{3})$, and the eigenvalues for λ and v are determined by the equations

$$\left.\begin{aligned} \frac{\lambda^2-1}{\lambda}=-\sqrt{3}\cot\lambda v a,\qquad \frac{\lambda^2+\beta}{\lambda}=-i(1+\beta)\cot\beta\lambda v a,\\ \frac{\lambda^2+\bar{\beta}}{\lambda}=+i(1+\bar{\beta})\cot\bar{\beta}\lambda v a. \end{aligned}\right\} \tag{23.28}$$

If λ is a solution of (23.28), then also $-\lambda$, $\bar{\lambda}$, $\beta\lambda$ and $\bar{\beta}\lambda$ are solutions. There exists a real solution which may be found from the equations

$$\cosh\sqrt{3}\,\lambda v a = 2\sec\lambda v a-\cos\lambda v a,\qquad \lambda=\frac{\sinh\sqrt{3}\,\lambda v a-\sqrt{3}\sin\lambda v a}{\cosh\sqrt{3}\,\lambda v a-\cos\lambda v a}. \tag{23.29}$$

The other solutions which may be generated from these do not lead to expressions different from (23.27). The first eigenvalue for $\lambda v a$ is a trifle to the right of $3\pi/2$. The form of the free surface corresponding to (23.17) is given by

$$\left.\begin{aligned} \eta(x,t)=\frac{\sigma}{g}\,C\Big\{&\frac{-2}{\lambda^2-1}\cos\lambda v(x-a)+\\ &+\Big[\frac{1}{\lambda+1}\,e^{\frac{1}{2}\sqrt{3}\lambda v x}+\frac{1}{\lambda-1}\,e^{-\frac{1}{2}\sqrt{3}\lambda v x}\Big]\cos\frac{1}{2}\lambda v(x-2a)+\\ &+\Big[\frac{1}{\lambda+1}\,e^{-\frac{1}{2}\sqrt{3}\lambda v(x-2a)}+\frac{1}{\lambda-1}\,e^{\frac{1}{2}\sqrt{3}\lambda v(x-2a)}\Big]\cos\frac{1}{2}\lambda v x\Big\}\sin(\sigma t+\tau). \end{aligned}\right\} \tag{23.30}$$

Note that (23.24) and (23.27) both give only symmetric modes. MACDONALD (1896) states that antisymmetric modes, if they exist, cannot be represented in the assumed forms (23.20) or (17.33).

VINT (1923) has succeeded in finding an infinite number of modes of motion in an inverted four-sided pyramid, each of whose sides makes a 45° angle with the horizontal. We refer to the original paper for the exact formulas.

Additional solutions have been obtained by inverse methods by SEN (1927) and by STORCHI (1949, 1952). STORCHI's result, although restricted to two-dimensional motion, is neat. Suppose that the form of the free surface is given as $\eta(x,t)=\eta(x)\sin(\sigma t+\tau)=F'(x)\sin(\sigma t+\tau)$, where $F(x)$ is analytic. Then, since $\eta(x)=\sigma g^{-1}\varphi(x,0)$ and $\varphi_y(x,0)=v\varphi(x,0)$,

$$f'(x-i0)=\varphi_x(x,0)-i\varphi_y(x,0)=\varphi_x(x,0)-iv\varphi(x,0)=\frac{g}{\sigma}\,[F''(x)-ivF'(x)]$$

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and

$$f(x - i0) = \frac{g}{\sigma}\,[F'(x) - i\nu F(x)] + \text{const}.$$

We may take the constant as zero and write

$$f(x + iy) = \frac{g}{\sigma}\,[F'(x+iy) - i\nu F(x+iy)], \tag{23.31}$$

where $F(z)$ is the analytic function determined by $F(x)$. From this we have

$$\left.\begin{aligned} \varphi(x,y) &= \frac{g}{2\sigma}\{F'(x+iy) + F'(x-iy) - i\nu[F(x+iy) + F(x-iy)]\},\\ \psi(x,y) &= \frac{-g}{2\sigma}\{i[F'(x+iy) + F'(x-iy)] - \nu[F(x+iy) + F(x-iy)]\}.\end{aligned}\right\} \tag{23.32}$$

Any streamline, defined by $\psi =$ real const, can now be taken as determining a possible basin shape corresponding to the assumed standing-wave profile. STORCHI applies the procedure to several special choices of F. An obvious disadvantage of this method, as well as of SEN'S, is that only one mode of motion is obtained for a resulting basin shape.

γ) Waves in movable basins. In several preceding sections, especially 19 and 22γ, we considered the wave motion occurring in the presence of an oscillating body when the fluid is exterior to the body. One may attempt analogous problems when the fluid is situated inside the body. Such problems occur in many situations of practical interest, for example, the sloshing of oil in a partly filled compartment of a tanker and the sloshing of fuel in an airplane or rocket. In each of these cases interest centers upon the dynamics of the whole system as well as upon the effect upon the walls of the container. A further interest in such problems arises from the interpretation of the experiments on standing waves, referred to earlier, carried out by HONDA and MATSUSHITA (1913), SASAKI (1914), and KIRCHHOFF and HANSEMANN (1880). The results were intended for comparison with theoretical prediction of standing waves in fixed basins. The waves were actually generated by oscillating the basin and finding the frequencies at which resonance appeared to occur.

We shall not consider the most general motions of the basin consistent with linearization of the free surface conditions, but shall limit ourselves here to a particular problem with small horizontal oscillations. In Sect. 26α small vertical oscillations of the basin will be considered. The general problem of motion of a body containing fluid with a free surface has been treated by MOISEEV (1953) and NARIMANOV (1956, 1957). However, both are primarily concerned with small oscillations. KREIN and MOISEEV (1957) have also considered certain mathematical aspects of this problem. OKHOTSIMSKII (1957) and RABINOVICH (1957) have considered the special case when the fluid is situated in a vertical, or almost vertical cylinder; NARIMANOV also gives special attention to this case. (Publication of the work of these three authors was apparently delayed; it is stated that, for the most part, it was carried out independently of and prior to MOISEEV'S papers.) A particular problem, the one discussed below, was treated by SRETENSKII (1951) and later by MOISEEV (1952a, b, 1953). Two later papers by MOISEEV (1954a, b) apply the theory to engineering problems, especially ships. Waves resulting from a special type of forced oscillation of a rectangular tank have been studied by BINNIE (1941) and TAMIYA (1958). A problem somewhat related to those of this section is the motion of a freely floating body in a fixed

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bounded basin (there is now no dissipation of energy as in the problem treated at the end of Sect. 22γ). This problem has been dealt with by PERZHNYANKO (1956) and MOISEEV (1958).

Waves in a basin with elastic restoring force. Consider the configuration shown in Fig. 30. The coordinate system OXY is fixed, the system $\bar{O}\bar{X}\bar{Y}$ moves with the carriage. Let $x_0(t) = O\bar{O}$, $u_0 = \dot{x}_0$. The bottom of the fluid is at $\bar{y} = -h$, the side walls at $\bar{x} = \pm a$. The motion will be taken as two-dimensional. Denote the mass of the carriage, per unit width, by m_c, that of the fluid by m_f and the total by $m = m_c + m_f$. Let the spring constant be mk^2. We suppose as usual that the motion may be described by a velocity potential $\Phi(x, y, t)$. Following the notation at the end of Sect. 2, let $\bar{\bar{\Phi}}(\bar{x}, \bar{y}, t)$ describe the motion relative to the basin, i.e.

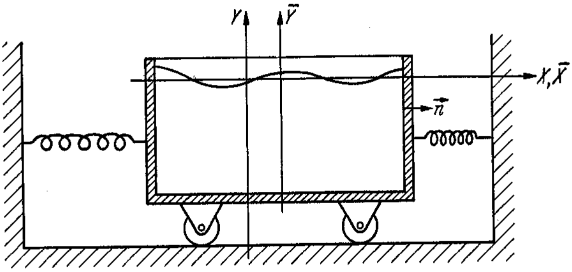

Fig. 30.

$$\left.\begin{aligned} &\Phi(x, y, t) \\ &= \bar{\bar{\Phi}}(\bar{x}, \bar{y}, t) + u_0 \bar{x}. \end{aligned}\right\} \quad (23.33)$$

We shall assume that x_0 and u_0 are both small, and of the same order as $\bar{\bar{\Phi}}$, i.e., in the notation of Sect. 10α, we assume expansions of the form

$$\left.\begin{aligned} x_0 &= \varepsilon x_0^{(1)}, \qquad u_0 = \varepsilon u_0^{(1)}, \\ \bar{\bar{\Phi}} &= \varepsilon \bar{\bar{\Phi}}^{(1)} + \varepsilon^2 \bar{\bar{\Phi}}^{(2)} + \cdots. \end{aligned}\right\} \quad (23.34)$$

We omit the formal details of substitution of the perturbation series in the exact boundary conditions. They lead to the following linearized boundary conditions for $\bar{\bar{\Phi}}$:

see errata

$$\left.\begin{aligned} &\bar{\bar{\Phi}}_{tt}(\bar{x}, 0, t) + g \bar{\bar{\Phi}}_{\bar{y}}(\bar{x}_y, 0, t) + \ddot{u}_0 \bar{x} = 0, \\ &\bar{\bar{\Phi}}_{\bar{x}}(\pm a, \bar{y}, t) = 0, \\ &\bar{\bar{\Phi}}_{\bar{y}}(\bar{x}, -h, t) = 0. \end{aligned}\right\} \quad (23.35)$$

The pressure, after discarding higher-order terms, is given by

$$p = -\varrho \Phi_t = -\varrho [\bar{\bar{\Phi}}_t + \dot{u}_0 \bar{x}]. \quad (23.36)$$

The motion of the carriage is determined by the equation

$$m_c \ddot{x}_0 = \int p \cos(n, \bar{x})\, ds - mk^2 x_0, \quad (23.37)$$

where the integral is taken over the wetted surface when the system is at rest. Substitution of (23.36) gives

$$\left.\begin{aligned} m \ddot{x}_0 &= -\varrho \int \bar{\bar{\Phi}}_t\, ds - mk^2 x_0 \\ &= -\varrho \int_{-h}^{0} \int_{-a}^{a} \bar{\bar{\Phi}}_{tx}\, dx\, dy - mk^2 x_0. \end{aligned}\right\} \quad (23.38)$$

[Eq. (23.38) is also a direct consequence of conservation of momentum.]

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The velocity potential $\bar{\bar{\Phi}}$ and the displacement x_0 must be determined together from Eqs. (23.35) and (23.38) and either initial conditions or the further assumption that the motion is harmonic in t.

As a preliminary we shall first suppose that the basin motion, i.e. x_0, is given, so that only (23.35) need be satisfied. One may try separation of variables and express $\bar{\bar{\Phi}}$ in the form

$$\bar{\bar{\Phi}} = \sum T_n(t)\, X_n(\bar{x})\, Y_n(\bar{y}). \tag{23.39}$$

LAPLACE's equation and the last two condition of (23.35) are satisfied by

$$\left.\begin{aligned} X_{2n} Y_{2n} &= \cos\frac{2n}{2a}\pi\bar{x}\cosh\frac{2n\pi}{2a}(\bar{y}+h),\\ X_{2n+1} Y_{2n+1} &= \sin\frac{2n+1}{2a}\pi\bar{x}\cosh\frac{2n+1}{2a}\pi(\bar{y}+h). \end{aligned}\right\} \tag{23.40}$$

In order to find the corresponding T_n, expand x in a Fourier series:

$$x = \sum_{n=0}^{\infty}(-1)^n\frac{8a}{(2n+1)^2\pi^2}\sin\frac{2n+1}{2a}\pi x \tag{23.41}$$

and substitute (23.39) and (23.41) in the first condition of (23.35):

$$\left.\begin{aligned} &\sum_{n=0}^{\infty}\left[\ddot{T}_{2n}\cosh\frac{2n}{2a}\pi h + T_{2n}\, g\,\frac{2n\pi}{2a}\sinh\frac{2n}{2a}\pi h\right]\cos\frac{2n}{2a}\pi x +\\ &\quad+\sum_{n=0}^{\infty}\left[\ddot{T}_{2n+1}\cosh\frac{2n+1}{2a}\pi h + T_{2n+1}\, g\,\frac{2n+1}{2a}\pi\sinh\frac{2n+1}{2a}\pi h +\right.\\ &\quad\left.+\dddot{x}_0(-1)^n\frac{2a}{(2n+1)^2\pi^2}\right]\sin\frac{2n+1}{2a}\pi x = 0. \end{aligned}\right\} \tag{23.42}$$

Let us set

$$\sigma_n^2 = g\pi\frac{n}{2a}\tanh\frac{n}{2a}\pi h,\quad b_{2n+1} = -(-1)^n\frac{2a}{(2n+1)^2\pi^2}\operatorname{sech}\frac{2n+1}{2a}\pi h. \tag{23.43}$$

Then Eq. (23.42) yields the infinite set of differential equations

$$\left.\begin{aligned} &\ddot{T}_{2n} + \sigma_{2n}^2 T_{2n} = 0,\\ &\ddot{T}_{2n+1} + \sigma_{2n+1}^2 T_{2n+1} = b_{2n+1}\dddot{x}_0. \end{aligned}\right\} \tag{23.44}$$

The solution of the first set, $T_{2n} = A_{2n}\cos(\sigma_{2n}t + \tau_{2n})$, is independent of the motion of the basin and yields the symmetric modes of oscillation in a fixed basin. The solution to the second set may also be found by elementary methods, but will not be given here. However, we note that, if x_0 is harmonic, it confirms that resonance occurs at the frequencies of the asymmetric modes for a fixed basin.

We now turn to the joint solution of (23.35) and (23.38). Substitute (23.39) into (23.38). Then, after evaluating the integral, one finds

$$m\ddot{x}_0 + \frac{4a\varrho}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sinh\frac{2n+1}{2a}\pi h\,\dot{T}_{2n+1} + m k^2 x_0 = 0. \tag{23.45}$$

The Eqs. (23.44) and (23.45) taken together may now be used to determine x_0 and the T_n. If we formulate an initial-value problem by requiring, say,

$$x_0(0) = c_0,\quad \dot{x}_0(0) = 0,\quad \bar{\bar{\Phi}}_{\bar{y}}(\bar{x},\bar{y},0) = 0,\quad \bar{\bar{\Phi}}_t(\bar{x},\bar{y},0) = 0, \tag{23.46}$$

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then the T_{2n} are all zero and the T_{2n+1} and x_0 must be determined from the differential equations. As usual, one looks for a solution in the form

$$x_0 = c\,\mathrm{e}^{-i\omega t}, \qquad T_{2n+1} = d_{2n+1}\,\mathrm{e}^{-i\omega t}, \tag{23.47}$$

where both c and d_{2n+1} may, of course, be complex. Substitution in (23.44) and (23.45), followed by elimination of d_{2n+1}, yields the following equation for determining ω:

$$\omega^2 - k^2 = \frac{32 a^2 \varrho}{\pi^3 m}\,\omega^4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \tanh\frac{2n+1}{2a}\pi h \frac{1}{\omega^2 - \sigma_{2n+1}^2}. \tag{23.48}$$

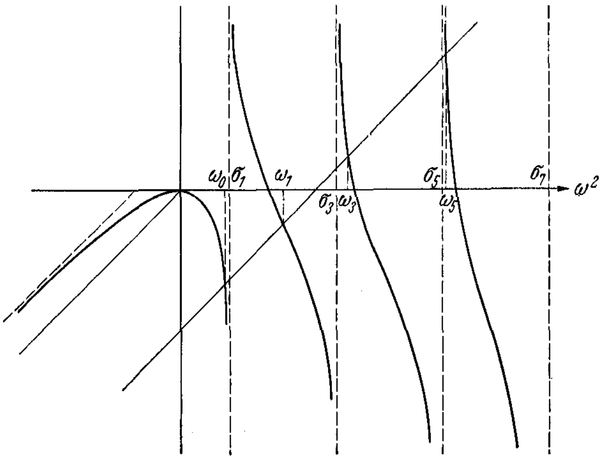

Fig. 31.

One may find the solutions graphically by plotting each side of the equation as function of ω^2. Fig. 31 gives a qualitative idea of the distribution of solutions $\omega_0, \omega_1, \ldots$. As $n \to \infty$, $\omega_{2n+1}^2 - \sigma_{2n+1}^2 \to 0$; this fact, which can be proved analytically and which seems clear from Fig. 31, would not have been so evident if we had divided (23.48) by ω^4 before plotting. A point of importance is that there is

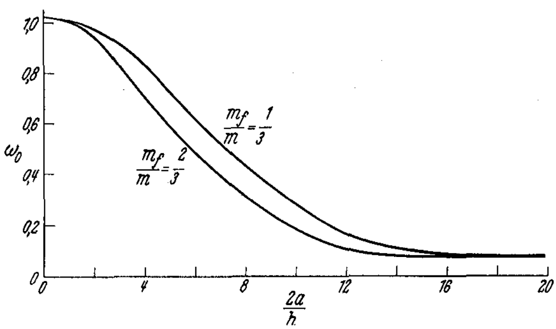

Fig. 32.

no intersection for $\omega^2 < 0$; as a result the motion is stable. This may be proved as follows. Since $x^{-1}\tanh x \leq 1$, the right hand side of (23.48), for $\omega^2 < 0$, is greater than or equal to

$$\frac{32 a^2 \varrho}{\pi^3 m}\,\omega^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \frac{2n+1}{2a}\pi h = \frac{2 a h \varrho}{m}\,\omega^2 = \frac{m_f}{m}\,\omega^2 > \omega^2. \tag{23.49}$$

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Hence the line $\omega^2 - k^2$ lies below the left-hand branch of the curve for $k^2 \geqq 0$. The eigenvalues ω_i depend upon the parameters k^2, $2a/h$ and $2\varrho a h/m = m_f/m$. Fig. 32 from MOISEEV (1953) shows the dependence of the fundamental mode ω_0 upon $2a/h$ for two values of m_f/m and $k^2 = 1$.

The general solution for x_0 and T_{2n+1} is

$$x_0(t) = \operatorname{Re} \sum_{s=0}^{\infty} c_s \, e^{-i\omega_s t}, \qquad T_{2n+1}(t) = \operatorname{Re} \sum_{s=0}^{\infty} d_{2n+1,s} \, e^{-i\omega_s t}. \tag{23.50}$$

The solution of the initial-value problem formulated in (23.46) will not be completed. It involves solution of infinite sets of linear equations. Approximate solutions can be obtained by considering only a finite number of equations and variables.

The general theory of stability of such systems is discussed in MOISEEV's 1953 paper. In an earlier papers (1952b) he studies the special case of a basin containing fluid and serving as the bob of a pendulum. If the suspension is by a parallelogram linkage, so that the container moves parallel to itself, the motion is always stable; if the suspension is by a rod rigidly attached to the container, the motion may be, under certain circumstances, unstable.

The last cited paper by MOISEEV describes briefly the results of an experiment with a pendulum; the measured and computed fundamental frequencies for the two systems of suspension agreed with 0.1%.

see errata

24. Gravity waves in the presence of surface tension. Apparently the first one to investigate the theory of waves in a fluid acted upon by both gravity and surface tension was KELVIN (1871a, b). However, many of the essential features had been discovered earlier through observation by RUSSELL (1844) and others; references may be found in KELVIN's papers. A good account of the classical researches of KELVIN and RAYLEIGH may be found in LAMB (1932, § 265 to 272). Also, Chap. XX of RAYLEIGH's *Theory of Sound* (Cambridge 1929; Dover, N. Y., 1945) contains an exposition of many of his own fundamental researches on surface-tension phenomena, including waves.

The chief mathematical complication added by the action of surface tension is a somewhat more elaborate dynamical boundary condition at an interface or free surface. The difference of primary physical interest lies in the existence of a minimum wave velocity and of two wave lengths with the same velocity. Many of the special problems considered in preceding sections can also be solved when surface tension is acting. However, there has been little motivation for carrying through such analyses for wave motion associated with solid boundaries, since it has been recognized that the additional forces would be small. A further difficulty also appears when the solid boundary pierces the surface, for an additional boundary condition is required at the intersection. As a result, most of the investigations have dealt with waves analogous to those considered in Sects. 14α, β and δ, 15, and 22β. In fact, the complex velocity potential for the Cauchy-Poisson initial-value problem including the effect of surface tension, has already been given in Eqs. (22.45) and (22.46). A topic of particular geophysical interest, the stability of an interface, will be dealt with in Sect. 26. Waves on the surface of a viscous fluid, including surface tension, are considered in Sect. 25.

B o u n d a r y c o n d i t i o n s. The linearized conditions to be satisfied at the interface of two fluids have already been given in Eqs. (10.7) and (10.8) (we recall that subscript 1 refers to the lower fluid). If one eliminates η from these two

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equations and makes use of the fact that LAPLACE's equation is satisfied on either side of the boundary, one has the following condition:

$$\Delta \Phi_1 = 0 \quad \text{for} \quad y < 0, \qquad \Delta \Phi_2 = 0 \quad \text{for} \quad y > 0. \tag{24.1}$$

$$\eta_t(x, z) = \Phi_{1y}(x, 0, z, t) = \Phi_{2y}(x, 0, z, t), \tag{24.2}$$

$$\varrho_1 \left[\Phi_{1tt}(x, 0, z, t) + g\,\Phi_{1y} + \frac{T}{\varrho_1}\Phi_{1yyy}\right] = \varrho_2 \left[\Phi_{2tt} + g\,\Phi_{2y}\right]. \tag{24.3}$$

If the upper fluid is absent, one sets ϱ_2 and Φ_2 equal to zero and may, of course, drop the subscript.

If the motion is two-dimensional one may introduce a stream function Ψ and a complex potential $F(z, t) = \Phi + i\Psi$ and express (24.2) and (24.3) as follows:

$$\eta_t(x) = \operatorname{Im} F_1'(x - i\,0) = \operatorname{Im} F_2'(x + i\,0), \tag{24.4}$$

$$\operatorname{Re} \varrho_1 \left\{F_{1tt}(x - i\,0) + i\,g\,F_1' - i\frac{T}{\varrho_1}F_1'''\right\} = \operatorname{Re} \varrho_2 \{F_{2tt}(x + i\,0) + i\,g\,F_2'\}. \tag{24.5}$$

If the upper fluid is absent and if the lower fluid is infinitely deep, one may extend the reasoning which led up to LEVI-CIVITA's differential equation (22.25) to derive the following one which must be satisfied for all z:

$$F_{tt}(z, t) + i\,g\,F' - i\frac{T}{\varrho}F''' = 0. \tag{24.6}$$

Furthermore, if the fluid is of constant depth h, CISOTTI's equation (22.30) may also be extended to include the effect of surface tension:

$$\left.\begin{aligned} &F_{tt}(z + i\,h, t) + F_{tt}(z - i\,h) + i\,g\,[F'(z + i\,h) - F'(z - i\,h)] - \\ &\quad - i\frac{T}{\varrho}[F'''(z + i\,h) - F'''(z - i\,h)] = 0 \quad \text{for} \quad -2h < y < 0. \end{aligned}\right\} \tag{24.7}$$

Elementary solutions. Let us suppose first that only one fluid is present, and in addition that

$$\Phi(x, y, z, t) = \varphi(x, y, z)\cos(\sigma t + \tau).$$

Then φ must be a potential function satisfying

$$-\sigma^2\varphi(x, 0, z) + g\,\varphi_y + \frac{T}{\varrho}\varphi_{yyy} = 0. \tag{24.8}$$

Just as in Sect. 13α, we may separate out the y-variable and obtain the following expressions:

infinite depth:

$$\varphi(x, y, z) = A\,e^{my}\varphi(x, y) \tag{24.9}$$

where

$$\Delta_2\varphi + m^2\varphi = 0$$

and

$$\sigma^2 = g\,m + \frac{T}{\varrho}m^3;$$

depth h:

$$\varphi(x, y, z) = A\cosh m_0(y + h)\,\varphi(x, z), \tag{24.10}$$

where

$$\Delta_2\varphi + m_0^2\varphi = 0$$

and

$$\sigma^2 = \left(g\,m_0 + \frac{T}{\varrho}m_0^3\right)\tanh m_0 h.$$

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One may also with no difficulty construct solutions analogous to (13.3) and (13.4), namely

$$\varphi(x, y, z) = A\left[m\left(1 - \frac{T}{\varrho g} m^2\right)\cos m y + \frac{\sigma^2}{g}\sin m y\right]\varphi(x, z) \tag{24.11}$$

and

$$\varphi(x, y, z) = A \cos m_i (y + h)\, \varphi(x, z) \tag{24.12}$$

for infinite and finite depth, respectively, where m_i in (24.12) must satisfy

$$\sigma^2 = \left(-g m_i + \frac{T}{\varrho} m_i^3\right)\tan m_i h$$

and $\varphi(x, z)$ must be a solution of

$$\Delta_2 \varphi - m^2 \varphi = 0.$$

Unfortunately, the set of function

$$\{\cosh m_0 (y + h),\ \cos m_i (y + h)\}$$

is no longer orthogonal in general, so that the convenience of general solutions like (16.3) is lost.

It is not necessary to repeat the computations of Sect. 13 since they remain unaltered. Essentially the only change is in the relation between the frequency σ and the wave number m. Here the fact of predominant physical interest is that for small values of m the relation is controlled chiefly by the gravitational constant g and for large values of m by T/ϱ.

If one forms two-dimensional progressive waves by superposing the standing-wave solutions obtained from (24.9) and (24.10), a further significant physical fact appears: the wave velocity now has a minimum for some value of $m > 0$, except for very shallow depth. These facts are displayed graphically in Fig. 11 and further information is given in the associated discussion (the curves were computed for water at 20° C and $h = \infty$ or 1 cm). Formulas for the position of the minimum and various associated values are given for infinite depth in the following table; the numerical values are for water at 20° C ($T = 72.8$ dynes/cm, $\varrho = 0.998$ gm/cm³):

$$\left.\begin{aligned} m_m &= \sqrt{\varrho g/T} = 3.66\ \text{cm}^{-1}, \\ \lambda_m &= 2\pi\sqrt{T/\varrho g} = 1.71\ \text{cm}, \\ c_m &= \sqrt[4]{4g\,T/\varrho} = 23.1\ \text{cm/sec}, \\ \sigma_m &= \sqrt[4]{4\varrho g^3/T} = 84.8\ \text{radians/sec} = 13.5\ \text{cycles/sec}. \end{aligned}\right\} \tag{24.13}$$

see errata

When $h \leqq \sqrt{3\,T/2\varrho g}$ there is no longer a minimum value of c for $m > 0$; in this case c increases monotonically with m. The critical depth for water is about 0.33 cm. Except in this latter case every value of c has associated with it waves of two different lengths, each of which travels with velocity c. KELVIN suggested that the shorter waves, whose behavior is controlled chiefly by surface tension be called "ripples". The suggestion has been followed for the most part (French: "rideaux"; German: "Rippeln" or "Kräuselwellen"; Russian: "ryabi"), but they are frequently also called "capillary waves" in contrast with "gravity waves".

The relation between σ and m was subjected to a rather thorough experimental investigation by MATTHIESSEN (1889). He made measurements with water, mercury, alcohol, ethyl ether and carbon disulfide with frequencies ranging from 8.4 to roughly 2000 cycles per second. Agreement between theory and measurement is generally within 5% with the greatest discrepancies occurring near the minima. RAYLEIGH (1890) and MICHIE SMITH (1890) were apparently the first to use the theoretical relation as a means of experimental determination of T, and it has become one of the standard experimental procedures. For more recent developments and further references see BROWN (1936) and TYLER (1941).

Solutions for standing or progressive interfacial waves, analogous to those considered in Sect. 14δ, can be found by application of the same methods. Since the analysis is similar we give only the relation between σ and m. If the two fluids fill the whole space, with their interface at $y=0$, then

$$\sigma^2 = \frac{\varrho_1 - \varrho_2}{\varrho_1 + \varrho_2} g\, m + \frac{T}{\varrho_1 + \varrho_2} m^3. \tag{24.14}$$

If the lower fluid is of depth d_1, the upper of depth d_2 and the interface at $y=0$, then

$$\sigma^2 = \frac{(\varrho_1 - \varrho_2)\, m\, g + T\, m^3}{\varrho_1 \coth m\, d_1 + \varrho_2 \coth m\, d_2}. \tag{24.15}$$

In both (24.14) and (24.15) $\sigma^2 > 0$ if

$$\varrho_2 < \varrho_1 + \frac{T\, m^2}{g}; \tag{24.16}$$

thus the motion may be stable even when the lower fluid is less dense than the upper one. This is not true in the absence of surface tension, as inspection of (14.29) and (14.30) shows.

The analogue of the next example of Sect. 14δ is somewhat more complex, for two surface tensions are necessary. Let T be the surface tension at the free surface $y=0$, and T_{12} that at the interface $y=-d_1$; let the rigid bottom be at $y=-h=-d_1-d_2$. Then the relation analogous to (14.31) is

$$\left.\begin{aligned}&\left(\frac{\sigma^2}{g\, m}\right)^2 [\varrho_1 \coth m\, d_1 \coth m\, d_2 + \varrho_2] - \\ &\quad - \frac{\sigma^2}{g\, m} \varrho_1 \left[\left(1 + \frac{T+T_{12}}{\varrho_1 g} m^2\right) \coth m\, d_2 + \varrho_1 \left(1 + \frac{T}{\varrho_2 g} m^2\right) \coth m\, d_1\right] + \\ &\quad + \left(1 + \frac{T}{\varrho_2 g} m^2\right) \left[\varrho_1 \left(1 + \frac{T+T_{12}}{\varrho_1 g} m^2\right) - \varrho_2 \left(1 + \frac{T}{\varrho_2 g} m^2\right)\right] = 0.\end{aligned}\right\} \tag{24.17}$$

The assumption of $d_1 = \infty$ no longer results in any notable simplification of the equation. However, one may show that the solutions $\sigma^2/g m$ are always real, and that they are positive if

$$\varrho_2 < \varrho_1 + \frac{T_{12}}{g} m^2. \tag{24.18}$$

This is the same condition for stability as was found in (24.16) (and is still necessary as well as sufficient). Much of the rest of the pure gravity-wave analysis of Sect. 14δ may be carried through. Thus, if σ_1 is the larger and σ_2 the smaller root of (24.17) for a given m, then one may establish the inequality

$$\left.\begin{aligned}&0 < \frac{\sigma_2^2}{g\, m} < \left(1 + \frac{T}{\varrho_2 g} m^2\right) \tanh m\, d_2 < \frac{\sigma_1^2}{g\, m} < \\ &\quad < \left(1 + \frac{T}{\varrho_2 g} m^2\right) \left(1 + \frac{T+T_{12}}{\varrho_1 g} m^2\right) \min\left\{1, \frac{\varrho_1}{\varrho_2} \tanh m\, h\right\}.\end{aligned}\right\} \tag{24.19}$$

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If η and η_{12} are the profiles of the free surface and interface, respectively, then one finds, analogously to (14.34),

$$\frac{\eta_{12}}{\eta} = \cosh m d_2 - \frac{g m}{\sigma_2}\left(1 + \frac{T}{\varrho_2 g} m^2\right) \sinh m d_2; \tag{24.20}$$

again, it follows from (24.18) that this ratio is positive for the larger and negative for the smaller of the two roots of (24.17). The discussion of the nature of the motion associated with the root σ_2 may be taken directly from Sect. 14δ; however, the upper bound for the velocity c_2 of a progressive wave of wave number m is now given by

$$c_2^2 = \frac{\sigma_2^2}{m^2} = \frac{\sigma_2^2}{g m} \cdot \frac{g}{m} < \frac{g}{m}\left(1 + \frac{T}{\varrho_2 g} m^2\right) \tanh m d_2 < g d_2 \left(1 + \frac{T}{\varrho_2 g} m^2\right). \tag{24.21}$$

Let us turn next to the situation in which the two fluids are moving and look for possible steady motions. Assume each fluid to move to the left with mean velocity c_i and let $F_i(z)$, $i = 1, 2$, be the complex velocity potentials. We again look for solutions in the form [cf. Eq. (14.36)]

$$F_i(z) = - c_i z + f_i(z), \quad i = 1, 2, \tag{24.22}$$

where f_i is assumed small with respect to $c_i z$. Then the linearized boundary conditions corresponding to (14.37) are

$$\left.\begin{aligned} \eta(x) = \frac{1}{c_1} \operatorname{Im} f_1(x - i0) &= \frac{1}{c_2} \operatorname{Im} f_2(x + i0), \\ \frac{\varrho_2}{c_2} \operatorname{Re}\{i g f_2(x + i0) + c_2^2 f_1'(x + i0)\} &= \frac{\varrho_1}{c_1} \operatorname{Re}\Big\{i g f_1(x - i0) + c_1^2 f_1'(x - i0) - \\ &\qquad - i \frac{T}{\varrho_1} f_1''(x - i0)\Big\}. \end{aligned}\right\} \tag{24.23}$$

If we look for a steady motion of the form

$$f_1 = a_1 e^{-imz}, \quad f_2 = a_2 e^{imz}, \tag{24.24}$$

then substitution in (24.23) yields

$$\frac{a_1}{c_1} = - \frac{\bar a_2}{c_2}$$

and

$$m(\varrho_1 c_1^2 + \varrho_2 c_2^2) = (\varrho_1 - \varrho_2) g + T m^2. \tag{24.25}$$

The last equation will not have a real solution for m, assuming $\varrho_1 > \varrho_2$, unless

$$4 g (\varrho_1 - \varrho_2) T \leqq (\varrho_1 c_1^2 + \varrho_2 c_2^2)^2. \tag{24.26}$$

There are then two solutions of the form (24.24). The effect of surface tension may be seen more clearly if one graphs each side of (24.25) and finds the intersections, if any. It will be shown in Sect. 26 that this type of motion is unstable if $|c_1 - c_2|$ becomes too large.

Singular solutions. The methods used in Sect. 13γ for finding source-type solutions can generally be extended to take account of surface tension. Aside from the algebraic complications the chief difficulties are associated with selecting the proper boundary conditions at infinity. For a stationary source of pulsating strength one may still impose a radiation condition as in (13.9) and obtain the correct solution. However, for the steadily moving source of constant strength the proper choice is no longer clear. Although it is possible to fall back upon arguments based upon considerations of group velocity, they are not

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completely convincing, so that it seems safer to formulate first an initial-value problem which can yield either of the two cases mentioned above as a limit when $t\to\infty$. First we give the velocity potential for a source of variable strength $m(t)$, $t\geqq 0$, moving on an arbitrary path $(a(t), b(t), c(t))$. The potential function Φ must satisfy the same conditions given on p. 491 except that 2 is now replaced by

$$\Phi_{tt}(x,0,z,t)+g\,\Phi_y(x,0,z,t)+\frac{T}{\varrho}\,\Phi_{yyy}(x,0,z,t)=0. \qquad (24.27)$$

There is no special difficulty involved in finding Φ. For infinite depth, it is as follows:

$$\left.\begin{aligned}\Phi(x,y,z,t)&=\frac{m(t)}{r(t)}-\frac{m(t)}{r_1(t)}+\\&+2\int_0^\infty dk\sqrt{gk+T'k^3}\int_0^t d\tau\, m(\tau)\sin\left[(t-\tau)\sqrt{gk+T'k^3}\right]e^{k[y+b(\tau)]}J_0(kR(\tau)),\end{aligned}\right\} \qquad (24.28)$$

where we have written T' for T/ϱ. One may similarly find the function analogous to (13.53) by replacing gk by $gk+T'k^3$. Knowledge of these functions allows one now to repeat, at least in part, the considerations of Sects. 22α and 22β.

For a stationary source at (a, b, c) with strength $m\cos\sigma t$, the velocity potential may be easily derived from (24.28). It is as follows:

$$\left.\begin{aligned}\Phi(x,y,z,t)=&\left[\frac{1}{r}+\frac{1}{r_1}+2\sigma^2\int_0^\infty\frac{1}{T'k^3+gk-\sigma^2}\,e^{k(y+b)}J_0(kR)\,dk\right]m\cos\sigma t+\\&+2\pi m\frac{\sigma^2}{g+3T'k_0^2}\,e^{k_0(y+b)}J_0(k_0R)\sin\sigma t,\end{aligned}\right\} \qquad (24.29)$$

where k_0 is the real solution of $\sigma^2=gk+T'k^3$. If the fluid is of depth h, then

$$\left.\begin{aligned}&\Phi(x,y,z,t)\\&=\left[\frac{1}{r}+\frac{1}{r_2}+2\int_0^\infty\frac{T'k^3+gk+\sigma^2}{T'k^3+gk-\sigma^2\coth kh}\cdot\frac{e^{kh}\cosh k(b+h)\cosh k(y+h)}{\sinh kh}J_0(kR)\,dk\right]\times\\&\times m\cos\sigma t+2\pi m\frac{T'k_0^3+gk_0+\sigma^2}{\sigma^2h+(3T'k_0^2+g)\sinh^2k_0h}\times\\&\times e^{k_0h}\sinh k_0h\cosh k_0(b+h)\cosh k_0(y+h)\cdot J_0(k_0R)\sin\sigma t,\end{aligned}\right\} \qquad (24.30)$$

where k_0 is the real root of

$$T'k^3+gk-\sigma^2\coth k\cdot h=0.$$

The velocity potential for a source moving in the direction Ox with constant velocity u_0 may also be obtained from (24.38) by a suitable limiting procedure, although the computation is somewhat more tedious. In a coordinate system moving with the source it is as follows for $h=\infty$:

see errata

$$\left.\begin{aligned}\varphi(x,y,z)=&\frac{m}{r}-\frac{m}{r_1}+\frac{4m}{\pi}\int_0^{\frac{1}{2}\pi}d\vartheta\,\mathrm{PV}\int_0^\infty dk\,\frac{g+T'k^2}{g+T'k^2-k u_0^2\cos^2\vartheta}\times\\&\times e^{k(y+b)}\cos[k(x-a)\cos\vartheta]\cos[k(z-c)\sin\vartheta]+\\&+4m\sum_{i=1,2}(-1)^{i-1}\int_0^{\vartheta_0}d\vartheta\,k_i(\vartheta)\,\frac{T'k_i^2+g}{T'k_i^2-g}\times\\&\times e^{k_i(y+b)}\sin[k_i(x-a)\cos\vartheta]\cos[k_i(z-c)\sin\vartheta],\end{aligned}\right\} \qquad (24.31)$$

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