

respectively. Define

$$r(\bar{x}) = \frac{1}{2}[u(\bar{x}) + b(\bar{x})], \quad s(\bar{x}) = \frac{1}{2}[u(\bar{x}) - b(\bar{x})], \quad (20.30)$$

so that now the top and bottom are given by

$$\bar{y} = -h + r(\bar{x}) + s(\bar{x}) \quad \text{and} \quad \bar{y} = -h + r(\bar{x}) - s(\bar{x}), \quad -a \leq \bar{x} \leq a, \quad (20.31)$$

respectively. The class of profiles in a form analogous to (20.23) is now given by

$$\bar{y} = -h + \varepsilon[r^{(1)}(\bar{x}) \pm s^{(1)}(\bar{x})], \quad -a \leq \bar{x} \leq a. \quad (20.32)$$

It is clear that as $\varepsilon \rightarrow 0$, the profiles approach the line segment $\bar{y} = -h, 0 \leq \bar{x} \leq a$, so that the perturbation procedure is allowable. The analysis leads to the linearized boundary condition

$$\varphi_y(x, -h \pm 0, t) = -c(t) r'(x - \int^t c(\tau) d\tau) \mp c(t) s'(x - \int^t c(\tau) d\tau), \quad \left. \begin{aligned} & -a \leq x - \int^t c d\tau \leq a. \end{aligned} \right\} (20.33)$$

The *slender-body approximation* is also consistent with the linearized free-surface condition. Let the body be defined by

$$(\bar{y} + h)^2 + \bar{z}^2 - r^2(\bar{x}) = 0, \quad |\bar{x}| < a, \quad h > 0, \quad (20.34)$$

in a coordinate system fixed in the body. If one considers the class of bodies defined by $\varepsilon r^{(1)}(\bar{x})$, then the appropriate condition to be satisfied by $\Phi^{(1)}$ is

$$\lim_{\varepsilon \rightarrow 0} \Phi_r^{(1)}(x - \int^t c(\tau) d\tau, -h + \varepsilon r^{(1)} \cos \vartheta, \varepsilon r^{(1)} \sin \vartheta) r^{(1)}(x - \int^t c d\tau) = -c r^{(1)} r^{(1)'}. \quad (20.35)$$

We note that the same problem may be approached by two linearized theories. For example, in approximating the flow about a hydrofoil, one may either consider it as a relatively deeply submerged body and satisfy the exact conditions on the surface, or consider it as a thin hydrofoil and use the conditions (20.33). Each method will have its own domain of excellence, but it is not proper in the present context to say that the thin-hydrofoil approximation is less exact than the other one, even though this is true in an unbounded fluid.

The *H-functions*. KOCHIN'S *H-function*, introduced in Sect. 19 α , may also be used effectively for the flows considered in the present section. The definition for three dimensions is identical in appearance with (19.14). For two dimensions (19.24) is replaced by

$$H(k) = \int_{C_1} e^{-ik\zeta} f'(\zeta) d\zeta. \quad (20.36)$$

However, the formulas for the force on the body are somewhat different. For three dimensions they are

$$\left. \begin{aligned} X &= -\frac{v^2 \rho}{2\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} |H(v \sec^2 \vartheta, \vartheta)|^2 \sec^3 \vartheta d\vartheta, \\ Y &= \rho g V - \frac{\rho}{8\pi^2} \int_0^\infty \int_{-\pi}^\pi |H(k, \vartheta)|^2 k d\vartheta dk + \\ &\quad + \frac{v^2 \rho}{4\pi^2} \int_{-\pi}^\pi \text{PV} \int_{-\infty}^1 \left| H\left(\frac{v(1-\lambda)}{\cos^2 \vartheta}, \vartheta\right) \right|^2 \frac{1-\lambda}{\lambda} \sec^4 \vartheta d\lambda d\vartheta, \\ Z &= -\frac{v^2 \rho}{2\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} |H(v \sec^2 \vartheta, \vartheta)|^2 \sec^4 \vartheta \sin \vartheta d\vartheta, \end{aligned} \right\} (20.37)$$

where V is the displaced volume of fluid and $v = g/c^2$. The two-dimensional formulas are

$$\left. \begin{aligned} X &= -\nu \varrho |H(v)|^2, \\ Y &= \varrho g A + \varrho c \Gamma - \frac{\varrho}{2\pi} \int_0^\infty |H(k)|^2 dk + \frac{\varrho v}{\pi} \text{PV} \int_{-\infty}^1 |H(v - kv)|^2 \frac{dk}{k}, \\ M &= -g \varrho A x_c - \varrho c \text{Re} \{i H'(0)\} - \varrho \text{Re} \left\{ \frac{1}{2\pi i} \int_0^\infty H'(k) \overline{H(k)} dk + \right. \\ &\quad \left. + \nu H'(v) \overline{H(v)} + \frac{\nu i}{\pi} \text{PV} \int_{-\infty}^1 H'(v - kv) \overline{H(v - kv)} \frac{dk}{k} \right\}, \end{aligned} \right\} \quad (20.38)$$

where A is the area of the profile, (x_c, y_c) are the coordinates of its centroid, and Γ is the circulation. The remarks made in Sect. 19 α concerning the use of the H -function apply also here.

Submerged circular cylinder. The appropriate linearization for the circular cylinder is the one associated with deep submergence. Hence, one must try to satisfy the exact boundary condition on the cylinder.

This problem has been treated by LAMB (1913; see also 1932, § 247), HAVELOCK (1927, 1929a, 1936), SRETENSKII (1938), who considers also finite depth, KOCHIN (1937) and HASKIND (1945 a), who applies KOCHIN's methods for finite depth. COOMBS (1950) considers the flow about a pair of submerged cylinders, and, as a preliminary, also about a single cylinder; numerical computations are carried though for two cases, one with the centers on a horizontal line and one with them on a vertical line. COOMB'S method has wider applicability than just to circular cylinders. In all the cited papers, with the exception of HAVELOCK's and COOMBS', the problem is solved by placing at the center of the circle a dipole modified to satisfy the free-surface condition, i.e., (13.45) with $\alpha = 0$ and $M = 2\pi c a^2$, where a is the radius and c the velocity [the c of (13.45) is taken as $-ih$, h the depth of the center]. This provides, of course, only an approximate solution, for in the presence of a free surface a dipole in a stream no longer generates a circle, as is testified to by the fact that the contour actually generated is subject to a moment. HAVELOCK (1927, 1929a) gave second approximations for drag and lift and later (1936b) a complete solution.

The problem may be treated by a combination of MILNE-THOMSON'S Circle Theorem (1956, p. 151) and a formula of KOCHIN'S. The former states that if $f(z)$ is the complex potential for a flow with its singularities all at a distance greater than a from the origin and with no solid boundaries, then

$$f(z) + \bar{f}\left(\frac{a^2}{z}\right) \quad (20.39)$$

is the complex potential for a flow with the same singularities but now with a circle of radius a and center at the origin situated in the fluid.

KOCHIN (1937) has proved that if $f(z)$ is a single-valued complex potential for a bounded contour C under a free surface, then

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{z - \zeta} d\zeta + \frac{1}{2\pi i} \int_{C_1} \overline{f(\zeta)} \left[\frac{1}{z - \bar{\zeta}} - 2i\nu e^{-i\nu z} \int_{\infty}^z \frac{e^{i\nu u}}{u - \bar{\zeta}} du \right] d\bar{\zeta}, \quad (20.40)$$

where C_1 is any contour in the lower half-plane containing C . The formula and its proof are almost identical with that given in (17.15). The first integral in

(20.40) represents a function regular everywhere outside C , the second integral a function regular everywhere in the lower half-plane. If one starts with a function $f(z)$ whose only singularities are contained inside C , then the operation

$$f + \mathfrak{R}\{f\}, \tag{20.41}$$

where

$$\left. \begin{aligned} \mathfrak{R}\{f\} &= \frac{1}{2\pi i} \int_{C_1} \overline{f(\bar{\zeta})} \left[\frac{1}{z - \bar{\zeta}} - 2i\nu e^{-i\nu z} \int_{\infty}^z \frac{e^{i\nu u}}{u - \bar{\zeta}} du \right] d\bar{\zeta}, \\ &= \frac{1}{2\pi i} \int_{C_1} \overline{f(z)} \left[\frac{1}{z - \bar{\zeta}} - 2\pi\nu e^{-i\nu(z-\bar{\zeta})} + 2i\nu \text{PV} \int_0^{\infty} \frac{e^{-iu(z-\bar{\zeta})}}{u - \nu} du \right] d\bar{\zeta} \end{aligned} \right\} \tag{20.42}$$

yields a complex potential function satisfying the free-surface condition and having the same singularities as f in the lower half-plane.

On the other hand, if one starts with a complex potential $f(z)$ whose only singularities are in the upper half-plane, then

$$f + \mathfrak{M}\{f\} \tag{20.43}$$

where

$$\mathfrak{M}\{f\} = \bar{f} \left(\frac{a^2}{z + ih} + ih \right) + ich, \quad a < h; \tag{20.44}$$

will be a complex potential for a flow about a circle of radius a and center $-ih$ and with the same singularities as f in the upper half-plane, the singularities of $\mathfrak{M}\{f\}$ being all inside the circle.

We start with the flow $f_0(z) = -cz$ representing a uniform flow from the right; the free-surface condition is satisfied for $y=0$ in a trivial manner. Now form the sequence

$$f_0, f_1 = \mathfrak{M}\{f_0\}, f_2 = \mathfrak{R}\{f_1\}, \dots, f_{2n+1} = \mathfrak{M}\{f_{2n}\}, f_{2n+2} = \mathfrak{R}\{f_{2n+1}\}, \dots \tag{20.45}$$

Then $f_0 + f_1, f_2 + f_3, f_4 + f_5, \dots$ each represent flows satisfying the boundary condition on the circle; hence, also their sum if the series converges. On the other hand, $f_0, f_1 + f_2, f_3 + f_4, \dots$ each represent flows satisfying the free surface condition, and, hence, also the sum if it exists.

Let us now consider the two operators \mathfrak{M} and \mathfrak{R} . \mathfrak{M} is always being applied to a function regular and bounded in the lower half-plane. Since $a^2(z + ih)^{-1} + ih$ maps the exterior of the circle on to the the interior, the maximum of $|\mathfrak{M}\{f\}|$ for z in the lower half-plane does not exceed that for $|f|$ within or on the circle. We write this as

$$|\mathfrak{M}\{f\}| \leq \|f\| \equiv \max_C |f|. \tag{20.46}$$

In particular,

$$\|f_{2n+1}\| \leq \|f_{2n}\|. \tag{20.47}$$

The operator \mathfrak{R} is always applied here to functions regular everywhere outside and on the boundary of the circle. Hence C_1 may be contracted to C and one can establish the following estimate for z in the lower half-plane:

$$\left. \begin{aligned} |\mathfrak{R}\{f\}| &\leq a \max_C |f| \left[\frac{1}{|z - \bar{\zeta}|} + 2\pi\nu e^{\nu(y+\eta)} + 2\nu e^{\nu(y+\eta)} |\text{Ei}(\nu|y + \eta)| \right] \\ &\leq a \left[\frac{1}{h-a} + 2\pi\nu e^{-\nu(h-a)} + 2\nu e^{-\nu(h-a)} \text{Ei}(\nu(h-a)) \right] \max_C |f| \\ &\leq K \|f\|, \end{aligned} \right\} \tag{20.48}$$

where in the second inequality h must be large enough that $\nu(h-a) > 0.4$. For fixed values of νa one may select h/a large enough so that K is as small as one wishes, in any case, less than 1. Thus, in particular,

$$\|f_{2n}\| \leq K \|f_{2n-1}\| < \|f_{2n-1}\| \tag{20.50}$$

$$\leq K^n \|f_0\|,$$

$$\|f_{2n+1}\| \leq \|f_{2n}\| \leq K^n \|f_0\|.$$

From this it follows easily that the series

$$f_0 + f_1 + f_2 + \dots + f_{2n} + f_{2n+1} + \dots \tag{20.51}$$

with terms defined by (20.45) converges uniformly in the part of the lower half-plane exterior to $|z + ih| < a$.

One may extend the method to flows about more general cylinders by combining the operator \mathfrak{M} with another defined in terms of conformal mapping of the

given profile into a circle. The procedure carried through above is a natural generalization of the procedure used by HAVELOCK in his first two papers (1926, 1929a) to find the second approximation. However, in his later paper (1939b) he used a different procedure, one which has also been used by URSELL in analogous problems. This consists in expressing the potential as a

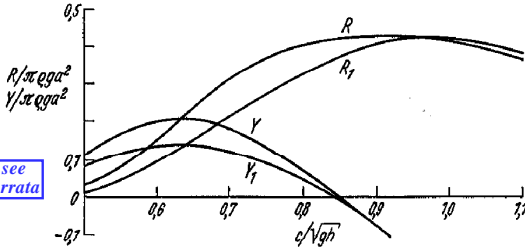


Fig. 19.

sum of multipoles situated at the center and, of course, already modified so as to satisfy the condition on the free surface and as $x \rightarrow \infty$. This leads to an infinite set of equations for the coefficients. The method is quite suitable for approximate computation.

After computation of $H(k)$, the formulas (20.38) can be used to find the force and moment. In the computation of H only the terms with odd indices contribute. This leads to a considerable saving in effort. For example, if one had approximated the flow by the first three terms of (20.51) and computed the force by integrating the pressure over the cylinder, the result would be the same as that obtained by using the H -function evaluated for f_1 alone, and without the need of finding f_2 . HAVELOCK has frequently made use of this device without specifically introducing the H -function. Fig. 19 from HAVELOCK (1936b) shows $R = -X$ and Y plotted in units of $\pi g \rho a^2$ with abscissa $1/\sqrt{\nu h}$ for $a/h = \frac{1}{2}$. The curves labelled R_1 and Y_1 give the result when only the first approximation is used, i.e.,

$$\left. \begin{aligned} H(k) &= 2\pi c a^2 k e^{-kh} \\ R_1 &= \pi \rho g a^2 \cdot \pi \left(\frac{a}{h}\right)^2 \left(\frac{2gh}{c^2}\right)^2 e^{-2gh/c^2}, \\ Y_1 &= -\pi \rho g a^2 \cdot \left(\frac{a}{h}\right)^2 \frac{c^2}{2gh} \left[1 + \frac{2gh}{c^2} - \left(\frac{2gh}{c^2}\right)^2 - \left(\frac{2gh}{c^2}\right)^3 e^{-2gh/c^2} \text{Ei}\left(\frac{2gh}{c^2}\right) \right]. \end{aligned} \right\} \tag{20.52}$$

Computation of M gives, on this approximation, the anomalous result

$$M_1 = h \left[1 - \frac{c^2}{gh} \right] \cdot R.$$

In the formula for Y the terms resulting from buoyancy and circulation are omitted. HAVELOCK (1928b) has investigated the form of the surface over a moving dipole, i.e., over a sphere to the same degree of approximation.

Some submerged three-dimensional bodies. The flow about submerged ellipsoids and bodies of revolution in general has obvious interest in connection with the wave resistance of submarines. As a result there is a considerable amount of both theoretical and experimental work available, and even some tables for computation of wave resistance. Most of the theoretical work does not go beyond the approximation in which one represents the body by the singularity distribution appropriate to an unbounded fluid, but with the potential function for the singularity modified to satisfy the conditions on the free surface and at $x = +\infty$. Thus, to find the flow about a submerged sphere one will in this approximation use a modified dipole with axis in the direction Ox and moment $\frac{1}{2}ca^3$. One should realize, however, that the boundary condition on the body appropriate to deep submergence has not been satisfied. The necessary refinements could be carried through for the sphere in a manner similar to that used for the circular cylinder. Since the sphere (and even more, the circular cylinder) is a poor shape to which to apply perfect-fluid theory, such a computation is of only moderate interest. Both POND (1951, 1952) and HAVELOCK (1952) have considered methods for improving the accuracy with which the boundary condition on bodies of revolution is satisfied. This is particularly important in estimating the moment about the transverse horizontal axis, but, as POND shows, of less importance for the wave resistance.

HAVELOCK (1931a) treated by the approximate method the wave resistance of prolate and oblate spheroids moving both along and at right angles to their axes. Later (1931b) he extended the results to general ellipsoids moving in the direction of the longest axis. WEINBLUM (1936) has considered bodies of revolution using the slender-body approximation, but satisfying it only in the approximate sense described above; his aim was to find forms of minimum wave resistance. WEINBLUM (1951) returned to the problem, taking up in particular numerical computation of the wave resistance for a given shape. Tables and graphs are given to facilitate the computation for certain classes of bodies. Experiments were made by WEINBLUM, AMTSBERG and BOCK (1936) on three forms at several depths. Presumably, more recent experiments exist whose results are not publicly available. A general survey of the theory may be found in BESSHO (1957).

If one has once computed the function $H(k, \vartheta)$ for a source and a dipole, it is usually straightforward to compute it for bodies generated by distributions of sources and dipoles, and hence to compute the force. Let S_1 be a surface containing a single submerged source of strength m at the point (a, b, c) , $b < 0$ [i.e., (13.36) multiplied by $-m$]; one finds

$$H(k, \vartheta) = 4\pi m e^{kb} e^{ik(a \cos \vartheta + c \sin \vartheta)}. \tag{20.53}$$

For a dipole of moment M in the direction Ox one finds

$$H(k, \vartheta) = 4\pi i M k e^{kb} e^{ik(a \cos \vartheta + c \sin \vartheta)} \cos \vartheta. \tag{20.54}$$

These may now be superposed as necessary for either discrete or continuous distributions. Thus, if we write $G(x, y, z; \xi, \eta, \zeta)$ for the function (13.36) with (a, b, c) replaced by (ξ, η, ζ) , and if we can express φ for some problem by

$$\varphi = \iint_S \gamma(\xi, \eta, \zeta) G(x, y, z, \xi, \eta, \zeta) d\sigma, \tag{20.55}$$

then [cf. Eq. (19.20)]

$$H(k, \vartheta) = -4\pi \iint_S e^{k\eta} e^{ik(\xi \cos \vartheta + \zeta \sin \vartheta)} \gamma(\xi, \eta, \zeta) d\sigma. \tag{20.56}$$

Prolate spheroid. We give an example of the preceding remarks. A prolate spheroid of major semi-axis a and minor semi-axes b moving with velocity c in the direction of its major axis can be represented in an unbounded fluid by a distribution of dipoles of moment density

$$\mu(\xi) = A c (a^2 e^2 - \xi^2), \quad |\xi| < a e, \tag{20.57}$$

where

$$A^{-1} = \frac{4e}{1-e^2} - 2 \log \frac{1+e}{1-e}, \quad e^2 = 1 - \frac{b^2}{a^2},$$

placed along the major axis between the two foci. Hence with the center at depth h one has in this approximation

$$\left. \begin{aligned} H(k, \vartheta) &= 4\pi i A c k e^{-kh} \cos \vartheta \int_{-ae}^{ae} (a^2 e^2 - \xi^2) e^{i k \xi \cos \vartheta} d\xi \\ &= 8 \sqrt{2} \pi i A c (ae)^{\frac{3}{2}} \frac{e^{-kh}}{k^{\frac{3}{2}} \cos^{\frac{3}{2}} \vartheta} J_{\frac{3}{2}}(a e k \cos \vartheta). \end{aligned} \right\} \tag{20.58}$$

Substituting in the first formula of (20.37) one finds

$$R = -X = +128\pi \rho v c^2 a^3 e^3 A^2 \int_0^{\frac{1}{2}\pi} e^{-2\nu h \sec^2 \vartheta} [J_{\frac{3}{2}}(a e \nu \sec \vartheta)]^2 \sec^2 \vartheta d\vartheta. \tag{20.59}$$

Fig. 20 from HAVELOCK (1931a) shows a graphical presentation of $R/\pi \rho a b^3$ for spheroids with various ratios of a/b and for $h = 2b$. In comparing the different

curves one should keep in mind the selected vertical scale; one based on displaced fluid, i.e. $R/\frac{4}{3}\pi \rho g a b^2$, would give the comparison a different aspect.

As mentioned earlier, it has been shown by both POND and HAVELOCK that this approximate treatment of the boundary condition on the body is inadequate for computation of the moment. Fig. 21 is from POND (1951, 1958) and shows the computed moment about

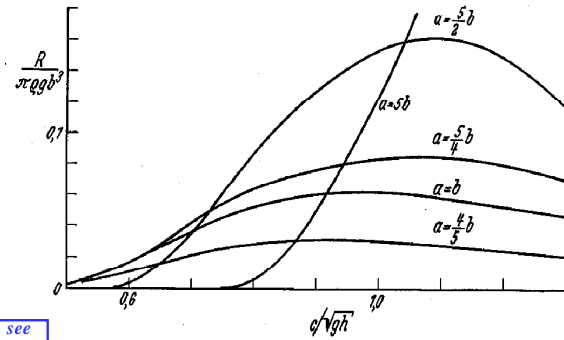


Fig. 20.

the center for a Rankine ovoid, i.e., for the body generated in an unbounded fluid by a source and sink of equal strengths moving in the direction of their axis. The dashed curves show the result with the approximate computation; the solid curves were computed by a method in which the boundary condition on the body is more closely satisfied. The length l of the body is 10.5 times the maximum diameter $d = 2b$. A positive moment is nose-up.

Slender bodies. It is known that, for bodies of revolution given in the form (20.34) the slender-body boundary condition (20.35) can be satisfied in an infinite fluid by a distribution of sources along the axis of strength density

$$\gamma(x) = \frac{1}{4} c \frac{d}{dx} r^2(x). \tag{20.60}$$

If one assumes that this same distribution of the modified sources (13.36) will satisfy approximately (20.35) then

$$\varphi(x, y, z) = \int_{-a}^a \gamma(\xi) G(x, y, z; \xi, -h, 0) d\xi \tag{20.61}$$

and

$$H(k, \vartheta) = -4\pi e^{-kh} \int_{-a}^a e^{ikh\xi \cos\vartheta} \gamma(\xi) d\xi. \tag{20.62}$$

From this one finds easily from (20.37)

$$\left. \begin{aligned} R &= -X \\ &= 16\pi \rho v^2 \int_0^{\pi/2} d\vartheta \sec^3\vartheta e^{-2\nu h \sec^2\vartheta} \int_{-a}^a \int_{-a}^a \gamma(\xi) \gamma(\xi') \cos[k(\xi - \xi') \sec\vartheta] d\xi d\xi' \\ &= 16\pi \rho v^2 \int_0^{\pi/2} \sec^3\vartheta e^{-2\nu h \sec^2\vartheta} [P^2 + Q^2] d\vartheta, \end{aligned} \right\} \tag{20.63}$$

where

$$P(\vartheta) = \int_{-a}^a \gamma(\xi) \cos(\nu \xi \sec \vartheta) d\xi, \quad Q(\vartheta) = \int_{-a}^a \gamma(\xi) \sin(\nu \xi \sec \vartheta) d\xi.$$

As mentioned earlier, WEINBLUM (1951) has published tables allowing one to compute R for γ -s representable as certain polynomials. An earlier paper (1936) considers the minimization of R among certain classes of polynomial γ -s. POND (1952) treats the necessary refinements to (20.61) in order to compute the moment. CUMMINS (1954) finds the additional effect of a train of waves on the surface.

Thin ships. Let the equation of the hull be given as in (20.22) by $z = \pm F(x, y)$ in a coordinate system moving with the ship which we take to move with constant velocity c in the direction Ox . If we assume that a steady state has been reached, the boundary condition for the hull appropriate to the thin-ship approximation is, from (20.26) with $\Phi(x, y, z, t) = \varphi(x - ct, y, z)$ and a change to a moving coordinate system,

$$\varphi_z(x, y, \pm 0) = \mp c F_x(x, y) \tag{20.64}$$

for $(x, y, 0)$ in S_0 , the centerplane section of the ship at rest. For $(x, y, 0)$ not in S_0 one has $\varphi_z(x, y, \pm 0) = 0$ from symmetry considerations. φ must, of course, also satisfy (20.1) and the condition of vanishing motion as $x \rightarrow \infty$.

The boundary conditions may be satisfied immediately by distributing sources (13.26) over S_0 . If we again denote the potential function in (13.36) by $G(x, y, z; \xi, \eta, \zeta)$, then, for infinitely deep fluid, the solution is

$$\varphi(x, y, z) = \frac{c}{2\pi} \iint_{S_0} G(x, y, z; \xi, \eta, 0) F_x(\xi, \eta) d\sigma. \tag{20.65}$$

This follows easily from known theorems in potential theory¹. (The part of G regular in $y \leq 0$ does not interfere with the satisfying of (20.64) since the z -derivative of these terms vanishes for $z = 0$.)

The quantity of chief interest is the resistance resulting from the waves. This may be computed by using again (20.56) and (20.37) (and remembering to

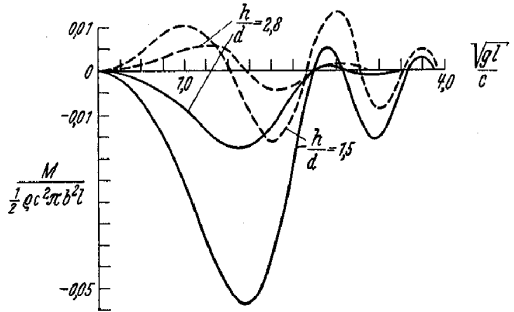


Fig. 21.

¹ See, e.g., O. D. KELLOGG: Foundations of potential theory, pp. 160–166. Berlin: Springer 1929.

take account of both halves of the hull), or by direct integration of the pressure over the hull after taking account of linearization, i.e.

$$R = 2 \rho c \iint_{S_0} \varphi_x(x, y, 0) F_x(x, y) dx dy. \quad (20.66)$$

If the latter form is used, only the single-integral term in G gives a non-vanishing contribution. In either case one finds immediately, again for infinitely deep fluid,

$$\left. \begin{aligned} R &= \frac{4g^2 \rho}{\pi c^2} \int_0^{\pi/2} \sec^3 \vartheta [P^2(\vartheta) + Q^2(\vartheta)] d\vartheta, \\ P &= \iint_{S_0} F_x(x, y) e^{\nu y \sec^2 \vartheta} \cos(\nu x \sec \vartheta) dx dy, \\ Q &= \iint_{S_0} F_x(x, y) e^{\nu y \sec^2 \vartheta} \sin(\nu x \sec \vartheta) dx dy. \end{aligned} \right\} \quad (20.67)$$

The result may be, and has been, put into a variety of different forms by change of variable and order of integration. We give one of them. Let $\lambda = \sec \vartheta$. Then one may verify that

$$\left. \begin{aligned} R &= \frac{4g^2 \rho}{\pi c^2} \iint_{S_0} dx dy \iint_{S_0} d\xi d\eta F_x(x, y) F_\xi(\xi, \eta) M(\nu(x - \xi), \nu(y + \eta)), \\ M(x, y) &= \int_1^\infty \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} e^{\lambda^2 y} \cos \lambda x d\lambda. \end{aligned} \right\} \quad (20.68)$$

This expression for R in terms of the hull form and velocity was first given by MICHELL (1898), but derivations by different methods have since been given by many other, e.g. HAVELOCK (1932, 1951), SRETENSKII (1937), KOCHIN (1937), LUNDE (1951), and TIMMAN and VOSSERS (1955). It is usually called "MICHELL'S integral".

Because MICHELL'S integral gives R directly in terms of the hull geometry it has been intensively investigated by several persons in order to throw light upon the influence of variations of hull form upon wave resistance. Foremost among these investigators has been HAVELOCK, who in a series of notable papers (1923, 1925 a, b, 1926a, 1932a, b, 1935) studied the effects of various systematic variations described by the titles of the papers. Much of this work is summarized in HAVELOCK (1926). In addition, there are numerous papers by G.P. WEINBLUM and W.C.S. WIGLEY devoted to comparison of experiment and theory. One can find surveys of much of this and related work, as well as further bibliography, in WIGLEY (1930, 1935, 1949), WEINBLUM (1950), HAVELOCK (1951), LUNDE (1957), and WEHAUSEN (1957), LUNDE'S 1951 paper contains derivations of practically all the general theoretical results, including the effect of finite depth, walls, and acceleration. TAKAO INUI (1954) has given an extensive survey of Japanese investigations on wave resistance and related topics, and in a later paper (1957) a complete survey.

In order to allow better exploitation of MICHELL'S integral much attention has been given in recent years to its numerical computation. One can find a general discussion in BIRKHOFF and KOTIK (1954), and various special proposals in KABACHINSKII (1947), REINOV (1951), GUILLOTON (1951) and WEINBLUM (1955). The last two papers both contain sets of tables to be used in evaluating MICHELL'S integral.

In making a comparison of the theoretically predicted wave resistance with measured wave resistance one must examine critically the experimental method for estimating the wave resistance. The standard method consists in measuring

the total resistance, estimating the part of the total resulting from the effects of viscosity, and attributing the difference to wave making. Thus the accuracy of the experimentally estimated wave resistance depends upon the accuracy of the estimated viscous resistance and upon the validity of the assumption that the two may be added. In the case of a very thin body this estimate can be made accurately and, in addition, the physical assumption in the thin-ship linearization is well realized. Fig. 22 from a report by WEINBLUM, KENDRICK and TODD (1952) shows a comparison between estimated and computed values of $R_w/\frac{1}{2}\rho c^2 S$ for a towed "friction plane" 21 feet long with parabolic ends and 3 foot draft. These experimental data present MICHELL'S integral in a most favorable light. For more ship-like forms the separation of viscous from wave-making resistance is more difficult and the compared values seldom show such striking quantitative agreement, although it is still fair in many cases. We call attention to the fact that MICHELL'S integral predicts the same wave-making resistance no matter in which direction the ship moves.

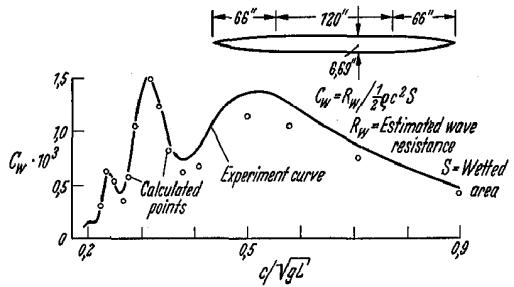


Fig. 22.

So far we have discussed vessels moving in an infinitely deep fluid. However, if for our function G we had taken (13.37) instead of (13.36), the same analysis would have led us to the following expression, first given by SRETENSKII (1937):

see errata

$$\left. \begin{aligned}
 R &= \frac{2\rho g c}{\pi} \int_{\mu_h}^{\infty} [P^2(\mu) + Q^2(\mu)] \sqrt{\frac{\mu}{\mu - \nu \tanh \mu h}} d\mu, \\
 P(\mu) &= \iint_{S_0} F_x(x, y) \frac{\cosh \mu(y+h)}{\cosh \mu h} \cos(x\sqrt{\nu \mu \tanh \mu h}) dx dy, \\
 Q(\mu) &= \iint_{S_0} F_x(x, y) \frac{\cosh \mu(y+h)}{\cosh \mu h} \sin(x\sqrt{\nu \mu \tanh \mu h}) dx dy.
 \end{aligned} \right\} (20.69)$$

Here μ_h is the nonzero solution of $\mu = \nu \tanh \mu h$ if such exists, i.e. if $c^2/g h > 1$; otherwise $\mu_h = 0$. As $h \rightarrow \infty$, $\mu_h \rightarrow \nu$ and one obtains one of the forms of MICHELL'S integral.

An expression for the wave resistance of a thin ship moving down the center of a rectangular canal was derived independently by SRETENSKII (1936, 1937) and KELDYSH and SEDOV (1937). The result may be found in LUNDE (1951).

One may naturally ask how the wave pattern illustrated in Fig. 1 for a moving source is related to that for a ship. In the thin-ship approximation, the ship is replaced by a distribution of sources on the centerplane, so that each infinitesimal area of the centerplane contributes to such a pattern according to its strength. However, in many large vessels the middle part of the ship is cylindrical, so that $F_x = 0$ in this region and only the bow and stern regions contribute a nonvanishing source density. Thus, if one replaces the ship by a single source in the bow region and a single sink in the stern region, the resulting wave pattern will approximate to that of a ship, the approximation being better at higher values of the Froude number c/\sqrt{gL} . Depending upon the value of c/\sqrt{gL} , the transverse wave systems

from the two singularities may either reinforce or partially cancel one another. When they are in phase, a larger amount of energy is being left behind in the wave system and the resistance curve shows a maximum, when they are out of phase a minimum, the so-called "humps and hollows" of the resistance curve; these show clearly in Fig. 22. Replacing the ship by a source and sink is, of course, a gross simplification. However, it serves to explain qualitatively certain aspects of a ship's wave pattern and resistance curve, and, in fact, can be given a certain

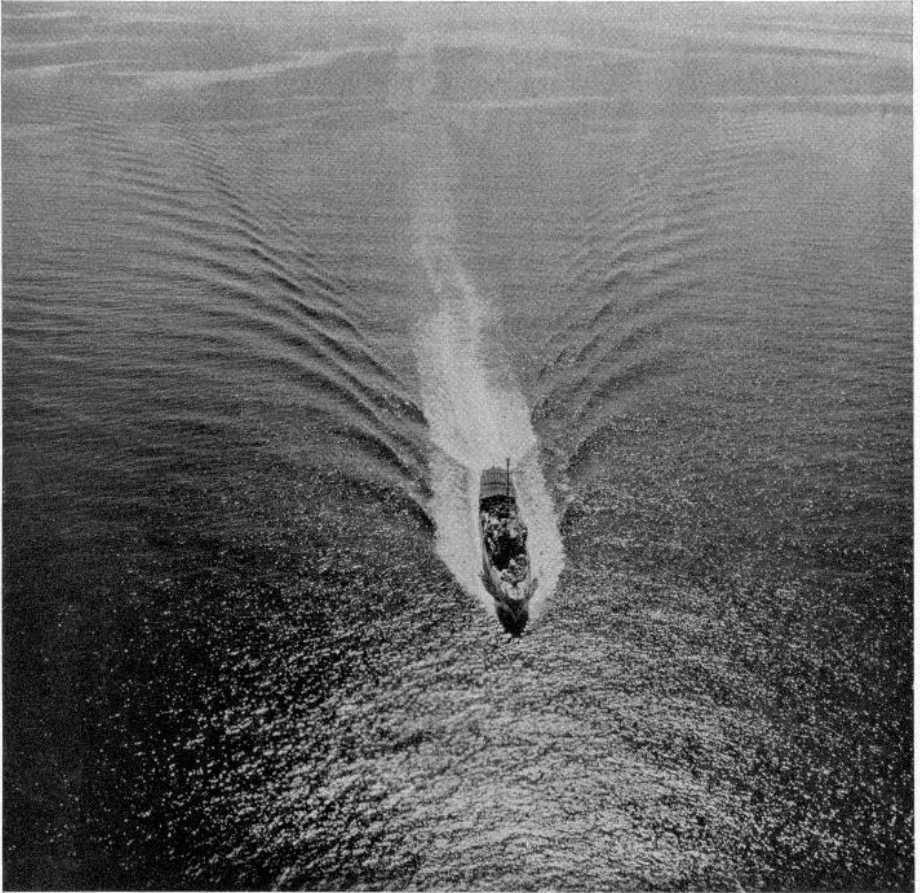


Fig. 23. Ship waves.

amount of validity as an approximate computation of MICHELL'S integral for sufficiently large c/\sqrt{gL} . For very large values of c/\sqrt{gL} , the wave length of the transverse waves along the path, $2\pi c^2/g$, becomes much larger than L and one may approximate the ship by a dipole. Many photographs of the wave pattern made by a fast motor boat fall into this class. The photograph reproduced in Fig. 23 shows clearly the diverging waves from the bow and stern, a third set possibly originating at the forward shoulder, and also the transverse waves, which presumably are here nearly in phase.

The angular opening of the wedge containing the wave pattern should be, according to (13.42) and Fig. 1, $38^\circ 56'$ in deep water. This is confirmed only approximately in photographs; difficulty in determining boundaries makes precise confirmation difficult. For ships moving in water of finite depth h the

angular opening changes as shown in Fig. 2, and for supercritical velocities, i.e. $c^2 > gh$, there are no transverse waves.

JINNAKA (1957) has recently published a brief survey of the theory of ship waves.

Thin hydrofoils. We take the hydrofoil as described in (20.29) and treat the problem two-dimensionally. Assuming constant velocity c and steady motion and taking our coordinate system moving with the hydrofoil, the boundary condition (20.33) becomes

$$\varphi_y(x, -h \pm 0) = -c r'(x) \mp c s'(x), \quad -a < x < a. \quad (20.70)$$

This problem has been treated very thoroughly by KELDYSH and LAVRENT'EV (1936). They follow a procedure quite analogous to that used in Sect. 19 α to find the waves generated by a vertical oscillator not in a wall. Distribute vortices of intensity $\gamma(x)$ and sources of strength $\sigma(x)$ along the line $-a < x < a$, $y = -h$, but taking them, of course, modified as in (13.43) in order to satisfy the free-surface condition and the conditions at infinity. To start with, we take $\sigma(x) = -2cs'(x)$. It then follows from the theorem of PLEMELJ-SOKHOTSII [cf. Eq. (17.18)] that

$$\varphi_y(x, -h + 0) - \varphi_y(x, -h - 0) = -2cs'(x), \quad (20.71)$$

a step toward satisfying (20.70). We now look for a complex potential in the form

$$f(z) = \int_{-a}^a [-2cs'(\xi) f_s(z; \xi - ih) + \gamma(\xi) f_v(z; \xi - ih)] d\xi, \quad (20.72)$$

where we have separated the source and vortex potentials in (13.43). The boundary condition (20.70) now yields an integral equation for $\gamma(x)$ in much the same manner that (17.18) was derived:

$$\left. \begin{aligned} \text{Im} \int_{-a}^a [-2cs'(\xi) f'_s(x - ih; \xi - ih) + \gamma(\xi) f'_v(x - ih; \xi - ih)] d\xi = cr'(x), \\ -a < x < a. \end{aligned} \right\} \quad (20.73)$$

Noting from the third expression in (13.43) that f_s and f_v are functions of the difference $x - \xi$, we define

$$\begin{aligned} 2\pi i f'_s(x - ih, -ih) &= i \left[\frac{1}{x} - \frac{1}{x - 2ih} \right] - 2\nu e^{-ivx} \int_{\infty}^x \frac{e^{ivt}}{t - 2ih} dt \\ &= H(x) + iJ(x), \\ 2\pi i f'_v(x - ih, -ih) &= \frac{1}{x} + \frac{1}{x - 2ih} - 2i\nu e^{-ivx} \int_{\infty}^x \frac{e^{ivt}}{t - 2ih} dt \\ &= K(x) + iI(x). \end{aligned}$$

Here, for example,

$$K(x) = \frac{1}{x} + \frac{x}{x^2 + 4h^2} + 2\nu \int_{\infty}^x \frac{2h \cos \nu(t - x) + t \sin \nu(t - x)}{t^2 + 4h^2} dt.$$

The integral equation (20.73) can now be written in the following form:

$$\int_{-a}^a \gamma(\xi) K(x - \xi) d\xi = -2\pi c r'(x) + \int_{-a}^a 2cs'(\xi) H(x - \xi) d\xi, \quad (20.74)$$

where the right-hand side is a known function.

This integral equation is the hydrofoil analogue of the thin-wing integral equation of airfoil theory:

$$\int_{-a}^a \gamma(\xi) \frac{1}{x-\xi} d\xi = -2\pi c r'(x). \tag{20.75}$$

In the latter equation the kernel is simpler and in addition only the function r' describing the camber and the angle of attack occurs on the right side. Since the wing thickness does not enter into the determination of γ in (20.75) it may be neglected, for only γ is needed to find the lift. The situation is clearly different for hydrofoils. Even a symmetric wing with zero angle of attack may have a circulation, and hence lift. This is a consequence, of course, of the presence of the free surface and the associated wave motion.

KOCHIN (1936) has also considered hydrofoils, but from a somewhat different viewpoint. He has essentially used the "deep-submersion" linearization described first in this section. Thus he must satisfy the exact boundary conditions on the wing as well as the Kutta-Joukowski condition. However, one cannot say here, as one could for an infinite fluid, that his method is more exact than that of KELDYSH and LAVRENT'EV. Their approximation is more accurate the thinner the wing, for a given submersion. KOCHIN's is more accurate the deeper the submersion, for a given wing.

Eq. (20.74) is not sufficient to determine $\gamma(x)$ uniquely. One must still add some further condition. We shall assume a finite velocity at the trailing edge, i.e.

$$\varphi_y(-a-0, -h) = \int_{-a}^a [\gamma(\xi)K(-a-\xi) - 2c s'(\xi)H(-a-\xi)] d\xi \text{ finite.} \tag{20.76}$$

KELDYSH and LAVRENT'EV propose solving the integral equation (without actually proving that a solution exists) by expanding K , H and γ in a power series in $\tau = a/\nu h$ and then determining recursively the coefficients. Let

$$\left. \begin{aligned} K(x) &= \frac{1}{x} + \frac{\tau}{a} \sum_{n=0}^{\infty} K_n(2\nu h) \tau^n \left(\frac{x}{a}\right)^n, \\ H(x) &= \frac{\tau}{a} \sum_{n=0}^{\infty} H_n(2\nu h) \tau^n \left(\frac{x}{a}\right)^n, \\ \gamma(x) &= \sum_{n=0}^{\infty} \gamma_n(x) \tau^n. \end{aligned} \right\} \tag{20.77}$$

Then (20.74) gives the following sequence of integral equations.

$$\left. \begin{aligned} \int_{-a}^a \gamma_0(\xi) \frac{d\xi}{x-\xi} &= -2\pi c r'(x), \\ \int_{-a}^a \gamma_1(\xi) \frac{d\xi}{x-\xi} &= \frac{1}{a} H_0 \int_{-a}^a 2c s'(\xi) d\xi = 0, \\ \dots \dots \dots \\ \int_{-a}^a \gamma_{n+1}(\xi) \frac{d\xi}{x-\xi} &= \frac{1}{a^{n+1}} H_n \int_{-a}^a 2a s'(\xi) (x-\xi)^n d\xi - \\ &\quad - \sum_{k+l=n} \frac{1}{a^{l+1}} K_l \int_{-a}^a \gamma(\xi) (x-\xi)^l d\xi, \\ \dots \dots \dots \end{aligned} \right\} \tag{20.78}$$

This procedure has the obvious advantage of reducing the solution to the airfoil integral equation for which an explicit solution satisfying the trailing-edge condition is known. If we denote temporarily the right-hand sides of the Eqs. (20.78) by $F_n(x)$, respectively, then the general solution is¹

$$\gamma_n(x) = \frac{1}{\pi^2 \sqrt{a^2 - x^2}} \left[\int_{-a}^a \frac{F_n(\xi) \sqrt{a^2 - \xi^2}}{\xi - x} d\xi + \pi \int_{-a}^a \gamma_n(\xi) d\xi \right], \tag{20.79}$$

where the value of $\int_{-a}^a \gamma_n d\xi$ is undetermined. In terms of the series expansion, condition (20.76) states that

$$\left. \begin{aligned} \sum_{n=0}^{\infty} \tau^n \int_{-a}^a \gamma_n(\xi) \frac{d\xi}{x - \xi} + \sum_{n=1}^{\infty} \tau^n \sum_{k+l=n-1} a^{-l-1} K_l \int_{-a}^a \gamma_k(\xi) (x - \xi)^l d\xi - \\ - \sum_{n=1}^{\infty} a^{-n-1} H_{n-1} \int_{-a}^a 2c s'(\xi) (x - \xi)^{n-1} d\xi \end{aligned} \right\} \tag{20.80}$$

must remain finite for $x \rightarrow -a$. We assume $s'(-a)$ finite. γ_k may possibly have a singularity of the form $1/\sqrt{a+x}$ near $x = -a$. However, the integral

$$\int_{-a}^a \frac{(x - \xi)^l}{\sqrt{a^2 - \xi^2}} d\xi = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (x - a \sin \alpha)^l d\alpha,$$

is a polynomial in x for $l \geq 0$. Thus the last two summations of (20.80) remain finite at $x = -a$. However, the first summations potentially contributes terms like

$$\int_{-a}^a \frac{d\xi}{(x - \xi) \sqrt{a^2 - \xi^2}} = \frac{\pi}{\sqrt{x^2 - a^2}},$$

In order to avoid this singularity at $x = -a$, we select the total circulation $\int_{-a}^a \gamma_n d\xi$ so that

$$\int_{-a}^a \gamma_n(\xi) d\xi = -\frac{1}{\pi} \int_{-a}^a \frac{F_n(\xi) \sqrt{a^2 - \xi^2}}{\xi + a} d\xi. \tag{20.81}$$

Substituting into (20.79), one finds finally

$$\gamma_n(x) = \frac{1}{\pi^2} \sqrt{\frac{a+x}{a-x}} \int_{-a}^a F_n(\xi) \sqrt{\frac{a-\xi}{a+\xi}} \frac{d\xi}{\xi-x}. \tag{20.82}$$

$\gamma(x)$ itself is given by the sum displayed in (20.77). Although the singularity at the trailing edge has been removed, there is still one at the leading edge; this occurs also in thin-airfoil theory and corresponds roughly to the fact that the conditions of linearization (i.e. of small disturbance) are not satisfied near the leading-edge stagnation point.

KELDYSH and LAVRENT'EV compute the integrals which will be necessary if $r(x)$ and $s(x)$ are given as polynomials and apply their computational method to a flat-plate and circular-arc airfoil at a small angle of attack.

¹ See, e.g., W. SCHMEIDLER: Integralgleichungen ..., pp. 55–56. Leipzig: Akademische Verlagsgesellschaft 1950.

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In order to find the force and moment on the wing it is convenient to fall back on the H -function. One finds easily

$$\left. \begin{aligned} H(k) &= e^{-kh} \int_{-a}^a [\gamma(\xi) - 2ic s'(\xi)] e^{-i\lambda s} ds, \\ |H(k)|^2 &= e^{-2hk} \int_{-a}^a \int_{-a}^a \{[\gamma(\xi)\gamma(x) + 4c^2 s'(\xi)s'(x)] \cos k(\xi - x) + \\ &\quad + 2c[\gamma(\xi)\sigma'(x) - \gamma(x)\sigma'(\xi)] \sin k(\xi - x)\} d\xi dx. \end{aligned} \right\} \quad (20.83)$$

Formulas (20.38) allow one to complete the calculation for special cases.

The theory analogous to that described above for fluid of finite depth h_0 has been carried through by TIKHONOV (1940). He has applied the method to calculate the lift and drag coefficients for a flat plate at a small angle of attack.

Rather than reproduce the graphical presentations of KELDYSH and LAVRENT'EV and TIKHONOV for the flat plate, we shall give instead the lift and drag coefficients for a submerged vortex. Here one may give relatively simple analytic expressions, and the qualitative behavior of the curves is similar to that of a flat plate. The formulas for lift L and drag D are as follows:

$$\left. \begin{aligned} D_\infty &= \rho v \Gamma^2 e^{-2\nu h}, \\ L_\infty &= \rho \Gamma c - \frac{\rho \Gamma^2}{4\pi h} + \frac{\rho \Gamma^2}{\pi} \nu e^{-2\nu h} \text{Ei}(2\nu h), \\ D_{h_0} &= \rho v \Gamma^2 \frac{\sinh^2 m_0 (h_0 - h)}{\nu h_0 - \cosh^2 m_0 h_0} \quad \text{if } \nu h_0 > 1, \quad = 0 \quad \text{if } \nu h_0 < 1, \\ L_{h_0} &= \rho \Gamma c - \frac{\rho \Gamma^2}{4\pi} \frac{1}{h_0 - h} + \frac{\rho \Gamma^2}{2\pi} \int_0^\infty (\nu + k) e^{-kh_0} \frac{\sinh 2k(h_0 - h)}{\nu \sinh k h_0 - k \cosh k h_0} dk, \end{aligned} \right\} \quad (20.84)$$

where m_0 is the real root of $m = \nu \tanh m h_0$. For finite depth the expression for D stems from the last term in (13.47). The dimensionless coefficients $C_D = D h / \rho \Gamma^2$ and $C_L = (L - \rho c \Gamma) h / \rho \Gamma^2$ are shown in Fig. 24a for infinite depth as functions of $c^2/g h$ and in Figs. 24b and c as functions of $c^2/g h_0$ for various values of $\beta = h/h_0$. For infinite depth C_L starts with a value $1/4\pi$ and tends asymptotically to $-1/4\pi$, crossing the axis at $c^2/g h = 2.47$. For finite depth the coefficients have a discontinuity at $c^2/g h_0 = 1$. As $c^2/g h_0 \rightarrow 0$, $C_D \rightarrow 0$, and as $c^2/h_0 \rightarrow 1$, $C_D \rightarrow \frac{3}{2}\beta(1 - \beta)^2$. For $c^2/g h_0 > 1$, C_L is always negative and increasing with a vertical asymptote at $c^2/g h_0 = 1$ and a horizontal one as $c^2/g h_0 \rightarrow \infty$ at $-\beta/4 \sin \beta \pi$; these curves start at $\frac{1}{4}\beta \cot \beta \pi$.

Further development of hydrofoil theory has taken place in several directions. HASKIND (1945a) has extended KOCHIN'S "deep-submersion" theory to water of finite depth. However, he does not discuss the steps necessary for fulfilment of the Kutta-Joukowski condition, as does KOCHIN. The lifting-line theory for airfoils of finite span has been extended to hydrofoils by WU (1954), BRESLIN (1957), and HASKIND (1956). PARKIN, PERRY and WU (1956) and LAITONE (1954, 1955) have investigated both theoretically and experimentally the effect of bringing a given hydrofoil so close to the surface that the infinitesimal-wave approximation breaks down completely. There exists also a considerable amount of work on flow about cavitating hydrofoils. However, since the effect of gravity is neglected, this work is not considered in the present article. Experimental data relevant to the theoretical development outlined above are scanty. Reports by BENSON and LAND (1942) and by LAND (1943) give results of an experimental investigation of the effect of depth of submersion. However, the investigations

were not designed to test the validity of the theory and do not, for example, include the region of maximum C_D . AUSMAN (1953), in connection with an experimental investigation of the pressure distribution on the upper surface of a hydrofoil, measured the lift coefficient and compared it with that predicted by the thin-hydrofoil theory. The theory failed when gh/c^2 became too small because the associated free surface over the hydrofoil no longer approximated infinitesimal waves, or, in other words, the thin hydrofoil was not thin enough

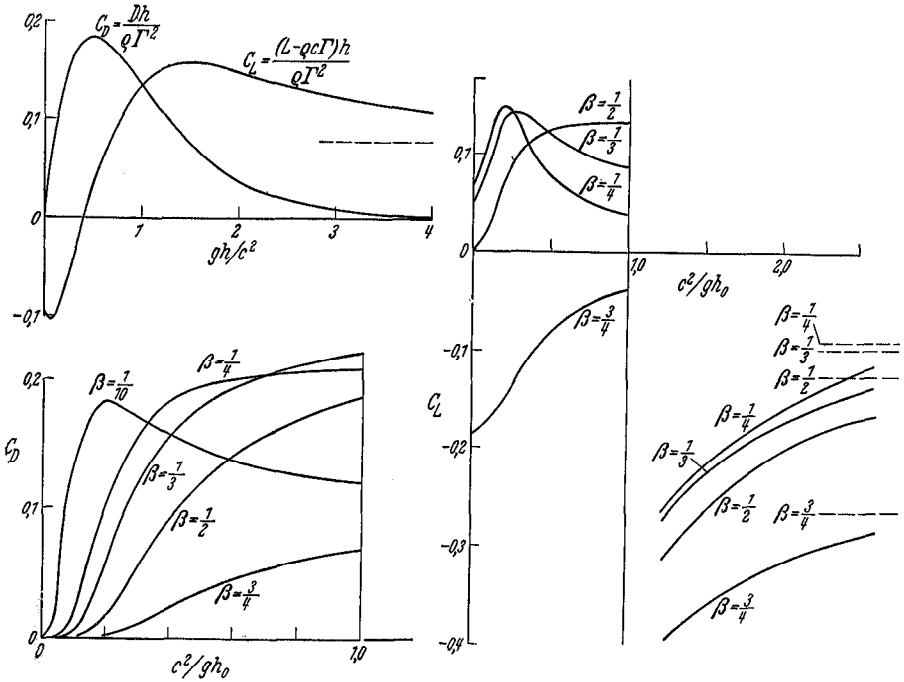


Fig. 24 a—c.

for these values of gh/c^2 for the theory to be applicable. It should also be emphasized that for small values of gh/c^2 the occurrence of cavitation on the upper surface must be taken into account for a complete theory. Recent measurements by NISHIYAMA (1959) show good agreement even for small values of gh/c^2 . A comprehensive survey of hydrofoil theory is given in a recent paper by NISHIYAMA (1957).

γ) *Planing surfaces.* The following discussion is limited to two-dimensional motion, for the theory of three-dimensional planing surfaces for flows with gravity does not appear to have been developed.

For the linearized problem it is natural to consider the planing surface or glider as an approximation to a flat plate moving along the surface of the undisturbed fluid, i.e. the curvature, angle of attack and vertical displacement are all assumed small. In order to formalize the perturbation procedure, let the planing surface be represented by

$$y = h + F(x), \quad |x| \leq a, \quad F(-a) = 0, \tag{20.85}$$

in coordinates fixed in space, and let the fluid have velocity $-c$ at $x = +\infty$. Thus, we are going to consider the flow to be a perturbation of a uniform flow.

First let us consider briefly in a qualitative fashion the exact solution. There will be a stagnation point A somewhere behind the leading edge and a jet will be thrown out ahead of the glider. We take it to be of thickness b and to make an angle β with OX . If $\Phi = -cx + \varphi(x, y)$ and $\Psi = -cy + \psi(x, y)$ are potential and stream function, respectively, we shall take the free surface ahead of the glider to be given by $\Psi = -bc$ and behind the glider by $\Psi = 0$ (see Fig. 25). Then b/a , and AL/a will all be functions of ga/c^2 and gh/c^2 . It will be assumed as one of the boundary conditions of the problem, in analogy with the Kutta-Joukowski condition, that the velocity is continuous at the trailing edge. It is obvious that the flow near the leading edge cannot conform to the requirement that it be a small perturbation of a uniform flow. However, we shall give arguments below to indicate that, except in the neighborhood of the leading edge, this effect is of the second order.

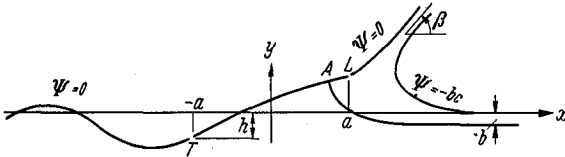


Fig. 25.

fluid. This problem can be solved exactly [see, e.g., MILNE-THOMSON (1956, § 12.26); A. E. GREEN (1935, 1936)]. The asymptotic expression, for small α , of both the ratios b/l and AL/l is $\frac{\pi}{2} \frac{\alpha^2}{1 + \cos \beta}$, i.e. they are both of the second order. We shall suppose that this relation continues to hold when gravity is acting.

We now carry through the perturbation procedure of Sect. 10 α [see especially Eq. (10.16)], writing

$$\left. \begin{aligned} \Phi &= -cx + \varepsilon\varphi^{(1)} + \dots, & \Psi &= -cy + \varepsilon\psi^{(1)} + \dots, & \eta &= \varepsilon\eta^{(1)} + \dots, \\ h + F(x) &= \varepsilon F^{(1)}(x) + \varepsilon h^{(1)} + \varepsilon^2 h^{(2)} + \dots, \\ b &= \varepsilon^2 b^{(2)} + \dots. \end{aligned} \right\} \quad (20.86)$$

Substitution in the exact boundary conditions then yields the following linearized conditions:

$$\left. \begin{aligned} \psi^{(1)}(x, 0) - \frac{c^2}{g} \psi_y^{(1)}(x, 0) &= 0, & |x| &> a, \\ \psi^{(1)}(x, 0) &= c(h^{(1)} + F^{(1)}(x)), & |x| &< a. \end{aligned} \right\} \quad (20.87)$$

The free surface is given by

$$\eta^{(1)}(x) = \frac{1}{c} \psi^{(1)}(x, 0) = \frac{c}{g} \varphi_x^{(1)}(x, 0), \quad |x| > a. \quad (20.88)$$

We require as usual that the disturbance vanish as $x \rightarrow \infty$. One will expect that the behavior near the leading edge will reflect in some manner the inconsistency of the exact solution with the notion of a small perturbation. It will turn out that it will be necessary to allow a singularity at the leading edge. (Almost the same situation exists in the thin-hydrofoil or thin-wing theory since the stagnation point near the leading edge also prevents the flow in that region from being a small perturbation of a uniform flow.) A singularity at the trailing edge, although mathematically possible, has been specifically proscribed. The strength of the singularity at the leading edge and the elevation h of the trailing edge will be determined as functions of ga/c^2 in the course of solving the problem.

The problem as formulated above has been considered, for infinite depth, by SRETENSKII (1933, 1940), SEDOV (1937), KOCHIN (1938), and MARUO (1951). HASKIND has extended SEDOV's analysis to finite depth (1943 a), and later (1955) has treated a glider moving on a wavy surface. YU. S. CHAPLYGIN (1940) has apparently carried through a fairly comprehensive numerical analysis for a flat plate making use of SEDOV's method of analysis [see SRETENSKII (1951, p. 83)]. SRETENSKII's papers are expounded in terms of a flat plate, but the method clearly has wider applicability, as he remarks in his first paper. SRETENSKII's 1940 paper gives the results of rather extensive calculations for flat plates. MARUO's paper is conceptually very similar to those of SRETENSKII, but his method is not quite as efficient for computation. However, he also gives computational results and includes a correction to take account of the failure of the linearized theory near the leading edge. More recently the problem has been considered again by authors unaware of the earlier work. SQUIRE (1957) has analyzed a gliding flat plate by a method similar to that used by SRETENSKII and MARUO. Certain integrals involved in this method have been tabulated by MILLER (1957). CUMBERBATCH (1958) has used a method similar to SEDOV's. Both authors add new results to the earlier work.

Both SEDOV and KOCHIN introduce the complex potential $f(z) = \varphi + i\psi$ and thereafter the function $f' + i\nu f, \nu = g/c^2$. Although the two methods are not by any means the same, they have much in common with the treatment of hydrofoils given above. Consequently, we shall outline below the method followed by SRETENSKII.

As a preliminary we need a result from Sect. 21 below. Suppose that a pressure distribution $p(x)$, which we take to be absolutely integrable, is given on the free surface. Then the complex velocity potential must satisfy

$$\operatorname{Re} \{f'(x + i0) + i\nu f(x + i0)\} = \frac{1}{\rho c} p(x), \tag{20.89}$$

and the free surface is given by

$$y = \eta(x) = \frac{1}{c} \psi(x, 0) = \frac{c}{g} \varphi_x(x, 0) - \frac{1}{\rho g} p(x). \tag{20.90}$$

The function $f(z)$ which satisfies (20.89) and which vanishes as $x \rightarrow \infty$ can be written in several forms, of which we select the following [see Eq. (21.38)]:

$$f(z) = \frac{1}{\pi i \rho c} \int_{-\infty}^{\infty} d\xi p(\xi) \operatorname{PV} \int_0^{\infty} \frac{-e^{-i\lambda(z-\xi)}}{\lambda - \nu} d\lambda - \frac{1}{\rho c} \int_{-\infty}^{\infty} p(\xi) e^{-i\nu(z-\xi)} d\xi. \tag{20.91}$$

The free surface is given by

$$\left. \begin{aligned} \eta(x) = \lim_{y \rightarrow 0} \frac{1}{\pi \rho c^2} \int_{-\infty}^{\infty} d\xi p(\xi) \operatorname{PV} \int_0^{\infty} \frac{\cos \lambda(x-\xi)}{\lambda - \nu} e^{\lambda y} d\lambda + \\ + \frac{e^{\nu y}}{\rho c} \int_{-\infty}^{\infty} p(\xi) \sin \nu(x-\xi) d\xi; \end{aligned} \right\} \tag{20.92}$$

the reason for leaving y explicitly in the formulas will appear below.

When a glider is moving on a free surface, the streamline $y = c^{-1}\psi(x, 0)$ will consist partly of free surface, where $p(x) = 0$, and partly of the wetted surface of the glider, where $p(x)$ is some unknown function.

Eq. (20.92) may then be written as the following integral equation for this unknown function $\phi(x)$:

$$\left. \begin{aligned} h + F(x) = \lim_{\nu \rightarrow -0} \frac{1}{\pi \rho c^2} \int_{-a}^a d\xi \phi(\xi) \text{PV} \int_0^\infty \frac{\cos \lambda(x - \xi)}{\lambda - \nu} e^{\lambda y} d\lambda + \\ + \frac{e^{\nu y}}{\rho c} \int_{-\infty}^\infty \phi(\xi) \sin \nu(x - \xi) d\xi, \quad |x| < a. \end{aligned} \right\} \quad (20.93)$$

Once $\phi(x)$ has been determined, one may substitute back into (20.92) in order to find the form of the free surface for $|x| > a$.

It is possible to work directly with (20.93), and this is the procedure followed by MARUO. However, SRETENSKII differentiates twice with respect to x and adds ν^2 times (20.93). This yields

$$\left. \begin{aligned} F''(x) + \nu^2 F(x) + \nu^2 h = \lim_{\nu \rightarrow -0} \frac{-1}{\pi \rho c^2} \int_{-a}^a d\xi \phi(\xi) \int_0^\infty (\lambda + \nu) \cos \lambda(x - \xi) e^{\lambda y} d\lambda \\ = \lim_{\nu \rightarrow -0} \frac{1}{\pi \rho c^2} \int_{-a}^a \phi(\xi) \left(\nu + \frac{\partial}{\partial y} \right) \frac{y}{(x - \xi)^2 + y^2} d\xi \\ = -\frac{\nu}{\rho c^2} \phi(x) - \lim_{\nu \rightarrow -0} \frac{1}{\pi \rho c^2} \frac{\partial}{\partial x} \int_{-a}^a \phi(\xi) \frac{x - \xi}{(x - \xi)^2 + y^2} d\xi \\ = -\frac{\nu}{\rho c^2} \phi(x) - \frac{1}{\pi \rho c^2} \frac{\partial}{\partial x} \text{PV} \int_{-a}^a \frac{\phi(\xi)}{x - \xi} d\xi. \end{aligned} \right\} \quad (20.94)$$

Although this last equation is a necessary condition for $\phi(x)$, it obviously cannot determine it uniquely, for the last term of (20.93), assuring vanishing of the disturbance far ahead of the glider, was lost in the formation of (20.94). Thus one still has need for (20.93). Eq. (20.94) is essentially the equation derived by SRETENSKII.

Let us now integrate (20.94) with respect to x from $x = -a$ to x , and denote

$$P(x) = \frac{1}{\rho c^2} \int_{-a}^x \phi(\xi) d\xi.$$

Then Eq. (20.94) becomes

$$F'(x) - F'(-a) + \nu^2 \int_{-a}^x F(\xi) d\xi + \nu^2 h(x+a) = -\nu P(x) - \frac{1}{\pi} \int_{-a}^a \frac{P'(\xi)}{x - \xi} d\xi, \quad (20.95)$$

where an additive constant has been discarded since h itself is an undetermined constant. Eq. (20.95) is just PRANDTL'S integro-differential equation for the circulation about an airfoil of finite span¹. Thus known methods for solving the airfoil equation can be carried over to the study of this equation. However, the solutions themselves cannot be taken over directly, for different boundary conditions are imposed: in the airfoil equation the unknown function is the circulation $\Gamma(x)$ and it is usually assumed that $\Gamma(-x) = \Gamma(x)$ and $\Gamma(-a) = \Gamma(a) = 0$;

¹ See, e.g., N.I. MUSKHELISHVILI: Singular integral equations, Chap. 17. Groningen: Noordhoff 1953.

in the present problem $P(x)$ is not necessarily symmetric and $P(-a) = P'(-a) = 0$, but $P(a)$ is not restricted except to be finite. The theory of the Prandtl integro-differential equation without the customary additional requirements associated with airplane wings has been developed by L. G. MAGNARADZE [Soobshch. Akad. Nauk Gruzin. SSR 3, 503–508 (1942)].

The equation can be solved by an extension of GLAUERT'S method¹. This is the method which has been used by both MARUO and SRETENSKII. However, each expands $P' = p$ rather than P in a Fourier series in order to obtain the correct behavior at the two end points. Introduce the new variables ϑ and γ by the equations

$$x = -a \cos \vartheta, \quad \xi = -a \cos \gamma$$

and assume the following expansion for $p(x)$:

$$\left. \begin{aligned} \frac{1}{\rho c^2} p(x) &= \frac{1}{\rho c^2} p(-a \cos \vartheta) = a_0 \tan \frac{1}{2} \vartheta + a_1 \sin \vartheta + \dots + a_n \sin n\vartheta + \dots \\ &= a_0 \sqrt{\frac{a-x}{a+x}} + a_1 \sqrt{a^2 - x^2} + \dots \end{aligned} \right\} \quad (20.96)$$

MARUO substitutes (20.96) into (20.93), SRETENSKII into (20.94). The latter, which seems less laborious, leads to

$$\left. \begin{aligned} a [F''(-a \cos \vartheta) + v^2 F(-a \cos \vartheta) + v^2 h] \sin \vartheta \\ = -va a_0 (1 - \cos \vartheta) - va \sum_{n=1}^{\infty} a_n \sin \vartheta \sin n\vartheta - \sum_{n=1}^{\infty} n a_n \sin n\vartheta. \end{aligned} \right\} \quad (20.97)$$

We shall not discuss SRETENSKII'S further steps to determine the coefficients a_n . However, they lead to expressions of the following kinds for the coefficients:

$$\left. \begin{aligned} a_{2n-1} &= A_{2n-1} a v^2 h + B_{2n-1} a v a_0 + C_{2n-1}, \\ a_{2n} &= B_{2n} a v a_0 + C_{2n}, \quad n = 1, 2, \dots, \end{aligned} \right\} \quad (20.98)$$

where A_n, B_n, C_n are functions of va . Substitution of the coefficients into (20.93) and into (20.93) differentiated once with respect to x and evaluated at $x = -a$ results in equations of the form

$$\left. \begin{aligned} v h &= Q_1 a_0 + R_1 v h + S_1, \\ F'(-a) &= Q_2 a_0 + R_2 v h + S_2, \end{aligned} \right\} \quad (20.99)$$

where Q_i, R_i, S_i are functions of va ; these equations may be used to determine vh and a_0 as functions of va . As long as $a_0 \neq 0$ there will be a singularity at the leading edge.

Once $p(x)$ has been determined approximately, one can compute the lift, drag and moment about, say, the center. To the order of approximation appropriate to the linear theory they are

$$\left. \begin{aligned} L &= \int_{-a}^a p(x) dx, \\ R &= \int_{-a}^a p(x) f'(x) dx, \\ M &= \int_{-a}^a p(x) x dx. \end{aligned} \right\} \quad (20.100)$$

¹ H. GLAUERT: The elements of airfoil and airscrew theory, Chap. XI. Cambridge 1943.

For the flat-plate glider it is possible to give the following asymptotic expressions for these quantities when $\nu a \rightarrow 0$.

$$\left. \begin{aligned} L &= \pi a \rho c^2 \alpha \left[1 - \nu a \left(\pi + \frac{4}{\pi} \right) \right] + O(\nu^2 a^2 \log \nu a), & R &= \alpha L, \\ M &= \frac{1}{2} \pi a^2 \rho c^2 \alpha \left[1 - \nu a \left(\pi + \frac{8}{3\pi} \right) \right] + O(\nu^4 a^4 \log \nu a). \end{aligned} \right\} \quad (20.101)$$

There were first given by SEDOV, but are also derived in the papers by KOCHIN and SRETENSKII.

Fig. 26 reproduces several of SRETENSKII's computed pressure distributions for a flat plate. The predictions of the linearized theory cannot, of course, be expected to be

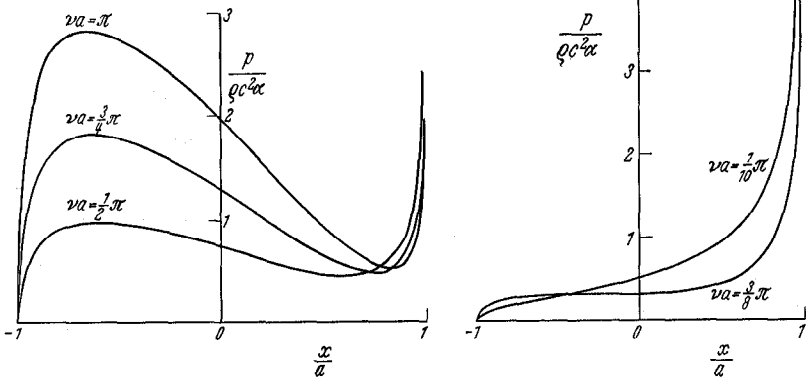


Fig. 26.

accurate near the leading edge. MARUO has corrected his computed points in this region by using the exact theory for a weightless fluid. Both MARUO and SRETENSKII give further computational results which we do not reproduce. MARUO (1959) has also provided experimental confirmation of the predicted pressure distributions.

21. Waves resulting from variable pressure distributions. In the situations considered up to now in this chapter the pressure at the free surface has been taken as constant. We now consider the result of allowing the pressure over the free surface to be a given function of both position and time. Otherwise the fluid is taken to be infinite in horizontal extent and to be either infinitely deep or of uniform depth h . The time variation in pressure will be limited to two cases. In Sect. 21 α a periodically varying pressure is considered; in Sect. 21 β the pressure is taken to move with uniform velocity; Sect. 21 γ gives some references to a combination of these two. In Sect. 22 waves from pressure distributions will be considered again in connection with initial-value problems. Since the methods for finding the velocity potential are similar in most respects to those used in finding the velocity potential for a source, we shall, with one exception, give the results without proof.

Just as in the cases of the stationary source of periodic strength and the moving source of constant strength treated in Sect. 13 γ , we must in the present

situation impose boundary conditions at infinity in order to ensure a unique solution. The imposed conditions, namely the radiation condition and the vanishing of the fluid motion far ahead, respectively, are selected as being physically reasonable. However, one may proceed differently, derive formulas analogous to (13.50) and (13.51) and then find the limit as $t \rightarrow \infty$. The resulting velocity potentials automatically satisfy the correct boundary conditions at infinity. This method has been used, for example, by G. GREEN (1948) in the two-dimensional problems considered in the following two sections, and also by STOKER (1953, 1954).

The theory of wave generation by pressure distributions has an obvious application in oceanographic problems. However, the theory was apparently first developed in an attempt to explain the wave pattern produced by a ship. We shall not attempt to disentangle the history of the subject. For the material covered in Sect. 21 β we call attention to a survey by J. K. LUNDE (1951 b) which also contains a useful bibliography.

α) *Pressure distributions periodic in time.* Three dimensions. The boundary conditions have already been given in Sect. 11. If Φ and p are represented by

$$\left. \begin{aligned} \Phi(x, y, z, t) &= \text{Re } \varphi(x, y, z) e^{-i\sigma t}, & p(x, y, z, t) &= p(x, z) e^{-i\sigma t}, \\ \varphi &= \varphi_1 + i\varphi_2, & p &= p_1 + ip_2, \end{aligned} \right\} \quad (21.1)$$

then the condition on the free surface may be written

$$\varphi_y(x, 0, z) - \frac{\sigma^2}{g} \varphi(x, 0, z) = \frac{i\sigma}{\rho g} p(x, z), \quad (21.2)$$

and the form of the surface is given by

$$\eta(x, z, t) = \text{Re} \left\{ \frac{i\sigma}{g} \varphi(x, 0, z) - \frac{1}{\rho g} p(x, z) \right\} e^{-i\sigma t}. \quad (21.3)$$

In addition, a radiation condition is assumed at infinity [see fifth Eq. (13.9)] and a condition appropriate to the depth of fluid. We shall also assume $p(x, z)$ to be absolutely integrable.

The velocity potential can be expressed as follows:

$$\left. \begin{aligned} \varphi(x, y, z) &= \frac{-i\sigma}{2\pi \rho g} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{p}(\xi, \zeta) d\xi d\zeta \text{PV} \int_{-\infty}^{\infty} \frac{k e^{ky}}{k - \nu} J_0(k \sqrt{(x-\xi)^2 + (z-\zeta)^2}) dk + \\ &+ \frac{\sigma \nu e^{\nu y}}{2\rho g} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{p}(\xi, \zeta) J_0(\nu \sqrt{(x-\xi)^2 + (z-\zeta)^2}) d\xi d\zeta, & \nu &= \frac{\sigma^2}{g}; \end{aligned} \right\} \quad (21.3)$$

and in cylindrical coordinates $x = R \cos \alpha$, $z = R \sin \alpha$ in the form

$$\left. \begin{aligned} \varphi(R, \alpha, y) &= \frac{-i\sigma}{2\pi \rho g} \int_0^{2\pi} \int_0^{\infty} \hat{p}(R', \alpha') R' dR' d\alpha' \text{PV} \int_{-\infty}^{\infty} \frac{k e^{ky}}{k - \nu} \times \\ &\times J_0(k \sqrt{R^2 + R'^2 - 2RR' \cos(\alpha' - \alpha)}) dk + \frac{\sigma \nu e^{\nu y}}{2\rho g} \int_0^{2\pi} \int_0^{\infty} \hat{p}(R', \alpha') \times \\ &\times J_0(\nu \sqrt{R^2 + R'^2 - 2RR' \cos(\alpha' - \alpha)}) R' dR' d\alpha'. \end{aligned} \right\} \quad (21.4)$$

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The addition theorem for J_0 allows one to write¹

$$J_0(k\sqrt{R^2 + R'^2 - 2RR'\cos(\alpha' - \alpha)}) = \sum_{n=0}^{\infty} \varepsilon_n J_n(kR) J_n(kR') \cos n(\alpha' - \alpha), \quad (21.5)$$

$$\varepsilon_0 = 1, \quad \varepsilon_n = 2, \quad n \geq 1.$$

If p is independent of α , one may derive easily

$$\varphi(R, y) = \frac{-i\sigma}{\rho g} \int_0^{\infty} p(R') R' dR' \text{PV} \int \frac{k e^{ky}}{k-y} J_0(kR) J_0(kR') dk + \left. \begin{aligned} &+ \frac{\pi\sigma\nu e^{\nu y}}{\rho g} J_0(\nu R) \int_0^{\infty} p(R') J_0(\nu R') R' dR'. \end{aligned} \right\} \quad (21.6)$$

The asymptotic form for large R of (21.6) is a relatively simple expression:

$$\varphi(R, y) \sim \frac{\pi\sigma\nu e^{\nu y}}{\rho g} \sqrt{\frac{2}{\pi\nu R}} e^{i(\nu R - \frac{\pi}{4})} \int_0^{\infty} p(R') J_0(\nu R') R' dR'. \quad (21.7)$$

We note in passing that the potential function (21.3) or (21.4) can also be obtained as a distribution of sources on the surface [see HUDIMAC (1953, p. 78)]. This may be easily verified as follows. In (13.17'') let $b=0$. Then, using (13.12), one obtains (substituting ξ, ζ for a, c)

$$2 \text{PV} \int \frac{k}{k-y} e^{ky} J_0(k\sqrt{(x-\xi)^2 + (z-\zeta)^2}) dk + i 2\pi\nu e^{\nu y} J_0(\nu\sqrt{(x-\xi)^2 + (z-\zeta)^2}).$$

A distribution over the plane $y=0$ of these sources of strength $+i\sigma p(\xi, \zeta)/4\pi\rho g$ yields (21.3) (we recall that a source of strength m behaves like $-m/r$ near the singularity).

The rate at which the pressure distribution does work upon the fluid can be calculated directly or by using Eq. (8.2). Consider the volume of fluid contained in a large cylinder of radius R_0 . Then, from (8.2), after appropriate linearization, the rate of increase of energy of the fluid is given by

$$\frac{dE}{dt} = \text{Re} \left\{ \int_0^{2\pi} \int_0^{R_0} p(R, \alpha, t) \Phi(R, \alpha, 0, t) R dR d\alpha + \int_0^{2\pi} \int_{-\infty}^0 \Phi_t(R_0, \alpha, y, t) \Phi_R(R_0, \alpha, y, t) R_0 dy d\alpha \right\}.$$

Now substitute (21.1) and take the average over a period, which will clearly be zero. The result may be written:

$$0 = \left[\frac{dE}{dt} \right]_{\text{av}} = \text{Re} \left\{ -\frac{1}{2} \iint p(R, \alpha) \bar{\varphi}_y(R, \alpha, 0) R dR d\alpha + \frac{1}{2} \rho \sigma \iint i \varphi_R(R_0, \alpha, y) \bar{\varphi}(R_0, \alpha, y) R_0 dy d\alpha \right\}. \quad (21.8)$$

¹ See G. N. WATSON: A treatise on the theory of Bessel functions, p. 353. Cambridge 1944.

The first integral gives the average rate W_{av} at which the pressure distribution is working upon the fluid. It must equal minus the second integral. If $p(R, \alpha) = p(R)$, then we may apply (21.7) to obtain a relatively simple expression for the average rate over the whole fluid:

$$W_{av} = \frac{\pi^2 \sigma v^2}{\rho g} \left| \int_0^\infty p(R') J_0(v R') R' dR' \right|^2. \quad (21.9)$$

To carry through the computation when p is not circularly symmetric is more complicated arithmetically, but can be carried through by use of (21.5).

One can find an investigation of the waves resulting from a doubly modulated pressure distribution over a rectangular domain,

$$p = A e^{-i\sigma t} \cos m x \cos n z, \quad |x| \leq a, \quad |z| \leq b,$$

in a paper of SRETENSKII (1956).

If the fluid is of uniform depth h , the expression for the velocity potential is

$$\varphi(x, y, z) = \frac{-i\sigma}{2\pi\rho g} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\xi, \zeta) d\xi d\zeta \text{PV} \int_0^\infty \frac{k \cosh k(y+h)}{k \sinh k h - \nu \cosh k h} \times \left. \begin{aligned} &\times J_0(k\sqrt{(x-\xi)^2+(z-\zeta)^2}) + \frac{\sigma}{2\rho g} \frac{m_0 \cosh m_0(y+h) \sinh m_0 h}{\nu h + \sinh^2 m_0 h} \times \\ &\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\xi, \zeta) J_0(m_0\sqrt{(x-\xi)^2+(z-\zeta)^2}) d\xi d\zeta, \end{aligned} \right\} \quad (21.10)$$

where, as usual, m_0 is the real solution of

$$m_0 \tanh m_0 h - \nu = 0.$$

Other forms of this expression similar to (21.4), (21.6) and (21.7) can be found with no difficulty. We give only the analogue of (21.9):

$$W_{av} = \frac{1}{2} \pi^2 \frac{\sigma \nu m_0}{\rho g} \frac{\sinh 2m_0 h}{\nu h + \sinh^2 m_0 h} \left| \int_0^\infty p(R') J_0(\nu R') R' dR' \right|^2. \quad (21.11)$$

The identities following (13.18) may be used to put both (21.10) and (21.11) into other forms.

Two dimensions. The derivation of the velocity potential will be carried through, at least in part, since it illustrates a nice application of the Plemelj-Sokhotskii formulas. Two complex units will be introduced, as described at the end of Sect. 11. That is, we shall write

$$\left. \begin{aligned} \Phi(x, y, t) &= \varphi_1(x, y) \cos \sigma t + \varphi_2(x, y) \sin \sigma t = \text{Re}_j \varphi e^{-i\sigma t}, \\ p(x, t) &= p_1(x) \cos \sigma t + p_2(x) \sin \sigma t = \text{Re}_j p e^{-i\sigma t}, \\ \varphi &= \varphi_1 + j \varphi_2, \quad p = p_1 + j p_2, \end{aligned} \right\} \quad (21.12)$$

and also introduce a stream function $\psi = \psi_1 + j \psi_2$ and a complex potential

$$f(z) = f_1(z) + j f_2(z), \quad f_k = \varphi_k + i \psi_k, \quad k = 1, 2. \quad (21.13)$$

Then the boundary condition on the free surface may be formulated as follows

$$\text{Im}_j \{f'(x - i0) + i \nu f(x - i0)\} = -\frac{\sigma}{\rho g} j p(x). \quad (21.14)$$

The definition of $g \equiv f' + i\nu f$ may be extended to the whole complex z -plane by reflection, i.e.

$$g(x + iy) = \overline{g(x - iy)}, \quad y > 0.$$

Then the condition (21.14) may be written in the form

$$\text{Im}_i \{f'(x \pm i0) + i\nu f(x \pm i0)\} = \pm \frac{\sigma}{\rho g} j p(x). \tag{21.15}$$

We shall suppose $p(x)$ to be absolutely integrable on the infinite interval. In addition, we shall suppose that either $p(x)$ satisfies a Hölder condition (and is hence continuous) on the whole infinite interval, or else that there are a finite number of segments $(-\infty, b_1)$, (a_2, b_2) , ..., (a_r, ∞) , $b_n < a_{n+1}$, such that $p(x)$ satisfies a Hölder condition on any closed segment interior to one of the above segments, and at an end-point may be expressed in the form

$$p(x) = \frac{q(x)}{(x - c)^\alpha}, \quad 0 \leq \alpha < 1, \quad c = a_i \text{ or } b_i,$$

where $q(x)$ satisfies a Hölder condition at the end. Here a Hölder condition means that for any pair of points x_1, x_2 , $p(x)$ satisfies

$$|p(x_1) - p(x_2)| < A |x_1 - x_2|^\mu, \quad \mu > 0.$$

In the first case f' will be assumed to have no singularities in the whole lower half-plane. In the second case the behavior of $f'(z)$ near an end-point c will be restricted so that it must satisfy

$$|f'(z)| < \frac{C}{|z - c|^\alpha}, \quad 0 \leq \alpha < 1.$$

As usual it will be assumed that $|f'|$ is bounded as $z \rightarrow \infty$ and that only outgoing waves are generated.

The solution of this boundary-value problem for the function $g(z) = f' + i\nu f$ is determined, up to an additive real constant which may be discarded here, by¹

see errata

$$f'(z) + i\nu f(z) = j \frac{\sigma}{\pi \rho g} \int_{-\infty}^{\infty} \frac{p(s)}{z - s} ds, \quad y < 0. \tag{21.16}$$

After integrating the differential equation and selecting the solution so as to represent outgoing waves at $x = \pm \infty$, one obtains finally [the derivation is similar to that of (13.28)]

see errata

$$f(z) = j \frac{\sigma}{\pi \rho g} e^{-i\nu z} \int_{-\infty}^{\infty} p(s) ds \int_{\infty}^z \frac{e^{i\nu \zeta}}{\zeta - s} d\zeta + \frac{\sigma}{\rho g} (1 + ij) e^{-i\nu z} \int_{-\infty}^{\infty} p(s) e^{i\nu s} ds, \tag{21.17}$$

where the path of integration for ζ is taken in the lower half-plane. The asymptotic form of the time-dependent velocity potential is given by

see errata

$$\text{Re}_i f(z) e^{-j\sigma t} \sim \frac{\sigma}{\rho g} e^{-i(\nu z \mp \sigma t)} \int_{-\infty}^{\infty} e^{i\nu s} [p_1(s) \mp i p_2(s)] ds \quad \text{as } x \rightarrow \pm \infty, \tag{21.18}$$

¹ See, e.g., N. I. MUSKHELISHVILI: Singular integral equations, §§ 43, 78. Groningen: Noordhoff 1953.

and the asymptotic form of the free surface by

$$\eta(x, t) \sim \text{Im}_i \left\{ \frac{\sigma^2}{\rho g^2} e^{-i(\nu x \mp \sigma t)} \int_{-\infty}^{\infty} e^{i\nu s} [\dot{p}_1 \mp i \dot{p}_2] ds \right\} \quad \text{as } x \rightarrow \pm \infty. \quad (21.19)$$

From this last expression one can easily derive the average rate at which the pressure system is transferring energy to the fluid:

$$\left. \begin{aligned} W_{av} &= \frac{1}{4} \frac{\sigma \nu}{\rho g} \left\{ \left| \int_{-\infty}^{\infty} (\dot{p}_1 - i \dot{p}_2) e^{i\nu s} ds \right|^2 + \left| \int_{-\infty}^{\infty} (\dot{p}_1 + i \dot{p}_2) e^{i\nu s} ds \right|^2 \right\} \\ &= \frac{1}{2} \frac{\sigma \nu}{\rho g} \left\{ \left[\int_{-\infty}^{\infty} \dot{p}_1 \cos \nu s ds \right]^2 + \left[\int_{-\infty}^{\infty} \dot{p}_1 \sin \nu s ds \right]^2 + \right. \\ &\quad \left. + \left[\int_{-\infty}^{\infty} \dot{p}_2 \cos \nu s ds \right]^2 + \left[\int_{-\infty}^{\infty} \dot{p}_2 \sin \nu s ds \right]^2 \right\}. \end{aligned} \right\} \quad (21.20)$$

The expression for $f(z)$ can be put into several different forms by changing variables and deforming the path of integration appropriately. Thus, if one introduces a new variable λ by $\nu(\zeta - z) = -\lambda(z - s)$ and deforms the resulting path to the x -axis, one obtains

$$f(z) = -j \frac{\sigma}{\pi \rho g} \int_{-\infty}^{\infty} \dot{p}(s) ds \text{PV} \int_0^{\infty} \frac{e^{-i k(z-s)}}{k - \nu} dk + \frac{\sigma}{\rho g} e^{-i\nu z} \int_{-\infty}^{\infty} \dot{p}(s) e^{i\nu s} ds. \quad (21.21)$$

A different deformation of the path leads to

$$\left. \begin{aligned} f(z) &= -j \frac{\sigma}{\pi \rho g} \int_{-\infty}^x \dot{p}(s) ds \int_0^{\infty} \frac{e^{-\mu(z-s)}}{\mu - i\nu} d\mu - j \frac{\sigma}{\pi \rho g} \int_x^{\infty} \dot{p}(s) ds \int_0^{\infty} \frac{e^{\mu(z-s)}}{\mu + i\nu} d\mu + \\ &\quad + \frac{\sigma}{\rho g} (1 + i\dot{\eta}) \int_{-\infty}^x \dot{p}(s) e^{-i\nu(z-s)} ds + \frac{\sigma}{\rho g} (1 - i\dot{\eta}) \int_x^{\infty} \dot{p}(s) e^{-i\nu(z-s)} ds. \end{aligned} \right\} \quad (21.22)$$

For fluid of depth h an expression for the complex velocity potential analogous to (21.10) and (21.21) is

$$\left. \begin{aligned} f(z) &= -j \frac{\sigma}{\pi \rho g} \int_{-\infty}^{\infty} ds \dot{p}(s) \text{PV} \int_0^{\infty} \frac{\cos k(z-s+i h)}{k \sinh k h - \nu \cosh k h} dk + \\ &\quad + \frac{\sigma}{\rho g} \frac{\sinh m_0 h}{\nu h + \sinh^2 m_0 h} \int_{-\infty}^{\infty} \dot{p}(s) \cos m_0(z-s+i h) ds. \end{aligned} \right\} \quad (21.23)$$

One will find both the two- and the three-dimensional case of a periodic pressure distribution over infinitely deep water discussed in LAMB (1904, pp. 387–393). STOKER (1957, Chap. 4) discusses in considerable detail the two-dimensional problem of waves generated by a periodic uniform pressure applied over a finite interval.

β) *Moving pressure distributions.* In this section we shall suppose that a fixed pressure distribution is moving with a constant velocity c . Thus the motion may be treated as time-independent in a coordinate system moving with the

pressure distribution. The boundary condition at the free surface is given by [see (11.3)]

$$\varphi_y(x, 0, z) + \frac{1}{\nu} \varphi_{xx}(x, 0, z) = \frac{c}{\rho g} p_x(x, z), \quad \nu = \frac{g}{c^2}, \tag{21.24}$$

and the form of the free surface by

$$\eta(x, z) = \frac{c}{g} \varphi_x(x, 0, z) - \frac{1}{\rho g} p(x, z). \tag{21.25}$$

In addition, we shall assume vanishing of the fluid motion far ahead, i.e. as $x \rightarrow +\infty$, and the usual conditions appropriate to infinite or finite depth. p will be assumed to be absolutely integrable and to vanish for sufficiently large values of $x^2 + z^2$; however, the latter condition can be weakened.

Results will be given without proof since their derivations are similar to those in Sects. 13 γ and 21 α . The results for two and three dimensions will be separated.

Three dimensions. The expression for the velocity potential for infinite depth of fluid can be given as follows:

$$\left. \begin{aligned} \varphi(x, y, z) = & \frac{1}{\pi^2 \rho c} \int_{-\infty}^{\infty} d\xi d\zeta p(\xi, \zeta) \int_0^{\frac{1}{2}\pi} d\vartheta \sec \vartheta \text{PV} \int_0^{\infty} dk \frac{k e^{ky}}{k - \nu \sec^2 \vartheta} \times \\ & \times \sin [k(x - \xi) \cos \vartheta] \cos [k(z - \zeta) \sin \vartheta] - \\ & - \frac{\nu}{\pi \rho c} \int_{-\infty}^{\infty} d\xi d\zeta p(\xi, \zeta) \int_0^{\frac{1}{2}\pi} d\vartheta \sec^3 \vartheta e^{-\nu \sec^2 \vartheta} \times \\ & \times \cos [\nu(x - \xi) \sec \vartheta] \cos [\nu(z - \zeta) \sec^2 \vartheta \sin \vartheta]. \end{aligned} \right\} \tag{21.26}$$

The rate at which the pressure distribution is transferring energy to the fluid is given by

$$W = - \int_{-\infty}^{\infty} \int p(x, z) \varphi_y(x, -0, z) dx dz. \tag{21.27}$$

This may be computed directly from (21.26). The first term gives no contribution since it is an odd function of $x - \xi$ [cf. the evaluation of (20.66)]. The final result may be expressed as follows:

$$\left. \begin{aligned} W = & \frac{\nu^2}{\pi \rho c} \int_0^{\frac{1}{2}\pi} d\vartheta \sec^5 \vartheta \int_{-\infty}^{\infty} dx dz \int_{-\infty}^{\infty} d\xi d\zeta p(x, y) p(\xi, \zeta) \times \\ & \times \cos (\nu \sec^2 \vartheta [(x - \xi) \cos \vartheta + (z - \zeta) \sin \vartheta]). \\ = & \frac{\nu^2}{\pi \rho c} \int_0^{\frac{1}{2}\pi} d\vartheta \sec^5 \vartheta [P^2(\vartheta) + Q^2(\vartheta)], \\ P(\vartheta) = & \int_{-\infty}^{\infty} \int dx dz p(x, z) \cos [\nu \sec^2 \vartheta (x \cos \vartheta + z \sin \vartheta)], \\ Q(\vartheta) = & \int_{-\infty}^{\infty} \int dx dz p(x, z) \sin [\nu \sec^2 \vartheta (x \cos \vartheta + z \sin \vartheta)]. \end{aligned} \right\} \tag{21.28}$$

If the pressure distribution is given in cylindrical coordinates, $p = p(R, \alpha)$, $x = R \cos \alpha$, $z = R \sin \alpha$, then one may express, say, $P(\vartheta)$ in the form

$$P(\vartheta) = \int_0^{\infty} dR \int_0^{2\pi} d\alpha R p(R, \alpha) \cos [\nu R \sec^2 \vartheta \cos (\alpha - \vartheta)] \tag{21.29}$$

and a similar formula for $Q(\vartheta)$. If p depends only upon R , then $Q(\vartheta) \equiv 0$ and

$$P(\vartheta) = \int_0^\infty 2\pi R p(R) J_0(\nu R \sec^2 \vartheta) dR. \tag{21.30}$$

If the fluid is of depth h , the velocity potential is given by

$$\begin{aligned} \varphi(x, y, z) = & \frac{1}{\pi^2 \rho c} \iint_{-\infty}^\infty d\xi d\zeta p(\xi, \zeta) \int_0^{\frac{1}{2}\pi} d\vartheta \sec \vartheta \times \\ & \times \text{PV} \int_0^\infty dk \frac{h \cosh k(y+h) \operatorname{sech} kh}{k - \nu \sec^2 \vartheta \tanh kh} \sin [k(x-\xi) \cos \vartheta] \cos [k(z-\zeta) \sin \vartheta] - \\ & - \frac{1}{\pi \rho c} \iint_{-\infty}^\infty d\xi d\zeta p(\xi, \zeta) \int_{\vartheta_0}^{\frac{1}{2}\pi} d\vartheta \sec \vartheta \frac{k_0 \cosh k_0(y+h) \operatorname{sech} k_0 h}{1 - \nu h \sec^2 \vartheta \operatorname{sech}^2 k_0 h} \times \\ & \times \cos [k_0(x-\xi) \cos \vartheta] \cos [k_0(z-\zeta) \sin \vartheta], \end{aligned} \tag{21.31}$$

where $k_0 = k_0(\vartheta)$ is the positive real root of

$$k - \nu \sec^2 \vartheta \tanh kh = 0, \quad \vartheta_0 < \vartheta < \frac{1}{2}\pi,$$

and

$$\vartheta_0 = \begin{cases} \text{Arc cos } \sqrt{\nu h} & \text{if } \nu h \equiv \frac{g h}{c^2} < 1, \\ 0 & \text{if } \nu h > 1. \end{cases}$$

The rate of transfer of energy may again be computed from (21.27) and again only the second integral gives a nonvanishing contribution. The result may be expressed in several forms analogous to (21.28) to (21.30):

$$\begin{aligned} W = & \frac{c}{\pi \rho g} \int_{\vartheta_0}^{\frac{1}{2}\pi} d\vartheta \frac{k_0^3 \cos \vartheta}{1 - \nu h \sec^2 \vartheta \operatorname{sech}^2 k_0 h} \iint_{-\infty}^\infty dx dz \iint_{-\infty}^\infty d\xi d\zeta p(x, z) p(\xi, \zeta) \times \\ & \times \cos (k_0 [(x-\xi) \cos \vartheta + (z-\zeta) \sin \vartheta]), \\ = & \frac{c}{\pi \rho g} \int_{\vartheta_0}^{\frac{1}{2}\pi} d\vartheta \frac{k_0^3 \cos \vartheta}{1 - \nu h \sec^2 \vartheta \operatorname{sech}^2 k_0 h} [P^2(\vartheta) + Q^2(\vartheta)], \\ P(\vartheta) = & \iint_{-\infty}^\infty p(x, z) \cos [k_0(x \cos \vartheta + z \sin \vartheta)] dx dy, \\ Q(\vartheta) = & \iint_{-\infty}^\infty p(x, z) \sin [k_0(x \cos \vartheta + z \sin \vartheta)] dx dy. \end{aligned} \tag{21.32}$$

In cylindrical coordinates formulas (21.29) and (21.30) carry over to the present situation with ν replaced by k_0 .

The asymptotic form of the free surface for either infinite or finite depth is much more complicated to analyze than for the stationary periodically oscillating pressure distribution of the preceding section. Although it is not strictly necessary to do so, it has been customary in this type of analysis to consider the special case of a "concentrated pressure point". To derive the velocity potential for the pressure point consider the pressure distribution defined by

$$p(R) = \begin{cases} \frac{p_0}{\pi R_0^2}, & R \leq R_0, \\ 0, & R > R_0. \end{cases} \tag{21.33}$$

Substitute in (21.26) or (21.31) and take the limit as $R_0 \rightarrow 0$. Then (21.26) becomes

$$\left. \begin{aligned} \varphi(x, y, z) = & -\frac{\dot{p}_0}{\pi^2 \dot{p} c} \int_0^{\frac{1}{2}\pi} d\vartheta \sec \vartheta \text{PV} \int_0^\infty dk \frac{k e^{ky}}{k - v \sec^2 \vartheta} \sin(kx \cos \vartheta) \cos(kz \sin \vartheta) - \\ & - \frac{v \dot{p}_0}{\pi \rho c} \int_0^{\frac{1}{2}\pi} d\vartheta \sec^3 \vartheta e^{y v \sec^2 \vartheta} \cos(vx \sec \vartheta) \cos(vz \sec^2 \vartheta \sin \vartheta). \end{aligned} \right\} \quad (21.34)$$

The velocity potential for the pressure point in fluid of finite depth is derived similarly. The equation representing the free surface may now be obtained immediately from (21.25):

$$\eta(x, z) = \frac{c}{g} \varphi_x(x, 0, z), \quad x^2 + z^2 > 0. \quad (21.35)$$

The velocity potential (21.34) is very similar to that of a submerged source in steady motion [see Eq. (13.36)] and the method sketched in Sect. 13 for the derivation of the asymptotic expression (13.42) can be carried over directly to the moving pressure point.

The result, expressed in cylindrical coordinates, is as follows:

$$\left. \begin{aligned} \text{for } 0 \leq \alpha < \pi - \text{Arc sin } \frac{1}{3} = \alpha_c: \\ \eta(R, \alpha) &= O(\nu R)^{-2}; \\ \text{for } \alpha = \alpha_c: \\ \eta(R, \alpha_c) &= -\frac{\dot{p}_0 v^2}{\pi \rho g} 2^{-\frac{1}{2}} 3^{\frac{1}{2}} \Gamma\left(\frac{1}{3}\right) (\nu R)^{-\frac{1}{3}} \sin\left(\frac{\sqrt{3}}{2} \nu R\right) + O((\nu R)^{-\frac{2}{3}}); \\ \text{for } \alpha_c < \alpha < \pi: \\ \eta(R, \alpha) &= \frac{\dot{p}_0 v^2}{\pi \rho g} \sqrt{\frac{2\pi}{\nu R}} \frac{1}{[1 - 9 \sin^2 \alpha]^{\frac{1}{4}}} \left\{ \sec^{\frac{5}{2}} \vartheta_1 \sin\left(\nu R \mu_1 - \frac{\pi}{4}\right) + \right. \\ & \quad \left. + \sec^{\frac{5}{2}} \vartheta_2 \sin\left(\nu R \mu_2 + \frac{\pi}{4}\right) \right\} + O((\nu R)^{-1}); \\ \text{for } \alpha = \pi: \\ \eta(R, \pi) &= -\frac{\dot{p}_0 v^2}{\pi \rho g} \sqrt{\frac{2\pi}{\nu R}} \sin\left(\nu R + \frac{\pi}{4}\right). \end{aligned} \right\} \quad (21.36)$$

The variables $\vartheta_1, \vartheta_2, \mu_1, \mu_2$ are the same ones defined following (13.42), where certain properties are also given. For the pressure point Fig. 1a is not quite accurate as a description of the wave crests in the region $|\pi - \alpha| < \alpha_c$ since the phase in (21.36) has been shifted by $\frac{1}{2}\pi$; the wave crests in Fig. 1a should be moved back a distance $\frac{1}{2}\pi$. Also, in the neighborhood of $\alpha = \alpha_c$ the expressions in (21.36) are inaccurate; in this region η may be expressed in terms of Airy functions [see URSELL (1960)].

The wave pattern resulting from a moving pressure distribution has been the subject of many investigations, starting apparently with KELVIN (1906). His aim was to explain the typical wave pattern found behind a ship. The procedure is quite reasonable as a method for obtaining a qualitative prediction of a ship's wave pattern, since a moving ship has associated with it a pressure distribution around the wetted hull. The obvious disadvantage of the method is that it gives no connection between the geometry of the hull and the wave pattern. For this the "thin ship" approximation of Sect. 20 β is better within its range of applicability. The single pressure point can be taken to represent approximately

a ship moving at high speed (more accurately, at high Froude number c/\sqrt{Lg} , where L is the length), say a fast motor boat.

For detailed investigation of the asymptotic expression one should refer to HOGNER (1923), PETERS (1949), BARTELS and DOWNING (1955) (who do not restrict themselves to a pressure point) and STOKER (1957, Chap. 8). The necessary modifications for finite depth have been made by HAVELOCK (1908) and TETURÔ INUI (1936) and are described qualitatively in the discussion following (13.42). One can find an exposition of the theory of waves generated by moving pressure distributions in a report by LUNDE (1951 b). Several papers by HAVELOCK (1909, 1914b, 1919, 1922) take up the wave resistance ($=W/c$) of a pressure distribution. HOGNER (1928) has also considered the wave resistance and gives essentially (21.28).

In the preceding considerations we have assumed that, as $x^2 + z^2 \rightarrow \infty$, $p(x, z)$ approached zero sufficiently quickly so that it might be represented as a Fourier integral. It is also possible to proceed somewhat differently, assume $p(x, y)$ periodic in one or both variables and use a Fourier series representation. This has been done, for example, by VOIT (1957a), who has considered for both infinite and finite depth a moving pressure distribution of the following form:

$$p(x, z) = \begin{cases} P(z) \sum_{n=1}^{\infty} a_n \cos n k x, & |z| < h \\ 0, & |z| > h. \end{cases}$$

The waves resulting from a pressure point moving parallel to beaches forming angles of 30° and 45° with the surface have been treated by HANSON (1926); in the same paper he also treats the waves formed by a pressure point moving over a two-layered fluid. A detailed investigation of this last topic is given in a paper of SRETENSKIĬ'S (1934).

Two dimensions. By introducing a stream function $\psi(x, y)$ and a complex potential $f = \phi + i\psi$, the free surface boundary condition can be put into a form analogous to (21.14), namely,

$$\text{Re} \{ f'(x - i0) + i\nu f(x - i0) \} = \frac{1}{\rho c} p(x), \quad \nu = \frac{g}{c^2}. \tag{21.37}$$

In addition, we assume $|f'|$ bounded as $z \rightarrow \infty$ and also $\lim_{x \rightarrow \infty} |f'| = 0$. We shall assume $p(x)$ subject to the same limitations as in Sect. 21 α .

One may apply the same method of analysis to derive the following forms for the complex velocity potential:

$$\left. \begin{aligned} f(z) &= \frac{e^{-i\nu z}}{\pi i \rho c} \int_{-\infty}^{\infty} ds \, p(s) \int_{\infty}^z \frac{e^{i\nu \zeta}}{\zeta - s} d\zeta \\ &= -\frac{1}{\pi i \rho c} \int_{-\infty}^{\infty} ds \, p(s) \text{PV} \int_0^{\infty} \frac{e^{-i\lambda(z-s)}}{\lambda - \nu} d\lambda - \frac{e^{-i\nu z}}{\rho c} \int_{-\infty}^{\infty} p(s) e^{i\nu s} ds \\ &= -\frac{1}{\pi i \rho c} \int_{-\infty}^x ds \, p(s) \int_0^{\infty} \frac{e^{-\mu(z-s)}}{\mu - i\nu} d\mu - \frac{1}{\pi i \rho c} \int_x^{\infty} ds \, p(s) \int_0^{\infty} \frac{e^{\mu(z-s)}}{\mu + i\nu} d\mu - \\ &\quad - \frac{2}{\rho c} e^{-i\nu z} \int_x^{\infty} p(s) e^{i\nu s} ds, \end{aligned} \right\} \tag{21.38}$$

where the path of integration for ζ in the first expression is taken in the lower half-plane. The rate at which the pressure distribution transfers energy to the fluid is easily found from formula (21.27) and the second expression for $f(z)$ to be

$$W = \frac{\nu}{\rho c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x) p(\xi) \cos \nu(x - \xi) dx d\xi. \tag{21.39}$$

If the fluid is of depth h , the complex velocity potential may be given in a form analogous to the second formula of (21.37):

$$f(z) = \left. \begin{aligned} & \frac{\nu c}{\pi \rho g} \int_{-\infty}^{\infty} ds p(s) \text{PV} \int_{-\infty}^{\infty} dk \frac{\sin k(z - s + ih) \operatorname{sech} kh}{k - \nu \tanh kh} - \\ & - \frac{\nu c}{\rho g} \int_{-\infty}^{\infty} p(s) \frac{\cos k_0(z - s + ih) \operatorname{sech} k_0 h}{1 - \nu h \operatorname{sech}^2 k_0 h} ds, \end{aligned} \right\} \tag{21.40}$$

where k_0 is the real positive root of

$$k_0 - \nu \tanh k_0 h = 0$$

and exists only if $\nu h = gh/c^2 > 1$; if $\nu h \leq 1$, the last term in (21.40) must be deleted. The rate at which the pressure is doing work upon the fluid is given by

$$W = \begin{cases} \frac{k_0^2 c}{\rho g} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x) p(\xi) \frac{\cos k_0(x - \xi)}{1 - \nu h \operatorname{sech}^2 k_0 h} dx d\xi & \text{if } \nu h > 1, \\ 0 & \text{if } \nu h < 1. \end{cases} \tag{21.41}$$

The absence of the second term in (21.40) and the vanishing of W when $\nu h < 1$ correspond to the absence of an infinite train of trailing waves. A similar phenomenon occurs behind a moving singularity in two dimensions [cf. (13.46) to (13.48) and the following remarks].

For either (21.37) or (21.39) the form of the free surface can be written down immediately from the formula

$$\eta(x) = \frac{1}{c} \psi(x, 0). \tag{21.41}$$

We shall not carry out the details. The asymptotic form of the surface behind a two-dimensional "pressure point", or also a distributed pressure, is much easier in two than in three dimensions and we again omit a detailed statement. However, the problem has been treated by KELVIN (1905) and is discussed in LAMB (1932, § 242 to 245), both for infinite and finite depth. It is also discussed in the paper of PETERS (1949) already cited in connection with the three-dimensional problem.

Derivations of the complex velocity potential may be found in the papers of SRETENSKII (1934, 1940), SEDOV (1936), KOCHIN (1939) and HASKIND (1943a) already cited in connection with planing surfaces. We refer also to papers of DEAN (1947) and TIMMAN and VOSSERS (1955).

γ) *Moving periodic pressure distributions.* It is clearly possible to combine the cases considered in Sects. 21 α and β and consider the waves resulting from a pressure distribution expressible in the form

$$p(x, z, t) = p_1(x - ct, z) \cos \sigma t + p_2(x - ct, z) \sin \sigma t,$$

where the coordinates are fixed in space. The resulting velocity potential will be analogous to (13.52) for the three-dimensional case, if one is dealing with a "pressure point".

We shall not reproduce the formulas here. However, the analogues of (13.49) and (13.53) for pressure distributions may be found in Eqs. (22.48) and (22.49) or in the cited report of LUNDE (1951b), and from these the required velocity potential may be found. For two-dimensional motion the details, carried out by this procedure, may be found in papers by KAPLAN (1957) and WU (1957).

22. Initial-value problems. In the special problems considered in Sects. 17 to 21, the dependence upon the time has been precipitated out, either by assuming the motion steady in a moving coordinate system or by assuming a harmonic dependence upon the time. In this section we shall consider motions in which the displacement and velocity of the surface are specified at some instant of time, say $t=0$, the motions of any solid boundaries are given for each instant $t \geq 0$ (except at the very end where freely floating bodies are considered) and the pressure distribution over the free surface is a given function for $t \geq 0$.

It is not usually possible in the most general situations to give explicit solutions for such problems. However, VOLTERRA (1934) has proved a uniqueness theorem and has shown how to reduce the problem to finding an appropriate GREEN'S function. His results were later rediscovered and extended to a wider class of problems by FINKELSTEIN (1957) [see also STOKER (1957, Chap. 6)]. However, the use of GREEN'S functions for initial-value problems extends back even earlier, at least to the papers of HADAMARD (1910, 1916) and BOULIGAND (1913). These theorems are discussed in Sect. 22 α .

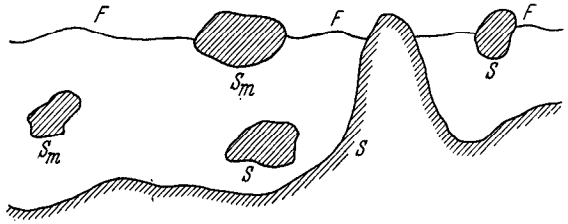


Fig. 27.

One of the classical problems in this category is associated with the names CAUCHY (1827) and POISSON (1815). In this problem the fluid is infinite in horizontal extent, without obstructions, and either infinitely deep or of uniform depth h . At the initial instant $t=0$, the form of the surface and its vertical velocity are given and one seeks the subsequent motion. Such problems have already been discussed at some length in Sect. 15. However, in that section interest centered upon investigation of certain aspects of the subsequent motion rather than upon obtaining the solution. In addition, the treatment of that section was limited to two-dimensional motion, although the methods could have been extended to three-dimensional motion.

The history of this problem, including an exposition of the methods used by various authors, is included in a paper by RISSER (1924, pp. 113–144). Another expository account can be found in VERGNE (1928, Chap. I). The problem is discussed here in Sect. 22 β .

In Sect. 22 γ several special initial-value problems are discussed.

α) *Some general theorems.* Let the fluid be bounded by the free surface F , fixed surfaces S and the surfaces of a finite number of bodies of bounded extent undergoing specified motions of small amplitude about equilibrium positions S_m (see Fig. 27). Let the pressure distribution on the free surface F , also be a given

function $\dot{p}(x, z, t)$. Furthermore, at time $t=0$ let the initial displacement and vertical velocity of the free surface be given functions

$$\eta(x, z, 0), \quad \eta_t = (x, z, 0). \quad (22.1)$$

The boundary conditions to be satisfied by the velocity potential $\Phi(x, y, z, t)$ are [see Eq. (11.4)]

$$\left. \begin{aligned} \Phi_{tt}(x, 0, z, t) + g\Phi_y(x, 0, z, t) &= -\frac{1}{\rho}\dot{p}_t(x, z, t) \quad \text{on } F, \\ \Phi_n &= 0 \quad \text{on } S, \\ \Phi_n &= V_n(t) \quad \text{on } S_m, \\ \Phi_t(x, 0, z, 0) &= -g\eta(x, z, 0) - \frac{1}{\rho}\dot{p}(x, z, 0) \quad \text{on } F, \\ \Phi_y(x, 0, z, 0) &= \eta_t(x, z, 0) \quad \text{on } F. \end{aligned} \right\} \quad (22.2)$$

Here F means that part of the plane $y=0$ occupied by fluid when everything is at rest. In addition, it will be assumed that, for each t , there is a bound B and a distance r_0 such that $|\Phi|$, $|\Phi_t|$, $|\text{grad } \Phi|$ and $|\text{grad } \Phi_t|$ are each less than B for $x^2 + y^2 + z^2 > r_0^2$.

Let us now suppose that it is possible to find a source function G of the following nature:

$$G(x, y, z; \xi, \eta, \zeta; t, \tau) = \frac{1}{v} + H(x, y, z; \xi, \eta, \zeta; t, \tau), \quad (22.3)$$

where as usual $r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2$ and H is harmonic for $y \leq 0$, and in addition G satisfies

$$\left. \begin{aligned} G(x, y, z; \xi, \eta, \zeta; t, t) &= G_t(x, y, z; \xi, \eta, \zeta; t, t) = 0, \\ G_n(x, y, z; \xi, \eta, \zeta; t, \tau) &= 0 \quad \text{for } (x, y, z) \text{ on } S \text{ and for all } t. \end{aligned} \right\} \quad (22.4)$$

This function has already been constructed in two cases. If there are no fixed boundaries and the fluid is infinitely deep, the function defined in (13.49) satisfies the conditions after slight modifications: replace (a, b, c) by (ξ, η, ζ) , set $m(t) = 1$, and extend the definition of Φ [of (13.49)] to negative t by $\Phi(x, y, z, -t) = \Phi(x, y, z, t)$. Then we take

$$G(x, y, z; \xi, \eta, \zeta; t, \tau) = \Phi(x, y, z; t - \tau) = G(x, y, z; \xi, \eta, \zeta; \tau, t).$$

Similarly, the function defined in (13.53) allows one to construct G when the fixed boundary consists of a horizontal bottom at $y = -h$. For the first G , FINKELSTEIN (1957, Appendix) has shown that G is $O(R^{-2})$ and G_R and G_y are $O(R^{-3})$ as $R \rightarrow \infty$, where $R^2 = (x - \xi)^2 + (z - \zeta)^2$; for the second G , FINKELSTEIN (1957, § 3) has shown that G , G_R and G_y are $o\left(\exp\left(\frac{-\pi}{2h} + \varepsilon\right)R\right)$ as $R \rightarrow \infty$ for arbitrary $\varepsilon > 0$.

Now apply GREEN'S theorem to the functions Φ_t and G and the region of fluid bounded by the surfaces S_m , the fixed boundaries S and a large sphere Ω of radius ρ and center at the origin, where ρ is chosen large enough to include all the surfaces S_m . Only parts of F , S and Ω will serve as bounding surfaces, and we shall call these parts F' , S' and Ω' , respectively. Then

$$\Phi_t(x, y, z, t) = \frac{1}{4\pi} \iint_{F'+S_m+S'+\Omega'} [G(\xi, \eta, \zeta; x, y, z; t, \tau) \Phi_{t\nu}(\xi, \eta, \zeta, t) - \Phi_t G_\nu] d\sigma, \quad (22.5)$$

where ν is the exterior normal. The right-hand side is actually independent of τ since τ enters only through the function H which is harmonic. The integral over