Let us define two averages, one for functions of x:

$$\bar{f}(t) = \int_{-\infty}^{\infty} f(x)\,\eta_R^2(x,t)\,dx \Big/ \int_{-\infty}^{\infty} \eta_R^2(x,t)\,dx,$$

and one for functions of k:

$$\bar{\Phi} = \int_{-\infty}^{\infty} \Phi(k)\,E(k)\,E^*(k)\,dk \Big/ \int_{-\infty}^{\infty} E(k)\,E^*(k)\,dk.$$

Then, assuming that the various quantities in question exist, one finds, using well known theorems on Fourier transforms[1],

$$\bar{x}_R(t) = \bar{x}_R(0) + \bar{\sigma}'\,t \tag{15.3}$$

and

$$\left.\begin{aligned}\overline{[x-\bar{x}_R(t)]^2} = \overline{[x-\bar{x}_R(0)]^2} + t\left\{\overline{\sigma'\left[i\log\frac{E^*}{E}\right]'} - \bar{\sigma}'\,\overline{\left[i\log\frac{E^*}{E}\right]'}\right\} + \\ + t^2\left\{\overline{\sigma'^2} - \bar{\sigma}'^2\right\}.\end{aligned}\right\} \tag{15.4}$$

Thus, on this definition the average position of η_R moves to the right with constant velocity $\bar{\sigma}'$ and the hump spreads according to a quadratic law. We note that the coefficient of t^2 is positive except if σ' is a constant, when it vanishes. It may become infinite, and, in fact, does so for infinitely deep water if the gravest modes are present, i.e., if $\int \eta_R\,dx \neq 0$. The coefficient of t vanishes if σ' is constant or if $[i\log E^*/E]'$ is constant; the latter will occur if $\eta(x,0)$ is either symmetric or antisymmetric about some point x_0, but this does not exhaust all possibilities. The sign of this term does not seem to be determined, so that the spread of the hump may conceivably decrease before starting to increase.

Investigations of the motion of the average position of the hump and of its spread give only a rather crude picture of its behavior. By other methods outlined below one may obtain further insight into the motion.

We begin by applying the analysis of the average motion to that part of η_R resulting from only a narrow band in its spectrum. Let

$$\eta_R(x,t;k_0,\varepsilon) = \mathrm{Re}\int_{k_0-\varepsilon}^{k_0+\varepsilon} \tfrac{1}{2}E(k)\,e^{-i(kx-\sigma(k)t)}\,dk. \tag{15.5}$$

We shall call this a *wave packet*. The average position satisfies

$$\bar{x}_R(t;k_0,\varepsilon) = \bar{x}_R(0;k_0,\varepsilon) + \bar{\sigma}'(k_0,\varepsilon)\,t,$$

where $\bar{\sigma}'(k,\varepsilon)$ is now the average of $\sigma'(k)$ over the narrow band $[k_0-\varepsilon, k_0+\varepsilon]$. The narrower the band, the closer $\bar{\sigma}'(k_0,\varepsilon)$ is to $\sigma'(k_0)$, assuming the latter continuous. As a limiting case we shall say that the wave packet resulting from an infinitesimal band about k_0 moves with velocity $\sigma'(k_0)$. It is customary to call $\sigma'(k)$ the *group velocity*. This is the same as the phase velocity $\sigma(k)/k$ only if $\sigma = ak$. A wave packet will spread with passage of time unless the two velocities are equal, for (15.4) is applicable to the wave packet with the restricted definition of average. As might be expected, the smaller the width of the band, the smaller the coefficient of t^2 and the smaller the rate of growth. However, as we shall see below, the initial spread may be wide for a narrow band.

The wave packet (15.5) may also be investigated by a different method. Let us expand $\sigma(k)$ in the first few terms of a Taylor series about k_0:

$$\sigma(k) = \sigma(k_0) + \sigma'(k_0)(k-k_0) + \tfrac{1}{2}\int_{k_0}^{k} \sigma''(\varkappa)(k-\varkappa)\,d\varkappa. \tag{15.6}$$

[1] See, e.g., S. Bochner and K. Chandrasekharan: Fourier transforms, Chap. IV, § 2. Princeton 1949.

We may then write

$$\left.\begin{aligned}\eta_R(x,t;k_0,\varepsilon) &= \operatorname{Re}\tfrac{1}{2}\,\mathrm{e}^{-i[k_0x-\sigma(k_0)t]}\Bigg\{\int\limits_{k_0-\varepsilon}^{k_0+\varepsilon} E(k)\,\mathrm{e}^{-i[x-\sigma'(k_0)t](k-k_0)}\,dk\;+\\ &\quad+\int\limits_{k_0-\varepsilon}^{k_0+\varepsilon} E(k)\,\mathrm{e}^{-i[x-\sigma'(k_0)t](k-k_0)}\left[\exp\left(-\tfrac{1}{2}\,i\,t\int\limits_{k_0}^{k}\sigma''(\varkappa)\,(k-\varkappa)\,d\varkappa\right)-1\right]dk\Bigg\}\\ &= \operatorname{Re}\tfrac{1}{2}\,\mathrm{e}^{-i[k_0x-\sigma(k_0)t]}M\big(x-\sigma'(k_0)\,t;\,k_0,\varepsilon\big)+R.\end{aligned}\right\}\tag{15.7}$$

Using the inequality $|\mathrm{e}^{iu}-1|\leqq|u|$, one finds

$$|R|\leqq\tfrac{1}{4}\,t\,\varepsilon^3\max_{|k-k_0|<\varepsilon}|E(k)|\cdot\max_{|k-k_0|<\varepsilon}|\sigma''(k)|.\tag{15.8}$$

The remainder can thus be made small by taking ε or t small enough. However, once ε is fixed, R will eventually become large as t increases. Let us suppose, however, that t and ε are small enough so that the first term determines the main features of the motion. The first factor represents a periodic wave of wave number k_0 moving with its phase velocity $\sigma(k_0)/k_0$. The second factor, determining the amplitude of the first, represents a profile being translated to the right with velocity $\sigma'(k_0)$. Thus one may say that the gross outline of the surface is moving to the right with the group velocity. One may see this more clearly if one assumes ε small enough so that we may take $E(k)$ as constant over the band width. Then

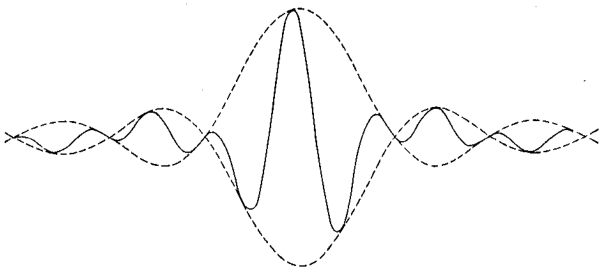

Fig. 8.

$$M\big(x-\sigma'(k_0)\,t;\,k_0,\varepsilon\big)=E(k_0)\,\frac{\sin\left(x-\sigma'(k_0)\,t\right)\varepsilon}{x-\sigma'(k_0)\,t},$$

and $\eta_R(x,t;k_0,\varepsilon)$ appears approximately as in Fig. 8. Here the dotted enveloping curves represent $\pm\frac{1}{2}M$ and move to the right with velocity $\sigma'(k_0)$, whereas the inscribed solid curves represent the first factor and move to the right with phase velocity $\sigma(k_0)/k_0$. The whole moves as a fixed pattern only if the two velocities are equal. Otherwise, assuming $\sigma'(k_0)<\sigma(k_0)/k_0$, the inscribed curves will progress through the wave packet, gradually disappearing at the right. For a very narrow band the packet will spread wide before its first zero on either side of the maximum.

A disadvantage of this last analysis is that it becomes less and less accurate as t becomes large. However, there exists another approximation to $\eta_R(x,t)$ for large values of t which helps to complete the picture. This ultimate behavior of η_R can to some extent be predicted from the analysis of the average motion of a wave band. If we think of η_R as made up of the contributions from a number of narrow wave bands, we know that each contribution is moving with the average group velocity of the band. Thus after some time we shall expect that these various contributions will have separated from one another, with the bands about the gravest modes, which travel fastest, having progressed the furthest. This prediction will be confirmed.

What is needed for this final approximation is an asymptotic expansion for large t. It is convenient to express η_R in the slightly altered form

$$\eta_R(x,t)=\tfrac{1}{2}\int\limits_{-\infty}^{\infty}E(k)\,\mathrm{e}^{-i\left[k\frac{x}{t}-\sigma(k)\right]t}\,dk\tag{15.9}$$

and to consider it as depending upon the two parameters x/t and t. Then for each value of x/t we shall give an expansion for large values of t. For a derivation of the expansion we refer to Stoker (1957, § 6.8) or Erdélyi (1956, § 2.9).

Let the functions $k_r(x/t)$, $r=1, 2, \ldots, n$, be defined by

$$\sigma'(k_r) = x/t; \tag{15.10}$$

i.e. we allow the possibility of several roots. In the situation of interest to us there will be either one or two roots, or none. The asymptotic expression for η_R is then given by

$$\left.\begin{aligned} \eta_R(x,t) = \operatorname{Re}\sum_r \frac{1}{2} E(k_r)\left[\frac{2\pi}{t\,|\sigma''(k_r)|}\right]^{\frac{1}{2}} \mathrm{e}^{-i[k_r x - \sigma(k_r)t - \frac{1}{4}\pi \operatorname{sgn}\sigma''(k_r)]} + \\ + \operatorname{Re}\sum_r \frac{1}{2} E(k_r)\frac{1}{\sqrt{3}}\Gamma\left(\frac{1}{3}\right)\left[\frac{6}{t\,|\sigma'''(k_r)|}\right]^{\frac{1}{3}} \mathrm{e}^{-i[k_r x - \sigma(k_r)t]} + O(t^{-\frac{2}{3}}), \end{aligned}\right\} \tag{15.11}$$

where the first summation is over all values of r for which $\sigma''(k_r) \neq 0$ and the second over all k_r for which $\sigma''(k_r) = 0$ but $\sigma'''(k_r) \neq 0$; further terms would be necessary for values of r for which both vanish but this will not occur in our examples. If some $k_r = 0$, then the corresponding term must be multiplied by $\frac{1}{2}$. For a value of x/t for which no solution to (15.10) exists, it is easy to show by a change of variables in (15.9), say $u = k\,x/t - \sigma(k)$, and integration by parts that $\eta_R(x,t) = O(t^{-1})$.

Let us examine in some detail the implications of one term of (15.11), say $r=1$, for the motion of η_R; if several terms are present for a given value of x/t one must superpose the resultant motions.

If x/t is held constant while t increases, then clearly one must set $x = \sigma'(k_1)\,t$, i.e. we are examining η_R from the standpoint of an observer moving with group velocity $\sigma'(k_1)$. Since the coefficient of the harmonic term is $t^{-\frac{1}{2}}$ times a function
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of k_1, which is being held constant, the gross outline of η_R will appear constant in form, but decreasing in amplitude because of $t^{-\frac{1}{2}}$. However, just as in the analysis of (15.7), there is a harmonic of wave number k_1 moving through the gross outline with phase velocity $\sigma(k_1)/k_1$. The amplitude of the gross outline is proportional to $E(k_1)$, but also depends now upon $\sigma''(k_1)$, in contrast to the situation for small t according to (15.7).

If the value of x/t is such that $\sigma''(k_1) = 0$, then one must examine a term from the second summation in (15.11). It is evident that the interpretation is the same except that σ''' occurs in place of σ'' and that the amplitude decreases more slowly because of the $t^{-\frac{1}{3}}$. This situation can happen, for example, in the case of gravity waves in water of depth h for $x = t\sqrt{gh}$. Then $k_1(\sqrt{gh}) = 0$, $\sigma''(0) = 0$, and $\sigma'''(0) = -h^2\sqrt{gh}$. This also occurs for combined gravity-capillary waves when the curve $\sigma'(k)$ has a minimum.

The approximation (15.11) to η_R will obviously be very poor for a value x/t such that $\sigma''(k_r)$ is near to zero for some r unless t is extremely large. It is shown elsewhere[1] how an Airy function may be used to modify the relevant term in the second summand to give a useful asymptotic expansion for k_r near a zero of σ''.

If x/t is fixed at a value for which (15.10) has no solution, then for an observer moving with this velocity the disturbance of the surface is very small, for it has been dying out as t^{-1}. The first term of the expansion may, of course, be com-

[1] H. Jeffreys and B. Jeffreys: Methods of mathematical physics, 3rd ed., § 17.09. Cambridge 1956. — See also C. Chester, B. Friedman and F. Ursell: Proc. Cambridge Phil. Soc. 53, 599–611 (1957).

puted as indicated above. This situation will occur for a disturbance in water of depth h if $x/t > \sqrt{gh}$. It will also occur when surface tension is taken into account for $x/t < \sigma'_{\min}$.

The asymptotic expansion (15.11) may also be used in a different fashion. Let us fix our attention upon one value of x and let t increase. Then x/t will decrease and the value $k_1(x/t)$ associated with the point x at a given moment will also change; for pure gravity waves it will increase. The observer stationed at x will then observe waves of continually increasing wave number (decreasing wave length) moving by with phase velocities appropriate to their lengths. The amplitudes at a given instant will depend upon the first factor. The gross outline of the waves will pass the observer at the group velocity appropriate to the wave number present at the moment, and, of course, the amplitude is decreasing as $t^{-\frac{1}{2}}$. In the case of a disturbance on water of depth h, if the observer is initially far from the hump, then even for large enough values of t for the asymptotic expansion to be valid the value of x/t may be greater than $\sqrt{gh}$. Then the observer will see practically no disturbance until the gravest modes begin to reach him. We note again that he must anticipate the arrival of a given wave number by its group velocity, not phase velocity, for it is the former which controls the amplitude. In the case of combined gravity-capillary waves, when t is large enough one will have $x/t < \sigma'_{\min}$ and the disturbance will be negligible.

It is also possible to find an asymptotic expansion for $\eta_R(x, t)$ for x/t fixed and large x. It turns out to be the same as (15.11) with $O(t^{-\frac{3}{2}})$ replaced by $O(x^{-\frac{3}{2}})$. This expansion allows one, so to speak, to take snapshots of the right-hand end of η_R at different instants of time. If we fix t and let x increase, x/t increases also and $k_1(x/t)$ decreases for pure gravity waves. Thus the wave length increases as one moves to the right; the observed amplitude will depend upon the first factor. For gravity waves on water of depth h, if x is large enough, $x/t > \sqrt{gh}$ and the disturbance will be small of order x^{-1}.

Finally, we use the asymptotic expansion to investigate the motion of a particular phase of $\eta_R(x, t)$, say a zero, for large t. Such a point will be determined by

$$\alpha(x, t) \equiv k_1 x - \sigma(k_1)\, t = \text{const},$$

where, as usual, $k_1 = k_1(x/t)$; solving for x gives $x = x(t)$. One may find $\dot{x}(t)$ from

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$$\dot{x}(t) = -\frac{\alpha_t}{\alpha_x} = -\frac{-k_1'\frac{x}{t^2} - \sigma(k_1) + \sigma'(k_1)\frac{x}{t^2}\,t}{k_1 + k_1'\frac{x}{t} - \sigma'(k_1)\,k_1'} = \frac{\sigma(k_1)}{k_1}.$$

Thus a particular phase travels with the phase velocity of the harmonic component associated with it at the moment. However, if the group and phase velocities are different, it is then moving at a different velocity from a point just keeping pace with waves of a given wave number. In particular, for gravity waves it is moving faster, hence moves into region of lower wave number and higher velocity and is accelerating. Acomputation of $\ddot{x}$ bears this out:

$$\ddot{x}(t) = -\frac{1}{t\,k_1\,\sigma''}\left[\frac{\sigma}{k_1} - \sigma'\right]^2,$$

for this is always positive for gravity waves. The right-hand side is, of course, a function of x and t. For deep-water gravity waves the function $x(t)$ may easily be found from the earlier equation:

$$x = \frac{\sigma(k_1)}{k_1}\,t - \frac{a}{k_1} = 2\sigma'(k_1)\,t - \frac{a}{k_1} = 2\,\frac{x}{t}\,t - \frac{4a}{g}\,\frac{x^2}{t^2}$$

or

$$x(t) = \frac{g\,t^2}{4a}\,.$$

Hence $\ddot{x} = g/2a$ and for large t the acceleration is constant. If the depth is finite, the computation is no longer simple, although it is possible to show that $x(t)$ varies from $x(t) = t\sqrt{gh}$ for a phase associated with $k=0$ to $x(t) = A\,t^2$ for a phase associated with very large k.

Fig. 9 is taken from a paper of KELVIN's (1907), and shows the computed values of $\eta(x,t)$ for an initial displacement given by

$$\eta(x,0) = \frac{[1+(1+x^2)^{\frac{1}{2}}]^{\frac{1}{2}}}{2^{\frac{3}{2}}(1+x^2)^{\frac{3}{2}}}\,[2-(1+x^2)^{\frac{1}{2}}]$$

and for $t/\pi^{\frac{1}{2}} = \frac{1}{2}$, 1, $\frac{3}{2}$, 4, 8 (the units have been chosen so that $g=4$). The description of the behavior of $\eta_R(x,t)$ outlined in the preceding paragraphs can be easily

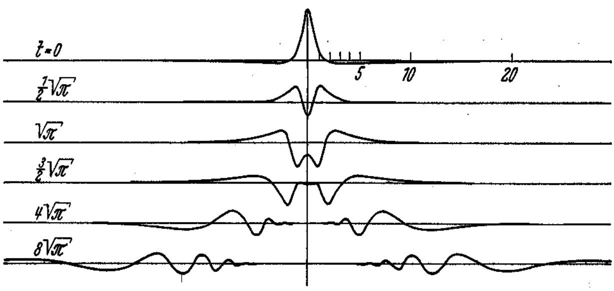

Fig. 9.

verified qualitatively by inspection of the successive snapshots of $\eta_R(x,t)$ GREEN (1909) has shown that if one estimates the wave length at any maximum as double the distance between the two including zeros, then the position is very close to that which would be estimated by using the group velocity (cf. HAVELOCK, 1914, p. 37).

Fig. 10 from a report by J.E. PRINS (1956; also 1958b) shows measured time histories taken at various distances from the center of an initial rectangular hump of length $2L$ and height Q in water of depth h for specific values shown in the figure. In general, the features of the motion described above were well verified by this experimental investigation.

We assemble here the expressions for $\sigma(k)$ and $k\sigma'/\sigma$ for a number of cases of water waves.

1. Deep-water gravity waves:

$$\sigma(k) = \sqrt{g\,k}\,, \qquad \frac{k\,\sigma'}{\sigma} = \frac{1}{2}\,.$$

2. Gravity waves at the interface of two fluids, each of infinite vertical extent:

$$\sigma(k) = \sqrt{\frac{\varrho_1-\varrho_2}{\varrho_1+\varrho_2}\,g\,k}\,, \qquad \frac{k\,\sigma'}{\sigma} = \frac{1}{2}\,.$$

3. Gravity waves in water of depth h:

$$\sigma(k) = \sqrt{g\,k \tanh k\,h}\,, \qquad \frac{k\,\sigma'}{\sigma} = \frac{1}{2}\left[1 + \frac{2\,k\,h}{\sinh 2\,k\,h}\right].$$

4. Gravity waves for a layer of thickness d of one fluid over a deep layer of a heavier one:

$$\sigma_1(k) = \sqrt{g\,k}\,, \qquad \frac{k\,\sigma_1'}{\sigma_1} = \frac{1}{2}\,,$$

$$\sigma_2(k) = \sqrt{\frac{\varrho_1 - \varrho_2}{\varrho_1 \coth k\,d + \varrho_2}\,g\,k}\,,$$

$$\frac{k\,\sigma_2'}{\sigma_2} = \frac{1}{2}\left[1 + \frac{2\,\varrho_1\,k\,d}{\varrho_1 \sinh 2\,k\,d + 2\,\varrho_2 \sinh^2 k\,d}\right].$$

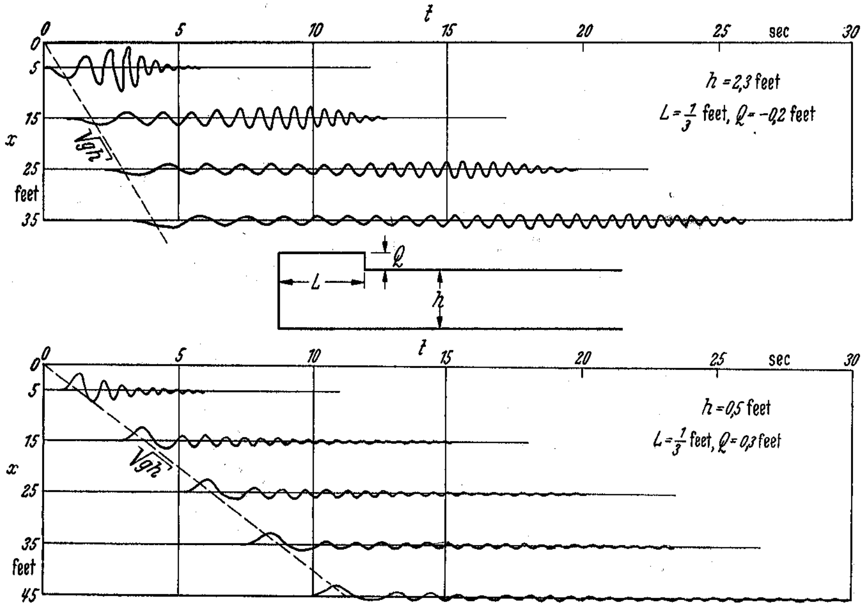

Fig. 10.

5. Waves at a free surface of a deep fluid with both gravity and surface tension acting:

$$\sigma(k) = \sqrt{g\,k + \frac{T\,k^3}{\varrho}}\,, \qquad \frac{k\,\sigma'}{\sigma} = \frac{1}{2}\,\frac{1 + 3\,T\,k^2/\varrho\,g}{1 + T\,k^2/\varrho\,g}\,.$$

6. Waves at a free surface of a fluid of depth h with both gravity and surface tension acting:

$$\sigma(k) = \sqrt{\left(g\,k + \frac{T\,k^3}{\varrho}\right) \tanh k\,h}\,,$$

$$\frac{k\,\sigma'}{\sigma} = \frac{1}{2}\left[1 + \frac{2\,h\,k}{\sinh 2\,h\,k} + 2\,\frac{T\,k^2/\varrho\,g}{1 + T\,k^2/\varrho\,g}\right].$$

In cases 1 to 4 σ'' is always negative if $k>0$. In case 5 it crosses the k-axis at $k = [g\,\varrho\,T^{-1}\,\tfrac{1}{3}(2\sqrt{3} - 3)]^{\frac{1}{2}}$ and becomes positive. In cases 1 to 4 $\sigma' < \sigma/k$ for $k > 0$. In case 5 $\sigma' < \sigma/k$ for $0 < k < \sqrt{g\,\varrho/T}$; then σ' crosses σ/k at the minimum of the latter and thereafter remains larger. (Note that σ' always passes through a stationary value of σ/k, passing from beneath to above in going through a minimum, and the reverse at a maximum.) We shall not discuss 6 in detail. For see errata $h > h_c = \sqrt{3T/2\varrho g}$, σ/k has a minimum for some k_0, $0 < k_0 < \sqrt{\varrho g/T}$ and σ' a minimum to the left of this. For $h \leqq h_c$, σ/k is an increasing function, starting at $\sqrt{gh}$ for $k=0$, and σ' is also increasing, $\sigma' > \sigma/k$ for $k>0$, $\sigma'(0) = \sqrt{gh}$. Fig. 11 shows graphs of σ, σ/k and σ' for 1, 3, 5, and 6 (the scales were chosen for convenience).

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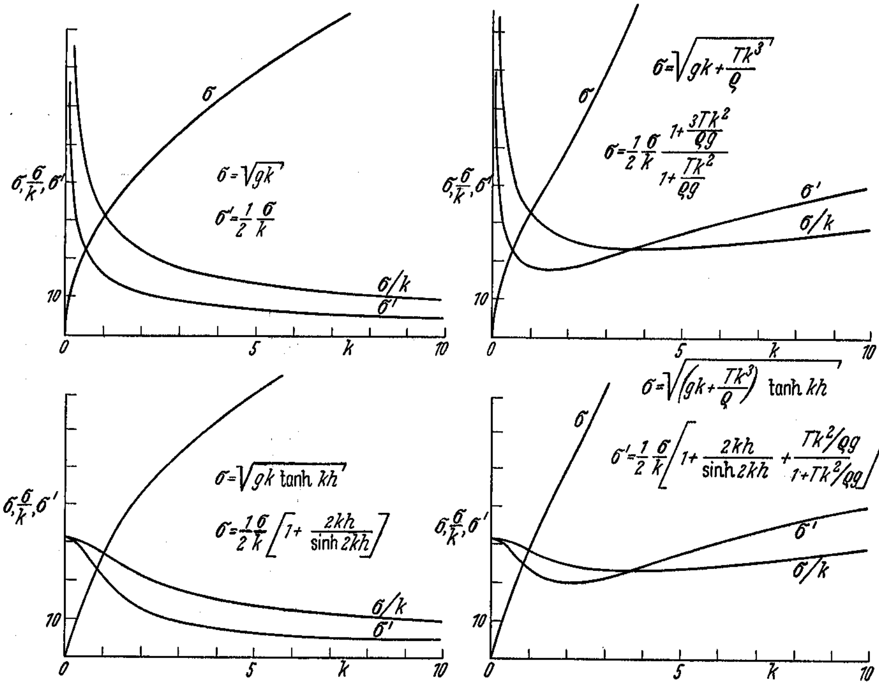

Fig. 11.

One may also take $\lambda = 2\pi/k$ as the independent variable, and then express the phase velocity c and group velocity U as functions of λ. An easy computation shows that

$$\lambda \frac{dc}{d\lambda} = c - U.$$

This equation has a simple interpretation in the geometry of the curve for $c(\lambda)$, as was shown by LAMB (1932, p. 382): For a given value of λ, U is the intercept on the vertical axis of the tangent to the $c(\lambda)$ curve at the point $(\lambda, c(\lambda))$. One value of U may correspond to more than one value of λ, as, for example, in the case of gravity-capillary waves. See HAVELOCK (1914, § 11).

β) The propagation of energy. It seems intuitively clear that as long as the right-moving part of an initial hump keeps its integrity the energy associated with the motion will in some sense move with the hump. We wish to consider in what sense this is true. We limit ourselves in the following discussion to a single fluid of depth h, where h may become infinite. However, surface tension may act upon the free surface.

We first introduce the notion of energy density for a given value of x. It will be convenient to separate potential, kinetic and surface energy. Let

$$\left.\begin{aligned}\mathscr{V}(x,t)&=\varrho g\int_0^{\eta_R(x,t)} y\,dy=\tfrac{1}{2}\varrho g\,\eta_R^2(x,t),\\ \mathscr{T}(x,t)&=\tfrac{1}{2}\varrho\int_{-h}^{0}(\Phi_x^2+\Phi_y^2)\,dy=\tfrac{1}{2}\varrho\int_{-h}^{0}(\Phi\,\Phi_x)_x\,dy+\tfrac{1}{2}\varrho\,\Phi\,\Phi_y(x,0,t),\\ \mathscr{S}(x,t)&=\tfrac{1}{2}T\,\eta_{Rx}^2(x,t)\end{aligned}\right\}\tag{15.12}$$

be the densities of potential, kinetic and surface energies, respectively, where here Φ is the velocity potential corresponding to η_R.

These functions may now be treated in the same way as η_R was in Sect. 15α. We may ask for the average position of the distributions of the several densities. They are defined by

$$\left.\begin{aligned}\bar{x}_V(t)&=\int_{-\infty}^{\infty}x\,\mathscr{V}(x,t)\,dx\Big/\int_{-\infty}^{\infty}\mathscr{V}(x,t)\,dx,\\ \bar{x}_T(t)&=\int_{-\infty}^{\infty}x\,\mathscr{T}(x,t)\,dx\Big/\int_{-\infty}^{\infty}\mathscr{T}(x,t)\,dx,\\ \bar{x}_S(t)&=\int_{-\infty}^{\infty}x\,\mathscr{S}(x,t)\,dx\Big/\int_{-\infty}^{\infty}\mathscr{S}(x,t)\,dx,\end{aligned}\right\}\tag{15.13}$$

respectively. Since all three densities are non-negative, one avoids the difficulty met with in defining the average position of η_R. In fact, it is obvious that the definitions of $\bar{x}_R$ and $\bar{x}_V$ coincide, so that the conclusions concerning $\bar{x}_R$ can be applied immediately to $\bar{x}_V(t)$. In particular,

$$\bar{x}_V(t)=\bar{x}_V(0)+\bar{\sigma}'t.\tag{15.14}$$

Consider now $\bar{x}_T(t)$. First we note that, from GREEN's Theorem,

$$\begin{aligned}\int_{-\infty}^{\infty}\mathscr{T}(x,t)\,dx&=\tfrac{1}{2}\varrho\int_{-\infty}^{\infty}\Phi(x,0,t)\,\Phi_y(x,0,t)\,dx+\\ &+\lim_{\substack{x_1\to-\infty\\x_2\to+\infty}}\tfrac{1}{2}\varrho\int_{-h}^{0}[-\Phi(x_1,y,t)\,\Phi_x(x_1,y,t)+\Phi(x_2,y,t)\,\Phi_x(x_2,y,t]\,dy.\end{aligned}$$

From the assumed square-integrability of η_R, the limit vanishes. Use of the identity $x(\Phi_x^2+\Phi_y^2)=(x\Phi)_x\,\Phi_x+(x\Phi)_y\,\Phi_y-\Phi\Phi_x$ and GREEN's Theorem gives

$$\begin{aligned}\int_{-\infty}^{\infty}x\,\mathscr{T}(x,t)\,dx&=\tfrac{1}{2}\varrho\int_{-\infty}^{\infty}x\,\Phi(x,0,t)\,\Phi_y(x,0,t)\,dx-\\ &-\lim_{\substack{x_1\to-\infty\\x_2\to+\infty}}\tfrac{1}{4}\varrho\int_{-h}^{0}[\Phi^2(x_2,y,t)-\Phi^2(x_1,y,t)]\,dy+\\ &+\lim_{\substack{x_1\to-\infty\\x_2\to\infty}}\tfrac{1}{2}\varrho\int_{-h}^{0}[-x_1\,\Phi(x_1,y,t)\,\Phi_x(x_1,y,t)+x_2\,\Phi(x_2,y,t)\,\Phi_x(x_2,y,t)]\,dy,\end{aligned}$$

where again the last two limits vanish. A similar computation shows

$$\begin{aligned}\int_{-\infty}^{\infty}x^2\,\mathscr{T}(x,t)\,dx&=\tfrac{1}{2}\varrho\int_{-\infty}^{\infty}x^2\,\Phi(x,0,t)\,\Phi_y(x,0,t)\,dx+\tfrac{1}{2}\varrho\int_{-\infty}^{\infty}\int_{-h}^{0}\Phi^2(x,y,t)\,dx\,dy+\\ &+\lim_{\substack{x_1\to-\infty\\x_2\to\infty}}\tfrac{1}{2}\varrho\int_{-h}^{0}[-x_1^2\,\Phi(x_1,y,t)\,\Phi_x(x_1,y,t)+x_2^2\,\Phi(x_2,y,t)\,\Phi_x(x_2,y,t)]\,dy-\\ &-\lim_{\substack{x_1\to-\infty\\x_2\to\infty}}\tfrac{1}{2}\varrho\int_{-h}^{0}[-x_1\,\Phi^2(x_1,y,t)+x_2\,\Phi^2(x_2,y,t)]\,dy.\end{aligned}$$

Collecting these results we have

$$\left.\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{T}(x,t)\,dx &= \tfrac{1}{2}\varrho \int_{-\infty}^{\infty} \Phi(x,0,t)\,\Phi_y(x,0,t)\,dx,\\
\int_{-\infty}^{\infty} x\,\mathcal{T}(x,t)\,dx &= \tfrac{1}{2}\varrho \int_{-\infty}^{\infty} x\,\Phi(x,0,t)\,\Phi_y(x,0,t)\,dx,\\
\int_{-\infty}^{\infty} x^2\,\mathcal{T}(x,t)\,dx &= \tfrac{1}{2}\varrho \int_{-\infty}^{\infty} x^2\,\Phi(x,0,t)\,\Phi_y(x,0,t)\,dx +\\
&\quad + \tfrac{1}{2}\varrho \int_{-\infty}^{\infty}\int_{-h}^{0} \Phi^2(x,y,t)\,dx\,dy.
\end{aligned}\right\} \tag{15.15}$$

Since from (15.2),

$$\eta_R(x,t) = \tfrac{1}{2}\int_{-\infty}^{\infty} E(k)\,\mathrm{e}^{-i(kx-\sigma t)}\,dk$$

and

$$\Phi(x,y,t) = \frac{1}{2}\int_{-\infty}^{\infty} i\,\frac{\sigma(k)}{k}\,Y(y)\,E(k)\,\mathrm{e}^{-i(kx-\sigma t)}\,dk, \tag{15.16}$$

one finds easily

$$\Phi(x,0,t) = \frac{1}{2}\int_{-\infty}^{\infty} i\,\frac{\sigma(k)}{k}\coth kh\,E(k)\,\mathrm{e}^{-i(kx-\sigma t)}\,dk,$$

$$\Phi_y(x,0,t) = \tfrac{1}{2}\int_{-\infty}^{\infty} i\,\sigma(k)\,E(k)\,\mathrm{e}^{-i(kx-\sigma t)}\,dk.$$

One may now apply again, as in Sect. 15a, theorems on Fourier transforms to obtain

$$\left.\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{T}(x,t)\,dx &= \frac{1}{4}\pi\varrho\int_{-\infty}^{\infty} E(k)\,E^*(k)\,\frac{\sigma^2}{k}\coth kh\,dk,\\
\int_{-\infty}^{\infty} x\,\mathcal{T}(x,t)\,dx &= \frac{1}{4}\pi\varrho\int_{-\infty}^{\infty} i\,E(k)\,E^{*\prime}(k)\,\frac{\sigma^2}{k}\coth kh\,dk +\\
&\quad + \frac{1}{4}\pi\varrho\,t\int_{-\infty}^{\infty} E(k)\,E^*(k)\,\sigma'(k)\,\frac{\sigma^2}{k}\coth kh\,dk,\\
\int_{-\infty}^{\infty} x^2\,\mathcal{T}(x,t)\,dx &= \int_{-\infty}^{\infty} x^2\,\mathcal{T}(x,0)\,dx +\\
&\quad + \frac{1}{2}\pi\varrho\,t\int_{-\infty}^{\infty} i\,E(k)\,E^{*\prime}(k)\,\sigma'\,\frac{\sigma^2}{k}\coth kh\,dk +\\
&\quad + \frac{1}{4}\pi\varrho\,t^2\int_{-\infty}^{\infty} E(k)\,E^*(k)\,\sigma'^2(k)\,\frac{\sigma^2}{k}\coth kh.
\end{aligned}\right\} \tag{15.17}$$

If one uses the definition introduced earlier for average of a function of k, one now finds

$$\bar{x}_T(t) = \bar{x}_T(0) + t\,\frac{\overline{\sigma'\sigma^2 k^{-1}\coth kh}}{\overline{\sigma^2 k^{-1}\coth kh}} \tag{15.18}$$

and a rather unwieldy expression for $\overline{[x-\bar{x}_T(t)]^2}$, similar in character to (15.4). We note that if we are dealing with pure gravity waves, so that $\sigma^2 = gk\tanh kh$, then formulas (15.17) simplify considerably and become identical with those for V. In this case the potential and kinetic energies are equal and propagate with the same velocities.

We may now carry out similar calculations for $S(x, t)$. The corresponding formulas follow

$$\left.\begin{aligned}\int_{-\infty}^{\infty} \mathscr{S}(x, t)\, dx &= \frac{1}{4}\pi T \int_{-\infty}^{\infty} k^2 E(k)\, E^*(k)\, dk,\\ \int_{-\infty}^{\infty} x\, \mathscr{S}(x, t)\, dx &= \frac{1}{4}\pi T \int_{-\infty}^{\infty} i\, k^2 E^{*\prime}(k)\, E(k)\, dk + {}\\ &\quad + \frac{1}{4}\pi T t \int_{-\infty}^{\infty} k^2 \sigma'(k)\, E(k)\, E^*(k)\, dk,\\ \int_{-\infty}^{\infty} x^2 \mathscr{S}(x, t)\, dx &= \int_{-\infty}^{\infty} x^2 \mathscr{S}(x, 0)\, dx + \frac{1}{2}\pi T t \int_{-\infty}^{\infty} k^2 \sigma' E^{*\prime}(k)\, E(k)\, dk + {}\\ &\quad + \frac{1}{4}\pi T t^2 \int_{-\infty}^{\infty} k^2 \sigma'^2 E(k)\, E^*(k)\, dk,\end{aligned}\right\} \tag{15.19}$$

and

$$\bar{x}_S(t) = \bar{x}_S(0) + t\,\frac{\overline{k^2 \sigma'}}{\overline{k^2}} \tag{15.20}$$

and again a formula for $\overline{[x - \bar{x}_S(t)]^2}$ similar in character to (15.4).

One should note that the total potential, kinetic and surface energies associated with $\eta_R(x, t)$ each remain constant in time. If $T \neq 0$, then the mean positions of the three energy densities propagate with different velocities, each velocity being an average, in some sense, of σ'. If one considers a wave packet (15.5), then as the width 2ε of the band of wave numbers approaches zero the velocity of propagation of the individual energy densities will each approach $\sigma'(k_0)$, the group velocity.

Consider now the total energy density,

$$\mathscr{E}(x, t) = \mathscr{V}(x, t) + \mathscr{T}(x, t) + \mathscr{S}(x, t).$$

Making use of the form of $\sigma(k)$,

$$\sigma^2(k) = (g\, k + T\, k^3/\varrho) \tanh k\, h,$$

one finds

$$\left.\begin{aligned}\int_{-\infty}^{\infty} \mathscr{E}(x, t)\, dx &= \frac{1}{4}\pi\varrho \int_{-\infty}^{\infty} \left[g + \frac{\sigma^2}{k} \coth k\, h + \frac{T}{\varrho} k^2\right] E(k)\, E^*(k)\, dk\\ &= \frac{1}{2}\pi \int_{-\infty}^{\infty} [g\,\varrho + T\, k^2]\, E(k)\, E^*(k)\, dk,\\ \int_{-\infty}^{\infty} x\, \mathscr{E}(x, t)\, dx &= \frac{1}{2}\pi \int_{-\infty}^{\infty} [g\,\varrho + T k^2]\, i\, E^{*\prime}(k)\, E^*(k)\, dk + \frac{1}{2}\pi t \int_{-\infty}^{\infty} \sigma'(k)\, \times\\ &\quad \times [g\,\varrho + T\, k^2]\, E\, E^*\, dk,\\ \int_{-\infty}^{\infty} x^2 \mathscr{E}(x, t)\, dx &= \int_{-\infty}^{\infty} x^2 \mathscr{E}(x, 0)\, dx + \pi t \int_{-\infty}^{\infty} \sigma'(g\,\varrho + T\, k^2]\, i\, E^{*\prime} E\, dk + {}\\ &\quad + \frac{1}{2}\pi t^2 \int_{-\infty}^{\infty} \sigma'^2 [g\,\varrho + T\, k^2]\, E\, E^*\, dk,\end{aligned}\right\} \tag{15.21}$$

and

$$\bar{x}_E(t) = \bar{x}_E(0) + t\,\frac{\overline{\sigma'[\varrho g + T k^2]}}{\overline{\varrho g + T k^2}}. \tag{15.22}$$

At any instant t half of the total energy is kinetic energy and the other half is divided between potential and surface energy.

There is another way of considering the energy transported by surface waves which, at first glance, is different from the preceding treatment. Consider a fixed plane $x = \text{const}$. Then from the results in Sect. 8 one may compute the rate at which energy is being transported through this plane, the so-called *energy-flux*. Let us denote it by $\mathcal{F}(x, t)$. After appropriate linearization, formula (8.10) gives

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$$\mathcal{F}(x,t) = -\int_{-\infty}^{0} \varrho\, \Phi_t(x,y,t)\, \Phi_x(x,y,t)\, dy - T\,\eta_t(x,t)\,\eta_x(x,t). \tag{15.23}$$

The expression for the flux has an advantage over the expressions for mean positions considered above in that no strong restrictions upon η are required for it to exist. In fact, it can be computed for a single harmonic wave

$$\eta = A \sin(k x - \sigma t). \tag{15.24}$$

With

$$\Phi = -A\,\frac{\sigma}{k}\,\frac{\cosh k(y+h)}{\sinh k h}\cos(k x - \sigma t),$$

one finds by a straightforward calculation

$$\mathcal{F}(x,t) = A^2 T k \sigma \cos^2(k x - \sigma t) + A^2 \varrho\,\frac{\sigma^3}{2k^2}\coth k h\left[1 + \frac{2 k h}{\sinh 2 k h}\right]\sin^2(k x - \sigma t).$$

Averaging over a wavelength (or over a period, it makes no difference which), one finds

$$\left.\begin{aligned} \mathcal{F}_{\text{av}} &= A^2\,\frac{1}{4}\,\frac{\sigma}{k}\left\{2 T k^2 + \sigma^2 \varrho\,\frac{\coth k h}{k}\left[1 + \frac{2 k h}{\sinh 2 k h}\right]\right\}\\ &= \frac{1}{2} A^2 (g \varrho + T k^2)\,\sigma'(k). \end{aligned}\right\} \tag{15.25}$$

Thus the group velocity enters again in connection with energy propagation, even though no "group" is present. The energy density and average energy per wave length for (15.24) are

$$\left.\begin{aligned} \mathcal{E}(x,t) &= A^2\left\{\frac{1}{2}\varrho g \sin^2(k x - \sigma t) + \frac{1}{2} T k^2 \cos^2(k x - \sigma t) + \right.\\ &\quad \left. + \frac{1}{4}\varrho\,\frac{\sigma^2}{k}\coth k h\left[1 - \frac{2 k h}{\sinh 2 k h}\cos 2(k x - \sigma t)\right]\right\},\\ \mathcal{E}_{\text{av}} &= \tfrac{1}{2} A^2 (g \varrho + T k^2). \end{aligned}\right\} \tag{15.26}$$

If one is dealing with a composite wave, averaging over a wave length is possible only if the resulting wave is periodic. However, even without this restriction, one may compute both the average flux and average energy per unit length from

$$\left.\begin{aligned} \mathcal{F}_{\text{av}} &= \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\mathcal{F}(x,t)\,dx,\\ \mathcal{E}_{\text{av}} &= \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\mathcal{E}(x,t)\,dx. \end{aligned}\right\} \tag{15.27}$$

Then if a composite wave propagating to the right is given by

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$$\eta(x, t) = \sum_{j=-\infty}^{\infty} E_j \,\mathrm{e}^{-i(k_j x - \sigma t)} = \sum_{j=1}^{\infty} a_j \cos(k_j x - \sigma_j t) + b_j \sin(k_j x - \sigma_j t), \tag{15.28}$$

with

$$\Phi = \sum_{j=-\infty}^{\infty} i E_j \frac{\cosh k_j (y+h)}{\sinh k_j h} \frac{\sigma_j}{k_j} \mathrm{e}^{-i(k_j x - \sigma_j t)},$$

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where $E_j = E^*_{-j} = \frac{1}{2}(a_j + b_j)$, $k_{-j} = -k_j$, $\sigma_j = \sigma(k_j) = -\sigma_{-j}$, one finds

$$\left.\begin{aligned} \mathscr{E}_{\mathrm{av}} &= \frac{1}{2} \sum_{-\infty}^{\infty} |E_j|^2 \left[T k_j^2 + \varrho g + \varrho \frac{\sigma_j^2}{k_j} \coth k_j h\right] \\ &= \sum_{-\infty}^{\infty} |E_j|^2 [\varrho g + T k_j^2] = \frac{1}{2} \sum_{1}^{\infty} (a_j^2 + b_j^2) [\varrho g + T k_j^2] \end{aligned}\right\} \tag{15.29}$$

and

$$\left.\begin{aligned} \mathscr{F}_{\mathrm{av}} &= \sum_{-\infty}^{\infty} |E_j|^2 \left\{T k_j \sigma_j + \varrho \frac{\sigma_j^3}{2 k_j^2} \coth k_j h \left[1 + \frac{2 k_j h}{\sinh 2 k_j h}\right]\right\} \\ &= \sum_{-\infty}^{\infty} |E_j|^2 [\varrho g + T k_j^2]\, \sigma_j'. \end{aligned}\right\} \tag{15.30}$$

In order to obtain these relatively simple formulas in which the contributions from the individual harmonics are isolated, it is essential that the averages be taken. Otherwise, for $\mathscr{E}(x, t)$ or $\mathscr{F}(x, t)$ one obtains a complicated double summation, and the role of the group velocity is not apparent.

A similar analysis may be carried through for the right-moving initial hump (15.16). However, an average of either $\mathscr{F}$ or $\mathscr{E}$ computed according to (15.27) would vanish. Instead we take the total flux and total energy, respectively:

$$\mathscr{F}_{\mathrm{total}} = \int_{-\infty}^{\infty} \mathscr{F}(x, t)\, dx, \qquad \mathscr{E}_{\mathrm{total}} = \int_{-\infty}^{\infty} \mathscr{E}(x, t)\, dx. \tag{15.31}$$

The resulting formulas are analogous to (15.29) and (15.30):

$$\left.\begin{aligned} \mathscr{E}_{\mathrm{total}} &= \tfrac{1}{2}\pi \int_{-\infty}^{\infty} [g \varrho + T k^2]\, E(k)\, E^*(k)\, dk, \\ \mathscr{F}_{\mathrm{total}} &= \tfrac{1}{2}\pi \int_{-\infty}^{\infty} \sigma'(k)\, [g \varrho + T k^2]\, E(k)\, E^*(k)\, dk. \end{aligned}\right\} \tag{15.32}$$

If the last result is applied to a narrow wave band, such as (15.5), then one finds the limiting relationship

$$\lim_{\varepsilon \to 0} \frac{\mathscr{F}_{\mathrm{total}}}{\mathscr{E}_{\mathrm{total}}} = \sigma'(k_0).$$

In the first method of treating the propagation of energy, i.e. in terms of the motion of the mean position of the energy density, it was not surprising that σ' should appear, for it is a familiar property of Fourier transforms that taking the derivative of the transform is associated with multiplying the function by the variable. Thus, if

$$g(k) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{ikx}\, dx,$$

then

$$g'(k) = \int_{-\infty}^{\infty} i x f(x)\, \mathrm{e}^{ikx}\, dx,$$

In the cases considered above the transform contained $e^{i\sigma t}$ as a factor, and the derivative contained $\sigma' t$ in one summand. However, the appearance of σ' in the formulas for $\mathscr{F}_{\mathrm{av}}$ or $\mathscr{F}_{\mathrm{total}}$ seems in some ways coincidental: One makes a calculation, and after gathering and manipulating terms discovers that a certain combination of them indeed contains σ'. That this is not really coincidence is indicated by the following theorem for the case (15.21):

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} x\,\mathscr{E}(x,t)\,dx = \mathscr{F}_{\mathrm{total}}. \tag{15.33}$$

It may be proved as follows. From the definition of $\mathscr{E}(x,t)$

$$\int_{-\infty}^{\infty} x\,\mathscr{E}(x,t)\,dx = \int_{-\infty}^{\infty} x\left[\tfrac{1}{2}\varrho g\eta^2 + \tfrac{1}{2}T\eta_x^2 + \tfrac{1}{2}\varrho\int_{-h}^{0}(\Phi_x^2+\Phi_y^2)\,dy\right]dx.$$

Hence

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$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} x\,\mathscr{E}(x,t)\,dx = \int_{-\infty}^{\infty} x\left[\varrho g\eta\eta_t + T\eta_x\eta_{xt} + \varrho\int_{-h}^{0}(\Phi_x\Phi_{xt}+\Phi_y\Phi_{yt})\,dy\,dx.\right.$$

Integrating the second and third terms by parts and taking account of the assumed behavior of η and Φ at $\pm\infty$, one finds

$$\int_{-\infty}^{\infty} x\left[\varrho g\eta\eta_t - T\eta_{xx}\eta_t + \varrho\int_{-h}^{0}(\Phi_y\Phi_{yt} - \Phi_{xx}\Phi_t)\,dy\right]dx - \\ -\int_{-\infty}^{\infty}\left[T\eta_x\eta_t + \varrho\int_{-h}^{0}\Phi_x\Phi_t\,dy\right]dx.$$

Since $\Phi_{xx}+\Phi_{yy}=0$, one may express the third summand in the first integral as

$$\varrho\int_{-h}^{0}(\Phi_y\Phi_{yt}+\Phi_{yy}\Phi_t)\,dy = \varrho\int_{-h}^{0}(\Phi_y\Phi_t)_y\,dy = \varrho\,\Phi_y(x,0,t)\,\Phi_t(x,0,t) = \varrho\,\eta_t\,\Phi_t.$$

Hence the first integral may be written

$$\int_{-\infty}^{\infty} x\,\eta_t\left[\varrho g\eta - T\eta_{xx} + \varrho\,\Phi_t(x,0,t)\right]dx,$$

which vanishes, since the term in brackets is just the dynamical boundary condition at the free surface. The second integral above is just $\mathscr{F}_{\mathrm{total}}$, so that (15.33) is proved.

A similar line of reasoning allows one to establish the following relation between $\mathscr{E}$ and $\mathscr{F}$:

$$\frac{\partial\mathscr{E}(x,t)}{\partial t} = -\frac{\partial\mathscr{F}(x,t)}{\partial x}, \tag{15.34}$$

essentially an expression of the conservation of energy. Eq. (15.33) may also be derived from (15.34) by writing the latter in the form

$$\frac{\partial(x\mathscr{E})}{\partial t} = -\frac{\partial(x\mathscr{F})}{\partial x} + \mathscr{F}$$

and integrating.

Although (15.33) may explain the presence of σ' in the energy flux for a continuous spectrum and finite total energy, one is still left with the apparently paradoxical situation that even for (15.24), when only one frequency is present, σ' enters into the expression for $\mathscr{F}_{\mathrm{av}}$. One would expect the occurrence of σ' only if one were dealing not only with a specific value k but also with neighboring values. There is no useful analogue to (15.33) for the discrete spectrum, because there is no Fourier integral to connect in a natural way the mean position of a hump with σ'. However, if one approximates (15.24) or (15.28) by considering only the segment of η between $-L$ and L and taking $\eta=0$ outside this segment, then one has approximated η by η_L, where the latter has a continuous spectrum and finite energy. For η_L it is reasonable that $\sigma'(k)$ should enter into the energy propagation. The definitions adopted for $\mathscr{F}_{\text{av}}$ and $\mathscr{E}_{\text{av}}$ in (15.27) reflect this approximation of η by η_L and then a passage to the limit in such a way as to keep these quantities finite. Thus it is perhaps not surprising after all that σ' has entered into the computation of $\mathscr{F}_{\text{av}}$, for the method of averaging $\mathscr{F}$ and $\mathscr{E}$ is such that one replaces the discrete spectrum by a continuous one and then takes a limit. A different explanation of this paradoxical situation has been given by Rayleigh [*Theory of sound*, Vol. I p. 479]; generally it seems to be overlooked.

One should note that the definitions of velocity of propagation of mean positions of humps and energy distributions for finite total energy and of total or average energy flux all retain meaning even if the boundary condition at the free surface has not been linearized. The comparative simplicity of the formulas when the boundary condition is linearized and the occurrence in them of σ' both result from the special form of the spectrum, namely, $E(k,t)=E(k,0)\,e^{i\sigma(k)t}$, and the applicability of properties of Fourier transforms of convolutions.

For further information one may consult the monograph of Havelock (1914) already cited, papers by Bourgin (1936), Rossby (1945, 1947), Eckart (1948), Broer (1951), and Poincelot (1953, 1954), Jeffreys and Jeffreys, *Methods of mathematical physics* (3rd ed., Cambridge, 1956, pp. 511–518) and standard texts such as Lamb (1932, Sects. 236, 237, 240, 241) and Kochin, Kibel' and Roze (1948, Chap. 8, Sect. 8).

16. The solution of special boundary problems. In the next several sections we shall be considering a variety of problems, each associated with some special geometrical configuration.

In treating a particular boundary configuration one must first consider whether it is tractable at all by the theory of infinitesimal waves, i.e. whether it is possible to select a perturbation parameter ε satisfying the requirements mentioned in Sect. 10. On this basis, for example, it would appear unreasonable to try to apply infinitesimal-wave theory to the waves generated by a vertical circular cylinder moving with constant velocity, for the slope of free surfaces may be expected to become very large near the front of the cylinder. On the other hand, in certain similar situations, notably the theory of planing surfaces, it is possible to strain the theory to accomodate such a situation. The choice of parameter will be discussed in each individual case. We call attention to the fact that in many cases it is a consequence of the linearization procedure that the boundary condition on a solid boundary is no longer to be satisfied on the physical boundary, but instead on some neighboring surface. The same situation occurred earlier in linearizing the free-surface condition. This should not be considered as a further approximation, but rather as one consistent with the infinitesimal-wave approximation.

The methods for finding a solution to a boundary-value problem, once it has been properly formulated, seem to fall into two or possibly three groups. One method is a combination of separation of variables and expansion of the factors in Fourier-type series or integrals. This requires, of course, a geometric configuration related in a suitable way to the coordinate surfaces of a set of variables which allows separation and a complete set of associated elementary solutions to be used in the expansion. If a Fourier-series expansion is to be used, orthogonality of the elementary solutions is desirable.

If the motion is harmonic in time with frequency σ and if the fluid is of finite depth h, then the functions

$$\{\cosh m_0(y+h), \cos m_i(y+h)\} \tag{16.1}$$

occurring as factors in (13.2) and (13.4), in (13.6), and in (13.8) may be shown easily by direct computation to be orthogonal on the interval $0 \geqq y \geqq -h$. Completeness follows from known criteria[1]. However, both orthogonality and completeness are consequences of the general theory of Sturm-Liouville systems. The result may be used in the following way, for example. Suppose fluid occupies the region

$$x>0, \quad 0>y>-h, \quad 0<z<l,$$

and that the boundary conditions on the walls and bottom are

$$\left.\begin{aligned} \Phi_x(0,y,z,t) &= F(y,z)\cos\sigma t, \\ \Phi_z(0,x,y,t) &= \Phi_z(l,x,y,t) = 0, \\ \Phi_x(x,y,-h,t) &= 0. \end{aligned}\right\} \tag{16.2}$$

Then, by expressing $F(y,z)$ as a double series

$$\left.\begin{aligned} F(y,z) = {} & \sum a_{0q}\cosh m_0(y+h)\cos\frac{\pi q}{l}z \\ & + \sum\sum a_{pq}\cos m_p(y+h)\cos\frac{\pi q}{l}z \end{aligned}\right\} \tag{16.3}$$

(with appropriate restrictions upon F), one may construct a solution from the elementary solutions in (13.6). Further conditions relating to boundedness and behavior as $x\to\infty$ are necessary in order to ensure a unique solution, but will not be discussed here. The elementary solutions (13.8) can be used in a similar way for the region exterior to a vertical cylindrical boundary. Still other configurations are possible corresponding to the various coordinate systems allowing separation of $\Delta_2\varphi \pm m\varphi = 0$.

If the fluid is infinitely deep, it is possible to construct a Fourier-integral see errata expansion using the function.

$$\{e^{\nu y}, k\cos ky + \nu\sin ky\}, \quad \nu=\sigma^2/g, \quad 0<k<\infty. \tag{16.4}$$

In fact, HAVELOCK (1929b) has remarked that the usual Fourier-integral representation of a function may be altered to give

$$\left.\begin{aligned} f(y) = {} & \frac{2}{\pi}\int_0^\infty\int_{-\infty}^0 f(\eta)\,\frac{(k\cos ky+\nu\sin ky)\,(k\cos k\eta+\nu\sin k\eta)}{k^2+\nu^2}\,d\eta\,dk \\ & + 2\nu\,e^{\nu y}\int_{-\infty}^0 f(\eta)\,e^{\nu\eta}\,d\eta. \end{aligned}\right\} \tag{16.5}$$

If the problem is such that rectangular coordinates may be used conveniently, then (16.5) may be combined with a Fourier-series or Fourier-integral expansion in z and the elementary solutions (13.5) used to construct a solution analogous to (16.3). The necessary expressions in both rectangular and cylindrical coordinates can be found in the cited paper of HAVELOCK.

If the fluid is of bounded horizontal extent and is bounded by vertical surfaces which are constant-coordinate surfaces in one of the coordinate systems

[1] See, e.g., N. LEVINSON: Gap and density theorems. Amer. Math. Soc. Colloq. Publ. No. 27, Chap. I. New York 1940.

allowing separation of $\Delta_2\varphi \pm m\varphi = 0$, the various possible modes of motion of the fluid may be obtained as the solution of an eigenvalue problem of a classical type. If the container is of more general shape, it is more difficult to obtain explicit solutions. The problem will be discussed in Sect. 23.

The orthogonal functions (16.1) were associated with a single value of the frequency σ. It is possible to derive another result concerning orthogonality of solutions associated with different values of σ. Let $\varphi_1(x, y, z)\cos\sigma_1 t$ and $\varphi_2(x, y, z)\cos\sigma_2 t$, $\sigma_1 \neq \sigma_2$, be regular velocity potentials of harmonic oscillations of different frequencies. Furthermore, let any solid boundaries be fixed and, if the fluid is not bounded in extent, we suppose that $|\text{grad}\,\varphi| = O(R^{-1-\varepsilon})$ as $R^2 = x^2 + z^2 \to \infty$. Consider the fluid contained within a large cylinder Ω_R of radius R and above the plane $y = -R$. The fluid will be bounded partly by free surface F_R, partly by solid boundaries S_R, partly by the horizontal plane B_R and partly by the cylinder Ω_R. Applying GREEN'S theorem to the two potential function, one obtains

$$\left.\begin{aligned} 0 &= \iint\limits_{F_R+S_R+B_R+\Omega_R} (\varphi_1\varphi_{2n} - \varphi_{1n}\varphi_2)\, d\sigma \\ &= \iint\limits_{F_R} (\varphi_1\varphi_{2y} - \varphi_{1y}\varphi_2)\, d\sigma + \iint\limits_{B_R+\Omega_R} (\varphi_1\varphi_{2n} - \varphi_{1n}\varphi_2)\, d\sigma. \end{aligned}\right\} \tag{16.6}$$

As $R\to\infty$, the integral over $\Omega_R + B_R \to 0$, and one has

$$\iint\limits_F (\varphi_1\varphi_{2y} - \varphi_{1y}\varphi_2)\, d\sigma = 0. \tag{16.7}$$

From the free-surface condition

$$\varphi_{iy}(x, 0, z) = -\frac{\sigma_i^2}{g}\varphi_i(x, 0, z), \qquad i = 1, 2, \tag{16.8}$$

and (16.7) becomes

$$\frac{\sigma_1^2 - \sigma_2^2}{g}\iint\limits_F \varphi_1(x, 0, z)\,\varphi_2(x, 0, z)\, d\sigma = 0, \tag{16.9}$$

or simply

$$\iint\limits_F \varphi_1(x, 0, z)\,\varphi_2(x, 0, z)\, d\sigma = 0. \tag{16.10}$$

Hence φ_1 and φ_2 are orthogonal over the free surface of the fluid. This theorem can be used for certain initial-value problems in a manner analogous to that in which the orthogonality of (16.1) can be used for boundary-value problems. This will be done in Sect. 23 α.

A second method for solving special problems is the method of GREEN'S functions or source functions [cf. VOLTERRA (1934)]. In this method one constructs first a potential function of the form

$$\left.\begin{aligned} G(x, y, z; \xi, \eta, \zeta) &= \frac{1}{r} + G_0(x, y, z; \xi, \eta, \zeta), \\ r^2 &= (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2, \end{aligned}\right\} \tag{16.11}$$

such that G_0 is regular in $y<0$ and such that G satisfies the free-surface condition, conditions at infinity appropriate to the problem at hand, and, if the fluid is of finite depth, the boundary condition on the bottom. Such solutions are, of course, just the singular solutions derived in Sect. 13 γ. Next, if there are surfaces S in (or on) the fluid upon which certain further boundary conditions must be satisfied, we attempt to satisfy them by a distribution of the modified sources (16.11) over the surface(s) S:

$$\Phi(x, y, z, t) = \iint\limits_S \gamma(\xi, \eta, \zeta, t)\, G(x, y, z; \xi, \eta, \zeta; t)\, d\sigma. \tag{16.12}$$

Here γ is an unknown function which is to be determined from the boundary condition on S. In most problems this boundary condition consists in specifying Φ_n on S. Well known properties of surface distributions of sources then allow one to formulate an integral equation for γ:

$$\left.\begin{aligned}\Phi_n(x, y, z, t) = -2\pi\gamma(x, y, z, t) + \iint_S \gamma(\xi, \eta, \zeta, t)\, G_n(x, y, z; \xi, \eta, \zeta; t)\, d\sigma, \\ (x, y, z) \text{ on } S,\end{aligned}\right\} \quad (16.13)$$

where n is the exterior normal to the surface S (taken here as a closed surface). When it is convenient, one may also use distributions of dipoles.

It is also possible, and sometimes advantageous, to construct solutions satisfying given boundary conditions on a closed surface S by distributing the singular solutions on surfaces, lines or points completely inside S. Examples will occur later.

A third method of approach is to seek first, instead of $\Phi(x, y, z, t)$ or $f(z, t)$ the functions

$$\chi = \Phi_{tt} + g\Phi_y \quad \text{or} \quad F = f_{tt} + i g f'.$$

These functions satisfy a simpler condition on the plane $y = 0$:

$$\chi(x, 0, z, t) = 0 \quad \text{or} \quad \operatorname{Re} F(x - i0, t) = 0.$$

If the other boundary conditions are such that they can be formulated simply in terms of χ or F, the new problem may be simpler to solve. After finding χ or F, one must then solve a differential equation in order to obtain the desired solution Φ or f. This procedure is called the "reduction" method by WEINSTEIN (1949). It was apparently first introduced by LEVI-CIVITA and has since been much exploited by CISOTTI, KELDYSH,, KOCHIN, SEDOV, HASKIND, LEWY, STOKER and others. It has already been used in the derivation of (13.28) and will be applied in several other problems[1]. The solution of the reduced problem may, of course, be carried out by one of the two methods already described above, or any other one which is convenient.

The methods outlines above do not exhaust the possible ones for finding analytic solutions. However, they will occur frequently in the next several sections. Several of the special problems treated in the following sections can be solved by each of the three approaches. The choice of a particular one has been made either to illustrate a method or because it happens to be convenient. Techniques for finding numerical solutions will not be discussed.

17. Two-dimensional progressive and standing waves in unbounded regions with fixed boundaries. In this and the following section we shall consider situations in which the region occupied by fluid extends to infinity horizontally, the solid boundaries are fixed, but of more complicated shape than the simple flat bottom considered up to now, and the motion of the fluid at infinity is prescribed, or at least partly so. We shall assume that the velocity is bounded at all interior points of the fluid and also at the infinite limits of the fluid. The motion is taken to be periodic everywhere with period σ. Hence we shall assume (cf. Sect. 11) that

$$\Phi(x, y, t) = \varphi_1(x, y)\cos\sigma t + \varphi_2(x, y)\sin\sigma t = \operatorname{Re}\varphi\, e^{-i\sigma t}.$$

The restriction to standing or progressive waves can be properly applied only at $x = \pm\infty$. Thus, we shall look for solutions which at $x = \infty$ behave like

$$(A\cos m x + B\sin m x)\cos\sigma t$$

[1] The method is used also by MUSKHELISHVILI [Singular integral equations, Noordhoff, Groningen, 1953, § 74] to reduce a mixed boundary condition of more complicated type to a simple one.

or

$$A\cos(mx+\sigma t)+B\cos(mx-\sigma t),$$

and similarly at $x=-\infty$ if the fluid extends in that direction. As we shall see below, the coefficients cannot be chosen independently if φ remains bounded everywhere.

The parameter of linearization may be chosen as

$$\varepsilon=\max(Am, Bm).$$

If the solution φ is bounded everywhere, then as $\varepsilon\to 0$, $\varphi\to 0$ uniformly. However, if a singularity is allowed, then $\varphi\to 0$ uniformly only in a region excluding a neighborhood about the singularity. One may presume that the solution to the linearized problem loses physical significance within such a neighborhood. [It is assumed by STOKER (1947, p. 5) that singularities at the surface are associated with breaking of the waves.]

We shall discuss below two types of problems: obstacles in an infinite ocean and sloping beaches. For each type a special case will be discussed in some detail.

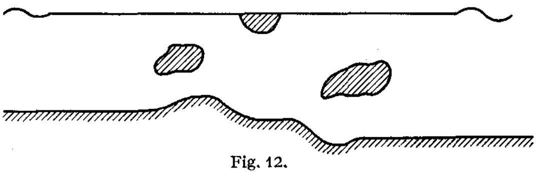

Fig. 12.

α) Obstructions in an infinitely long canal. Consider first the following situation. The fluid extends from $x=-\infty$ to $x=+\infty$; the bottom is given by $y=-h(x)$, where $h(x)=h_1>0$ for $x\geqq x_1$, $h(x)=h_2>0$ for $x\leqq x_2<x_1$; fixed obstacles may be present in the fluid or on the surface (see Fig. 12). The surface at $x=+\infty$ will be assumed to behave like

$$\eta=A_1\cos(m_1x+\sigma t+\alpha_1)+B_1\cos(m_1x-\sigma t+\beta_1)$$

and at $x=-\infty$ like

$$\eta=A_2\cos(m_2x+\sigma t+\alpha_2)+B_2\cos(m_2x-\sigma t+\beta_2).$$

A proof of the existence of a solution to this problem does not seem to exist for the general case. One would not expect a uniqueness theorem since no statement has been made about singularities or circulation. For infinite depth and a submerged body KOCHIN (1939) has proved the existence for sufficiently large values of m (the situation is slightly different, but the proof carries over). KREISEL (1949) has established the existence of a solution and its uniqueness in two cases. In the first case $h_1=h_2$, only obstacles on the bottom are allowed, Φ is assumed bounded, and a certain constant, defined in terms of the wavelength and the conformal mapping of the fluid region onto the strip $0<y<h_1$, must be less than 1. Included are theorems comparing the values of this constant for different types of obstructions. The second result allows a shallow obstruction in the surface, but requires a flat bottom and sufficiently long waves and again bounded Φ. ROSEAU (1952) has proved existence and uniqueness for no obstructions within the fluid, but for $h_1\neq h_2$; the curve joining the two ends is of a special sort. JOHN (1950, p. 78ff.) has proved uniqueness for a flat bottom and for a body in the free surface with the property that every vertical line intersects either the free surface or the body just once; certain regularity properties of Φ must also be assumed. If the body is convex and intersects the free surface perpendicularly, he is able to prove also existence of a solution.

Existence and uniqueness theorems have also been proved for several special configurations. In most of these cases explicit solutions are given. A vertical-line barrier extending from the free surface to a depth l in an infinite fluid has been considered by DEAN (1945), URSELL (1947) and HASKIND (1948). Both DEAN and URSELL, and also MARNYANSKII (1954), also consider a barrier extending from $-\infty$ to a distance l below the surface. JOHN (1948) has generalized both these problems to the case of a slanting barrier of slope $\pi/2n$, and obtained a more general solution even for the vertical barrier. DEAN (1948) and URSELL (1950) have also considered submerged circular cylinders in an infinitely deep fluid, and URSELL has established a uniqueness theorem for this case. A horizontal obstruction of finite width on the water surface (the "finite-dock problem") has been treated by RUBIN (1954), who proved existence of a solution by a variational method. Other references concerning the dock problem will be given below. BARTHOLOMEUSZ (1958) treats the long-wave approximation for reflection at a step in the bottom.

Reflection and transmission coefficients. If one assumes the existence of a solution to the general problem stated above, one may establish the form of the solution for $x > x_1$ and $x < x_2$ by using the completeness of the functions [cf. (16.1)]

$$\{\cosh m_0(y+h), \quad \cos m_n(y+h)\}$$

in the interval $-h \leqq y \leqq 0$ (cf. KREISEL 1949, pp. 26—29; JOHN 1948, p. 152). It is

$$\left.\begin{aligned} \Phi(x,y,t) &= [A_i \cos(m_0^{(i)} x + \sigma t + \alpha_i) + B_i \cos(m_0^{(i)} x - \sigma t + \beta_i)] \times \\ &\times \cosh m_0^{(i)}(y+h_i) + \sum_{n=1}^{\infty} (a_{in}\cos\sigma t + b_{in}\sin\sigma t)\exp(-m_n^{(i)}|x|)\cos m_n^{(i)}(y+h_i), \end{aligned}\right\} \tag{17.1}$$

where $i=1, 2$ and $\sigma^2 = g m_0^{(i)} \tanh m_0^{(i)} h_i = -g m_n^{(i)} \tan m_n^{(i)} h_i$.

Let us now apply the formula for dE/dt in Eq. (8.2) to the region of fluid bounded by the planes $x = c_2 < x_2$, $x = c_1 > x_1$, the bottom and any other obstructions, which we take to be between these two planes. Then, if $\varphi_x^2 + \varphi_y^2$ is bounded in the region considered,

$$\frac{dE}{dt} = \int_{-h_1}^{0} \varrho\, \Phi_t\, \Phi_x(c_1, y, t)\, dy - \int_{-h_2}^{0} \varrho\, \Phi_t\, \Phi_x(c_2, y, t)\, dy,$$

since on the "physical" boundaries [cf. Eq. (8.3)] either $p=0$ or $\Phi_n = 0$. Anticipating that we are interested only in the asymptotic values for $c_1 \to \infty$ and $c_2 \to -\infty$, we compute the above expression using only the first term in (17.1) and average over a period $2\pi/\sigma$:

see errata

$$\begin{aligned}\left[\frac{dE}{dt}\right]_{\text{av}} &= \pi\, m_0^{(1)} h_1 \left[1 + \frac{\sinh 2 m_0^{(1)} h_1}{2 m_0^{(1)} h_1}\right] [A_1^2 + B_1^2] - \\ &\quad - \pi\, m_0^{(2)} h_2 \left[1 + \frac{\sinh 2 m_0^{(2)} h_2}{2 m_0^{(2)} h_2}\right] [A_2^2 - B_2^2].\end{aligned}$$

Since the average energy in the region is constant,

see errata

$$m_0^{(1)} h_1 \left[1 + \frac{\sinh 2 m_0^{(1)} h_1}{2 m_0^{(1)} h_1}\right] [A_1^2 - B_1^2] = m_0^2 h_2 \left[1 + \frac{\sinh 2 m_0^{(2)} h_2}{2 m_0^{(2)} h_2}\right] [A_2^2 - B_2^2]. \tag{17.2}$$

This is, of course, a statement of the conservation of energy. If A_1 is given $\neq 0$ and $B_2 = 0$, then A_2, B_1 are uniquely determined. For suppose two solutions

Φ and Φ' are possible, both with the same A_1 and $B_2=0$, but one with A_2, B_1, the other with A_2', B_1'. Apply (17.2) to the difference $\Phi-\Phi'$:

$$-m_0^{(1)} h_1\left[1+\frac{\sinh 2m_0^{(1)} h_1}{2m_0^{(1)} h_1}\right](B_1-B_1^1)^2 = m_0^{(2)} h_2\left[1+\frac{\sinh 2m_0^{(2)} h_2}{2m_0^{(2)} h_2}\right](A_2-A_2^1)^2.$$

Each side must be zero since they differ in sign and are equal. Hence $A_2=A_2'$ and $B_1=B_1'$. This does not, of course, imply the uniqueness of Φ itself.

If $h_1=h_2$, then (17.2) simplifies in an obvious way:

$$A_1^2-B_1^2=A_2^2-B_2^2. \tag{17.3}$$

Here h may also be infinite.

Setting $B_2=0$ and fixing A_1 as above corresponds physically to giving the amplitude of an incoming wave far to the right. B_1 is then the amplitude of the see errata reflected wave and A_2 of the transmitted wave. The theorem of the preceding paragraph states that A_1 fixes them uniquely. We define $|B_1/A_1|$ as the *reflection coefficient* R and $|A_2/A_1|$ as the *transmission coefficient* T. They are uniquely see errata determined and $R^2+T^2=1$. Properly one should define both left and right coefficients since the channel is not symmetric. However, the uniqueness theorem implies that both have the same value [see KREISEL (1949) or MEYER (1955)]. One can clearly arrange the phases so that A_1 and A_2 have the same sign. If this is done, $\alpha_2-\alpha_1$ will be the *phase shift* caused by the obstacles.

KREISEL (1949) has proved several general theorems concerning the reflection coefficient if $h_1=h_2$. In particular, if there are no obstacles within the fluid, he determines upper and lower bounds for the reflection coefficient in terms of the conformal mapping $z(\zeta)$ of the infinite strip $0>\eta>-h$ onto the region occupied by fluid, with infinities corresponding. His bounds become closer as the wavelength increases. He gives, for example, asymptotic expressions as $m_0\to 0$ for the reflection coefficient from a horizontal reef of width a and height ε and from a flat plate in the surface of beam b, namely,

$$\frac{\varepsilon}{h}\,\frac{2m_0 h\,|\sin 2m_0 a|}{\sinh 2m_0 h'(1+2m_0 h/\sinh 2m_0 h)}$$

and

$$\frac{m_0 b}{1+2m_0 h/\sinh 2m_0 h}.$$

Other general considerations will be found in BIESEL and LE MÉHAUTÉ (1955).

An interesting special result of DEAN (1947) [see also URSELL (1950)] is that the reflection coefficient from a submerged circular cylinder in infinitely deep water vanishes. The proof may be briefly sketched. Let a be the radius and let the center be at $(0,-b)$, $b>a$. Let the velocity potential be written as a sum of an incoming wave and a diverging wave:

$$\Phi = A\nu e^{\nu y}\cos(\nu x+\sigma t)+\Phi_0;$$

and suppose that Φ_0 can be expressed as a sum of multipoles (13.31), starting with dipoles:

$$\Phi_0=\sum a_n\Phi_n^{(s)}(x,y,t)+b_n\Phi_n^{(a)}(x,y,t)+c_n\Phi_n^{(s)}\left(x,y,t+\frac{\pi}{2\sigma}\right)+d_n\Phi_n^{(a)}\left(x,y,t+\frac{\pi}{2\sigma}\right),$$

where $\Phi_n^{(s)}$ is the potential for the symmetric potential of order n and strength $Q=1$, and $\Phi_n^{(a)}$ that for the antisymmetric one. The boundary condition on the cylinder [using the notation of (13.31)],

$$\left.\frac{\partial \Phi_0}{\partial r}\right|_{r=a} = A\nu e^{-\nu b} e^{\nu a\cos\vartheta}[\{\sin(\nu a\sin\vartheta)\sin\vartheta - \cos(\nu a\cos\vartheta)\cos\vartheta\}\cos\sigma t + \\ + \{\cos(\nu a\sin\vartheta)\sin\vartheta + \sin(\nu a\sin\vartheta)\cos\vartheta\}\sin\sigma t],$$

gives the relation $a_n = -d_n$, $b_n = c_n$. The reflected wave at $+\infty$ from the anti-symmetric functions then just cancels that from the symmetric functions. They reinforce each other at $x = -\infty$. The phase change for $b/a = \frac{5}{4}$, $\sigma^2 a/g = \frac{4}{3}$ was computed numerically by both DEAN and URSELL and for this case was very close to 90°.

As mentioned above, straight-line barriers have been considered by DEAN (1945, 1946), URSELL (1947), HASKIND (1948), JOHN (1948), and LEVINE (1957). The last three authors use the reduction method, whereas the first two use a Fourier-integral method which leads to a singular integral equation. We shall treat this problem by the reduction method. DEAN and JOHN also treat barriers inclined at an angle $\pi/2n$. LEVINE and RODEMICH (1958) solve the vertical-barrier problem by several methods, including the cited ones, and then apply one of them to the problem of waves incident upon two parallel vertical barriers.

Vertical barrier. Let the barrier extend along the y-axis from $y=0$ to $y=-l$ and suppose an incoming wave is given at $x=+\infty$ as

$$\eta = A\cos(\nu x + \sigma t + \alpha), \qquad \sigma^2 = g\nu.$$

We shall look for a velocity potential Φ having the form

$$\Phi = -A\frac{g}{\sigma} e^{\nu y}\sin(\nu x + \sigma t + \alpha) + \varphi_1\cos\sigma t + \varphi_2\sin\sigma t$$

and satisfying the following boundary conditions on the free surface and the barrier:

see errata

$$\Phi_{tt} + g\,\Phi_y(x, 0, t), \quad |x| > 0 \quad \text{and} \quad \Phi_x(0, y, t) = 0, \quad 0 > y > -l.$$

As $x \to \pm\infty$, $\varphi_1\cos\sigma t + \varphi_2\sin\sigma t$ must represent outgoing waves. In the neighborhood of $(0, -l)$ it will be assumed that

$$\lim\,[x^2 + (y+l)^2]\,(\Phi_x^2 + \Phi_y^2) = 0 \quad \text{as} \quad (x, y) \to (0, -l).$$

In the neighborhood of the intersection of the barrier and the surface (0, 0) as well as in the region of fluid bounded away from the barrier, we shall assume $\Phi_x^2 + \Phi_y^2$ bounded. It should be noted, however, that this assumption excludes a large class of solutions of possible physical interest (cf. JOHN 1948).

If we introduce the stream functions Ψ, ψ_1, and ψ_2 corresponding to Φ, φ and φ_2 and the corresponding complex potentials F, f_1 and f_2, we have

$$F = \left(-\frac{Ag}{\sigma} i e^{-i(\nu z+\alpha)} + f_1\right)\cos\sigma t + \\ + \left(-\frac{Ag}{\sigma} e^{-i(\nu z+\alpha)} + f_2\right)\sin\sigma t = F_1\cos\sigma t + F_2\sin\sigma t$$

and the boundary conditions

see errata

$$\operatorname{Re}\{-\nu F_n + i F_n'\} = 0, \quad y = 0, \quad |x| > 0, \quad n = 1, 2,$$
$$\operatorname{Re} F_n' = 0, \quad x = 0, \quad 0 > y > l, \quad n = 1, 2.$$

After finding F_1 and F_2 satisfying these conditions, constants occurring in the solutions must be adjusted so that f_1 and f_2 satisfy the radiation conditions:

see errata

$$\lim_{x\to\pm\infty}(f_1'\pm\nu f_2)=0,\qquad \lim_{x\to\pm\infty}(f_2'\mp f_1)=0.$$

Consider the function

$$G_1=F_1'+i\nu F_1=e^{-i\nu z}(e^{i\nu z}F_1)'.$$

Then the boundary conditions imply that G_1 satisfies

$$\operatorname{Im} G_1=0 \quad \text{for} \quad y=0,\quad |x|>0,$$
$$\operatorname{Im} G_1'=0 \quad \text{for} \quad y=0,\quad |x|>0 \quad \text{and} \quad x=0,\quad 0>y>-l.$$

The function G_1 may be extended into the upper half-plane by defining $G_1(x+iy)=\bar{G}_1(x-iy)$ for $y>0$. Since we have assumed $|F'|\leqq B$ for $|z|>b>l$, we may conclude that $|G_1|<B+C|z|$ for $|z|>b$ and expand G_1 in a Laurent series

$$G_1(z)=cz+d+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots,\qquad |z|>b>l,$$

where all coefficients are real since $\operatorname{Im} G_1(x+i0)=0$. The condition $|F_1'|\to 0$ as $y\to-\infty$ implies $|G_1'|\to 0$ as $y\to-\infty$ and hence $c=0$. We may arbitrarily set $d=0$ by redefinition of Ψ_1. Further, we may show as follows that $a_1=0$. Consider a contour containing the obstruction and lying in the region $|z|>b$. Then

$$\oint G_1(z)\,dz=2\pi i\,a_1.$$

Let this contour be contracted onto the barrier. Then, from the assumed behavior of F_1 on the barrier, the integral vanishes; hence $a_1=0$. Thus

$$G_1(z)=\frac{a_2}{z^2}+\frac{a_3}{z^3}+\cdots.$$

Let us now exploit the boundary condition for G_1' by mapping the z-plane into a ζ-plane by the mapping

$$z=\sqrt{\zeta^2-l^2},$$

where the branch of the square root is chosen which makes $z\cong\zeta$ for large ζ. This maps the z-plane cut from $-il$ to il, i.e. along the barrier, onto the z-plane cut from $-l$ to $+l$, with infinities and upper and lower half-planes corresponding. Then $G_1'(z(\zeta))=H_1(\zeta)$ is analytic in the whole lower half-plane with a singularity only at $\zeta=0$, corresponding to $z=-il$, and $\operatorname{Im} H_1(\xi+i0)=0$. Since $H_1(\zeta)$ must agree with $G_1'(z)$ for large z, $H_1(\zeta)$ must have the form

$$H_1(\zeta)=\frac{b_3}{\zeta^3}+\frac{b_4}{\zeta^4}+\cdots,\qquad b_n \text{ real}.$$

The condition on Φ near the edge of the barrier, implies that $|z+il|\cdot|F_1'|\to 0$ as $z\to-il$, or $|\zeta^4 H_1(\zeta)|\to 0$ as $\zeta\to 0$ and hence that $b_n=0$, $n\geqq 4$. Thus

$$H_1(\zeta)=\frac{C_1}{\zeta^3},\qquad C_1 \text{ real},$$

or

$$G_1'(z)=\frac{C_1}{(z^2+l^2)^{\frac{3}{2}}}.$$

Integrating, and writing $D_1=C_1/l^2$,

$$G_1(z)=D_1\frac{z-\sqrt{z^2+l^2}}{\sqrt{z^2+l^2}}=D_1\frac{z}{\sqrt{z^2+l^2}}-D_1,$$

where the constant of integration has been chosen so as to make $G_1(z)$ behave like z^{-2} for large z. Then

$$F_1(z) = E_1\,\mathrm{e}^{-i\nu z} + D_1\,\mathrm{e}^{-i\nu z}\int_{i\infty}^{z} \frac{z-\sqrt{z^2+l^2}}{\sqrt{z^2+l^2}}\,\mathrm{e}^{i\nu z}\,dz,$$

where the path of integration will be taken around the right-hand side of the barrier. The boundary condition $\operatorname{Re} F_1'(0+iy)=0,\ 0<|y|<l$, relates E_1 and D_1 as follows. From $F_1(z)$:

$$F_1'(z) = \mathrm{e}^{-i\nu z}\left[-i\nu E_1 + D_1\frac{z}{\sqrt{z^2+l^2}} - i\nu D_1\int_{i\infty}^{z}\frac{z\,\mathrm{e}^{i\nu z}}{\sqrt{z^2+l^2}}\,dz\right].$$

Take the path of integration along the y-axis, so that the integral becomes

$$\int_{i\infty}^{z} = -i\int_{l}^{\infty}\frac{y\,\mathrm{e}^{-\nu y}}{\sqrt{y^2-l^2}}\,dy \mp \int_{l}^{y}\frac{y\,\mathrm{e}^{-\nu y}}{\sqrt{l^2-y^2}}\,dy$$

$$= -i\,l\,K_1(\nu l) \mp \int_{l}^{y}\frac{y\,\mathrm{e}^{-\nu y}}{\sqrt{l^2-y^2}}\,dy, \qquad x=\pm 0.$$

Hence

$$\operatorname{Re} F_1'(0+iy) = \mathrm{e}^{\nu y}\left[+\nu\operatorname{Im}E_1 - \nu\,l\,D_1\,K_1(\nu l)\right] = 0$$

or

$$\operatorname{Im}E_1 = +D_1\,l\,K_1(\nu l).$$

see errata Let $E_1 = e_1 + ilDK_1(\nu l)$. Then

$$F_1(z) = \mathrm{e}^{-i\nu z}\left[e_1 + i\,l\,D_1\,K_1(\nu l) + D_1\int_{i\infty}^{z}\frac{z-\sqrt{z^2+l^2}}{\sqrt{z^2+l^2}}\,\mathrm{e}^{i\nu z}\,dz\right],$$

We now compute the asymptotic expressions for $F_1(z)$ for $x\to\pm\infty$. If the path of integration is taken on a large arc of radius R in the first quadrant and then to z, and if R is allowed to become infinite, it follows from JORDAN's lemma that the integral may also be written

$$\int_{\infty}^{z}\frac{z-\sqrt{z^2+l^2}}{\sqrt{z^2+l^2}}\,\mathrm{e}^{i\nu z}\,dz.$$

Clearly,

$$F_1(z) \sim \mathrm{e}^{-i\nu z}\left[e_1 + i\,l\,D_1\,K_1(\nu l)\right] \quad \text{as} \quad x\to+\infty.$$

As $x\to-\infty$,

$$F_1(z) \sim \mathrm{e}^{-i\nu z}\left[e_1 + i\,l\,D_1\,K_1(\nu l) + D_1\int_{\infty}^{-\infty}\frac{z-\sqrt{z^2+l^2}}{\sqrt{z^2+l^2}}\,\mathrm{e}^{i\nu z}\,dz\right],$$

where the path of integration passes below the barrier. By completing this path by a large semicircle in the upper half-plane, which gives a zero contribution in the limit, and then contracting the contour about the barrier, one sees that

$$\int_{\infty}^{-\infty}\frac{z-\sqrt{z^2+l^2}}{\sqrt{z^2+l^2}}\,\mathrm{e}^{i\nu z}\,dz = +2\int_{-l}^{l}\frac{y\,\mathrm{e}^{-\nu y}}{\sqrt{l^2-y^2}}\,dy$$

$$= -2\nu\int_{-l}^{l}\mathrm{e}^{-\nu y}\sqrt{l^2-y^2}\,dy = -2\pi\,l\,I_1(\nu l).$$

Hence

$$F_1(z) \sim e^{-i\nu z}\left[e_1 + i\,l\,D_1\,K_1(\nu l) - 2\pi\,l\,D_1\,I_1(\nu l)\right] \quad \text{as} \quad x \to -\infty.$$

Similar expressions hold for $F_2(z)$ with constants e_2 and D_2.

For f_1 and f_2 we have the asymptotic expressions:

$$\left.\begin{aligned} f_1(z) &\sim e^{-i\nu z}\left[\frac{A\,g}{\sigma}\,i\,e^{-i\alpha} + e_1 + i\,l\,K_1(\nu l)\,D_1\right], \\ f_2(z) &\sim e^{-i\nu z}\left[\frac{A\,g}{\sigma}\,e^{-i\alpha} + e_2 + i\,l\,K_1(\nu l)\,D_2\right] \end{aligned}\right\} \quad \text{as} \quad x \to +\infty,$$

$$\left.\begin{aligned} f_1(z) &\sim e^{-i\nu z}\left[\frac{A\,g}{\sigma}\,i\,e^{-i\alpha} + e_1 + (i\,K_1 - 2\pi\,I_1)\,l\,D_1\right], \\ f_2(z) &\sim e^{-i\nu z}\left[\frac{A\,g}{\sigma}\,e^{-i\alpha} + e_2 + (i\,K_1 - 2\pi\,I_1)\,l\,D_2\right] \end{aligned}\right\} \quad \text{as} \quad x \to -\infty.$$

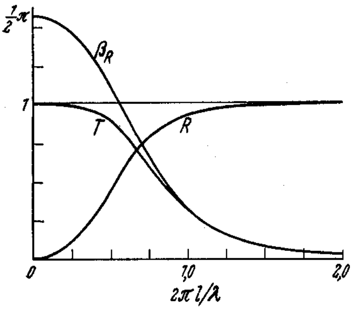

Fig. 13.

The radiation condition gives simultaneous equations for the determination of e_1, e_2, D_1 and D_2. The solution may be written

$$l(D_1 + i\,D_2) = -\frac{A\,g}{\sigma}\,i\,e^{-i\alpha}\,\frac{1}{\pi I_1 + i\,K_1},$$

$$e_1 + i\,e_2 = -\frac{A\,g}{\sigma}\,i\,e^{-i\alpha}\left(1 + \frac{\pi\,I_1}{\pi\,I_1 + i\,K_1}\right).$$

Substitution in the expressions for f_1 and f_2, and computation of $F_1\cos\sigma t + F_2\sin\sigma t$ give, after a somewhat tedious calculation, the following asymptotic expressions for Φ:

$$\left.\begin{aligned} \Phi &\sim \frac{A\,g}{\sigma}\,e^{\nu y}\left\{-\sin(\nu x + \sigma t + \alpha) + \frac{\pi\,I_1}{\sqrt{\pi I_1^2 + K_1^2}}\,\sin(\nu x - \sigma t - \alpha - \beta_R)\right\}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad x \to +\infty, \\ \Phi &\sim \frac{-A\,g}{\sigma}\,e^{\nu y}\,\frac{K_1}{\sqrt{\pi^2 I_1^2 + K_1^2}}\,\sin(\nu x + \sigma t + \alpha + \beta_T), \qquad x \to -\infty, \end{aligned}\right\} \tag{17.4}$$

see errata

where $\tan\beta_R = K_1/\pi I_1 = \cot\beta_T$, and $I_1 = I_1(\nu l)$, $K_1 = K_1(\nu l)$. Clearly the reflection and transmission coefficients are

$$R = \frac{\pi\,I_1}{\sqrt{\pi I_1^2 + K_1^2}}, \qquad T = \frac{K_1}{\sqrt{\pi I_1^2 + K_1^2}}. \tag{17.5}$$

R, T and $\beta_R = \frac{1}{2}\pi - \beta_T$ are shown in Fig. 13 as functions of $2\pi l/\lambda = \nu l$, The reflection coefficient is practically one if $l/\lambda \geqq \frac{1}{4}$.

One may now use the velocity potential to find the behavior of the fluid near the barrier, in particular, the water height and the pressure. The calculations will not be carried through, but may be found in HASKIND (1948). The elevation on either side of the barrier is given by

$$\eta(\pm 0, t) = A\left[\cos(\sigma t + \alpha) \mp \frac{1 + \nu l\,S(\nu l)}{\sqrt{\pi^2 I_1^2 + K_1^2}}\cos(\sigma t + \alpha + \beta_R)\right] \tag{17.6}$$

where

$$S(\nu l) = \frac{\pi}{2\nu l}\left[I_1(\nu l) + L_1(\nu l)\right] = \int_0^1 e^{\nu l y}\sqrt{1 - y^2}\,dy,$$

L_1 being a Struve function of imaginary argument[1]. Let the force and moment about the origin, per unit length of barrier, be denoted by X and M, the former being positive if directed along OX and the latter counterclockwise. Then

$$\left.\begin{aligned} X &= +2\varrho g A l X_0 \cos(\sigma t + \alpha + \beta_R), \\ M &= +2\varrho g A l^2 M_0 \cos(\sigma t + \alpha + \beta_R), \end{aligned}\right\} \tag{17.7}$$

where

$$X_0 = \frac{S}{\sqrt{\pi I_1^2 + K_1^2}}, \quad M_0 = \frac{1}{\nu l \sqrt{\pi I_1^2 + K_1^2}}\left(S - \frac{\pi}{4}\right).$$

HASKIND also computes the average force and moment per unit length of the barrier. The results are:

$$\left.\begin{aligned} -X_{av} &= \frac{1}{2}\varrho g A^2 \frac{\pi^2 I_1^2}{\pi^2 I_1^2 + K_1^2} = \frac{1}{2}\varrho g A^2 R^2, \\ -M_{av} &= \frac{1}{2}\varrho g A^2 l\left[S(-\nu l) - T(-\nu l) - \frac{\pi I_1(\nu l)}{2\nu l}\right]\frac{\pi I_1}{\pi_1^2 I_1^2 + K_1^2}, \end{aligned}\right\} \tag{17.8}$$

where

$$\nu l\, S(-\nu l) = \tfrac{1}{2}\pi\,[I_1(\nu l) - L_1(\nu l)],$$
$$T(-\nu l) = \tfrac{1}{2}\pi\,[I_0(\nu l) - L_0(\nu l)].$$

Fig. 14 displays all four functions in dimensionless form.

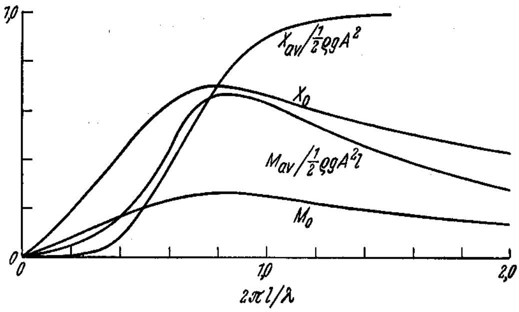

Fig. 14.

The method of integral equations. This method for finding solutions has been frequently used, especially by KOCHIN (1937, 1939, 1940) and his colleagues. One of its advantages is that approximate solutions to the integral

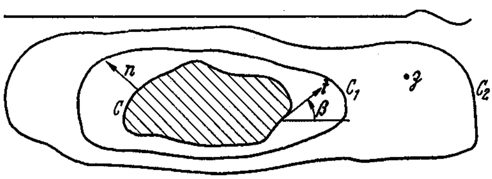

Fig. 15.

equation can frequently be found even when an explicit solution cannot be easily obtained. The following exposition follows approximately KOCHIN (1937) and KELDYSH and LAVRENT'EV (1937).

Consider a submerged obstacle whose contour C is given parametrically by $z = z(s)$ and is oriented counterclockwise. Let $\beta(s)$ be the angle between the tangent vector and the positive x-direction (see Fig. 15). We shall assume that

[1] G. N. WATSON: Bessel functions, p. 329; L_1 is tabulated in J. Math. Phys. **25**, 252—259 (1946).

as $x \to \infty$ the motion approximates to a standing wave:

$$\Phi(x, y, t) \sim A \frac{g}{\sigma} e^{\nu y} \cos(\nu x + \alpha) \cos(\sigma t + \tau), \qquad \nu = \frac{\sigma^2}{g}. \tag{17.9}$$

The other boundary conditions in terms of the complex potential $f(z) = \varphi(x, y) + i\psi(x, y)$ are

$$\left.\begin{aligned} \operatorname{Im}\{f'(x) + i f(x)\} &= 0, \\ \operatorname{Im}\{f'(z(s))\, e^{i\beta(s)}\} &= 0, \\ \lim_{y\to-\infty} |f'| &= 0. \end{aligned}\right\} \tag{17.10}$$

Write $f(z)$ in the form

$$f(z) = f_1(z) + \frac{A g}{\sigma} e^{-i(\nu z + \alpha)} = f_1(z) + a\, e^{-i\nu z}. \tag{17.11}$$

Then $f_1(z)$ must satisfy

$$\left.\begin{aligned} \lim_{x\to\infty} f_1(z) &= 0, \\ \operatorname{Im}\left[f_1'(z(s)) - i a \nu\, e^{-i\nu z(s)}\right] e^{i\beta(s)} &= 0, \end{aligned}\right\} \tag{17.12}$$

as well as the free surface condition and the condition as $y \to -\infty$.

We shall try to express $f_1(z)$ as a distribution of vortices over the contour C. However, the vortices are chosen so that the conditions on the free surface, at $x = \infty$ and at $y = -\infty$ are satisfied. As is apparent from the derivation of (13.28), the complex velocity potential for such vortices is given by

$$f_v(z; c) = \frac{\Gamma}{2\pi i}\left\{\log(z - c)(z - \bar{c}) - 2 e^{-i\nu z} \int_{\infty}^{z} \frac{e^{i\nu u}}{u - \bar{c}}\, du\right\}. \tag{17.13}$$

We set $\Gamma = 1$ and try to express $f_1(z)$ as follows:

$$f_1(z) = \int_C \gamma(s)\, f_v(z; z(s))\, ds, \tag{17.14}$$

where $\gamma(s)$ must be chosen so that the boundary condition on the body is satisfied.

In order to derive an integral equation for $\gamma(s)$, consider the following expression for $f_1'(z)$, a direct consequence of CAUCHY's integral:

$$f_1'(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f'(\zeta)}{z - \zeta}\, d\zeta - \frac{1}{2\pi i}\int_{C_2} \frac{f'(z)}{z - \zeta}\, d\zeta = g_1(z) + g_2(z).$$

The function $g_1(z)$ is regular everywhere outside C_1 and $g_2(z)$ is regular everywhere inside C_2. One may contract C_1 onto C and extend $g_2(z)$ analytically into the whole lower half-plane (or fluid strip if the depth is finite).

Consider now (for infinite depth; the finite-depth case is analogous) the following function:

$$g(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f_1'(\zeta)}{z - \zeta}\, d\zeta + \frac{1}{2\pi i}\int_{C_2} \overline{f_1'(\zeta)}\left[\frac{1}{z - \bar{\zeta}} - 2 i\nu\, e^{-i\nu z}\int_{\infty}^{z} \frac{e^{i\nu u}}{u - \bar{\zeta}}\, du\right] d\bar{\zeta}.$$

The first summand is identical with $g_1(z)$ and the second is also regular in the whole half-plane. $g(z)$ satisfies the same boundary conditions as $f_1'(z)$. Hence $f_1'(z) - g(z)$ is regular in the whole lower half-plane, satisfies the free-surface condition and vanishes as $y \to -\infty$ and $x \to +\infty$. The uniqueness argument used in the derivation of (13.28) shows that $f_1'(z) \equiv g(z)$. Thus we have

$$f_1'(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f_1'(\zeta)}{z-\zeta}\, d\zeta - \frac{1}{2\pi i}\int_{C_1} \overline{f_1'(\zeta)} \left[\frac{1}{z-\bar\zeta} - 2i\nu\, e^{-i\nu z}\int_\infty^z \frac{e^{i\nu u}}{u-\bar\zeta}\, du\right] d\bar\zeta. \quad (17.15)$$

Now contract C_1 to C. Then

$$[f_1'(\zeta) - i a\nu\, e^{-i\nu\zeta}]\, e^{i\beta} = v_t + i v_n = v_t$$

or

$$f_1'(\zeta) = v_t\, e^{-i\beta} + i a\nu\, e^{-i\nu\zeta}. \quad (17.16)$$

If one substitutes above, one finds that the contribution from the second summand in $f_1'(\zeta)$ vanishes and that, since $d\zeta/ds = e^{i\beta(s)}$,

$$\left.\begin{aligned} f_1'(z) &= \frac{1}{2\pi i}\int_C v_t(s)\left[\frac{1}{z-\zeta} - \frac{1}{z-\bar\zeta} + 2i\nu\, e^{-i\nu z}\int_\infty^z \frac{e^{i\nu u}}{u-\bar\zeta}\, du\right] ds \\ &= \int_C v_t(s)\, f_v'(z; z(s))\, ds. \end{aligned}\right\} \quad (17.17)$$

This identifies $\gamma(s)$ as the tangential velocity $v_t(s)$ at a point of the contour.

Let us now consider the effect of letting $z \to z(s')$, a point of the contour C. Then, according to the Theorem of PLEMELJ-SOKHOTSKII,

$$\left.\begin{aligned} \int_C \gamma(s)\, f_v'(z; z(s))\, ds &= \int_C \gamma(s)\, f_v'(z; z(s))\, e^{-i\beta(s)}\, dz(s) \to \tfrac{1}{2}\gamma(s')\, e^{-i\beta(s')} + \\ &\qquad + \mathrm{PV}\int_C \gamma(s)\, f_v'(z(s'); z(s))\, ds, \end{aligned}\right\} \quad (17.18)$$

whereas

$$f_1'(z) \to v_t(s')\, e^{-i\beta(s')} + i a\nu\, e^{-i\nu z(s')} = \gamma(s')\, e^{-i\beta(s')} + i a\nu\, e^{-i\nu z(s')}.$$

Hence we have the integral equation for $\gamma(s)$:

$$-\tfrac{1}{2}\gamma(s') + \mathrm{PV}\int_C \gamma(s)\, f_v'(z(s'); z(s))\, e^{i\beta(s')}\, ds = i A\sigma\, e^{-i[\nu z(s') - \beta(s') + \alpha]}. \quad (17.19)$$

This is really two integral equations. The imaginary part gives a singular integral equation of the first kind:

$$\mathrm{PV}\int_C \gamma(s)\, K(s', s)\, ds = -2\pi A\sigma\, e^{\nu y(s')} \cos[\nu x(s') - \beta(s') + \alpha]. \quad (17.20)$$

The real part gives a Fredholm equation of the second kind with continuous kernel:

$$-\frac{1}{2}\gamma(s') + \frac{1}{2\pi}\int_C \gamma(s)\, L(s', s)\, ds = 2\pi A\sigma\, e^{\nu y(s')} \sin[\nu x(s') - \beta(s') + \alpha]. \quad (17.21)$$

Here

$$f_v'(z(s'); z(s))\, e^{i\beta(s')} = \frac{1}{2\pi i}[K(s', s) + i L(s', s)]. \quad (17.22)$$

The kernel $K(s', s)$ is of the form

$$K(s', s) = \frac{1}{s'-s} + C(s', s), \quad (17.23)$$

where $C(s', s)$ is continuous; the first term comes from $e^{i\beta(s')}/[z(s') - z(s)]$. If the curve C is sufficiently smooth,

$$\lim_{s' \to s} \mathrm{Im} \frac{e^{i\beta(s')}}{z(s') - z(s)} = \frac{1}{\varrho(s)}, \quad (17.24)$$

where $\varrho(s)$ is the radius of curvature of C at $z(s)$.

If the obstacle consists of a smooth arc, an analogous argument leads to only the singular integral equation above, but with $\gamma(s)$ now identified with the jump in $v_t(s)$ as one goes from the left to the right side of the arc.

There does not seem to be a published proof that a solution to either integral equation exists for all ν. However, KOCHIN (1936, pp. 119—126) shows the existence of a solution for both sufficiently large and sufficiently small values of ν for the equation of the second kind when the body is completely submerged.

By adjusting the phases in (17.9) one may obtain two Φ's which may be added to give an outgoing progessive wave. The behavior as $x \to -\infty$ will then be a superposition of an incoming and an outgoing wave. However, one may also modify the preceding arguments in order to treat the pregressive-wave problem directly. One specifies, say, an incoming wave from the right, writes

$$\Phi(x, y, t) = \frac{A g}{\sigma} e^{\nu y} \cos(\nu x + \sigma t) + \Phi^*(x, y, t), \tag{17.25}$$

where Φ^* must satisfy the radiation condition, and tries to express the corresponding complex potential as a distribution of the vortices (13.28) since they already satisfy the radiation condition. We shall not dwell on the details except to remark that the problem leads to a pair of coupled integral equations since one needs a distribution not only of (13.20) as it stands, but also of the vortices obtained by replacing t by $t - \pi/2\sigma$. This method could have been applied, for example, to the problem of the vertical barrier considered above.

Dock problems. This term is generally applied to water-wave problems in which the obstruction is a horizontal plane of finite or semi-infinite extent, either submerged or lying on the surface. The solution for the semi-infinite dock in infinitely deep water was given by FRIEDRICHS and LEWY (1948), and at about the same time the same problem in water of finite depth was treated by A. HEINS (1948) who also allowed a restricted type of three-dimensional motion. The methods were quite different. Subsequently HEINS (1950) and GREENE and HEINS (1953) extended the treatment to submerged docks in water of finite and infinite depth. As was remarked earlier, RUBIN (1954) has shown the existence of a solution for the finite dock in infinitely deep water. SPARENBERG (1957) has deduced an integral equation of the second kind for this problem.

As an example, consider a submerged dock at depth b and extending from $x = -a$ to $x = a$. The integral equation (17.20) then becomes

$$\mathrm{PV}\int_{-a}^{a} \gamma(\xi)\, K(x, \xi)\, d\xi = -2\pi A \sigma e^{-\nu b} \cos(\nu x + \alpha), \tag{17.26}$$

where $K(x, \xi) = K(x - \xi)$ with

$$\left.\begin{aligned} K(x) &= \mathrm{Re}\left\{\frac{1}{x} - \frac{1}{x - 2ib} + i\frac{2\nu}{\pi} e^{-i\nu x} \int_{\infty}^{x} \frac{e^{-i\nu u}}{u - 2ib}\, du\right\} \\ &= \frac{1}{x} - \frac{x}{x^2 + 4b^2} - \frac{2\nu}{\pi} \int_{\infty}^{x} \frac{u \sin\nu(u - x) + 2b\cos\nu(u - x)}{u^2 + 4b^2}\, du. \end{aligned}\right\} \tag{17.27}$$

Without actually establishing the existence of a solution to (17.26), KELDYSH and LAVRENT'EV (1937) in treating the flow about thin hydrofoils (see Sect. 20β) propose an approximate method of solution by expanding $\gamma(x)$ and $K(x)$ in a series in $\tau = a/2b$:

$$\gamma(x) = \gamma_0(x) + \gamma_1(x)\,\tau + \cdots,$$
$$K(x) = \frac{1}{x} + a\sum_n K_n\left(\frac{x}{a}\right)^n \tau^{n+1}$$

and determining the $\gamma_n(x)$ recursively. In the problem treated by them the total vorticity was fixed by the Kutta-Joukowski condition, in the present problem the corresponding condition is still to be determined.

If the dock extends from $-\infty$ to 0, one may modify the earlier arguments so as to apply to an unbounded body and derive the integral equation

$$\mathrm{PV}\int_{-\infty}^{0} \gamma(\xi)\,K(x-\xi)\,d\xi = -2\pi A\,\sigma\,e^{-\nu b}\cos(\nu x + \alpha). \tag{17.28}$$

An integral equation of this form is known as a Wiener-Hopf integral equation and in many cases can be solved by use of Fourier transforms. It does not seem possible to expound the method briefly, so we refer to the paper of GREENE and HEINS (1953) where this problem is treated, but with the kernel expressed differently.

When the semi-infinite dock is on the surface, the dock may be considered as a limiting case of a beach in which the angle between the bottom and the free surface is 180°. Although waves on beaches are discussed in the next section, the methods which allow extension of the angle to 180° are also difficult and will not be considered there. They may be found in STOKER's *Water waves* (1957, § 5.4).

β) Waves on beaches. Let the fluid at rest be contained in the wedge defined by

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$$\tan\gamma \leqq \frac{-y}{x} \leqq 1, \quad x>0, \quad y>0,$$

i.e., the bottom is the plane $x\sin\alpha + y\cos\alpha = 0$. For such a body of fluid one may look for periodic waves which are either standing or progressive. The appropriate mathematical problem for standing wave is to find a velocity potential

$$\Phi(x,y,t) = \varphi(x,y)\cos(\sigma t + \tau) \tag{17.29}$$

satisfying

1. $\Delta\varphi = 0$,
2. $\varphi_y(x,0) - \dfrac{\sigma^2}{g}\,\varphi(x,0) = 0$,
3. $\varphi_x\sin\gamma + \varphi_y\cos\gamma = 0$ for $x\sin\gamma + y\cos\gamma = 0$,
4. $\lim\limits_{x^2+y^2\to\infty} \varphi_x^2 + \varphi_y^2 = 0$ for $x\sin\gamma + y\cos\gamma = 0$.

This problem, in both this form and the three-dimensional form to be considered in Sect. 18, has received intensive study in recent years (e.g., MICHE 1944, LEWY 1945, STOKER 1947, FRIEDRICHS 1948, ISAACSON 1948, 1950, WEINSTEIN 1949, PETERS 1950, 1952, ROSEAU 1952, LEHMAN 1954, BRILLOUËT 1957). In particular, the cited work of BRILLOUËT and Chap. 5 of STOKER's *Water waves* (1957) contain a general exposition of the mathematical theory. We shall restrict the present treatment to simple cases.

KIRCHHOFF (1879) was apparently the first one to treat the two-dimensional case. The problem was taken up again by MACDONALD (1896), POCKLINGTON (1921), and by HANSON (1926), who considered both the two and three-dimensional cases. All these authors restricted the solution to be bounded everywhere. This has the effect of excluding a physically important class of solutions with singularities at the origin. One may see this easily if $\gamma = 90°$, i.e. when there is a vertical cliff. A bounded solution is obviously $\varphi(x,y) = A\,e^{\nu y}\cos\nu x$, $\nu = \sigma^2/g$. This generates a standing wave behaving like $\cos\nu x$ at $x=\infty$. However, if we wish to construct a solution behaving, say, like an incoming wave at infinity we need also a standing-wave solution behaving like $\sin\nu x$ at infinity. No such solution exists which is bounded everywhere. However, as we shall see, it is possible to construct such a solution by allowing a singularity at the origin. If the two standing-wave solutions are used to construct an incoming progressive wave, the consequent loss of energy associated with the singularity is sometimes interpreted physically as representing loss of energy in breaking of the waves, at least when α is sufficiently small for this to happen. There is, of course, no a priori method of selecting the mathematical solution best representing the physical phenomena. The comparison between physical waves and mathematical solutions is discussed briefly in STOKER (1957, pp. 69—77).

KIRCHHOFF'S approach to the solution is interesting historically because of its similarity to the method used later by PETERS (1950) and ROSEAU (1951). His reasoning runs as follows, with a slight change in notation. Let $f(z)=\varphi+i\psi$ be the complex potential. Then

$$2\varphi(x,y)=f(x+iy)+\bar{f}(x-iy),$$
$$2i\psi(x,y)=f(x+iy)-\bar{f}(x-iy).$$

The free-surface condition becomes

$$i[f'(x)-\bar{f}'(x)]=\nu[f(x)+\bar{f}(x)],\qquad \nu=\sigma^2/g.$$

But then also

$$i[f'(z)-\bar{f}'(z)]=\nu[f(z)+\bar{f}(z)]. \tag{17.30}$$

The bottom must be a streamline. Hence

$$f(r\,e^{-i\gamma})-\bar{f}(r\,e^{i\gamma})=\text{const};$$

we may take this constant as 0. From this

$$\bar{f}(z)=f(z\,e^{-i2\gamma}). \tag{17.31}$$

Hence

$$\frac{d}{dz}[f(z)-f(z\,e^{-i2\gamma})]=-i\nu[f(z)+f(z\,e^{-i2\gamma})]. \tag{17.32}$$

This differential-difference equation must hold for all z for which $f(z)$ and $f(z\,e^{-i2\gamma})$ are both defined, namely for

$$-\gamma<\arg z<\gamma.$$

KIRCHHOFF'S formal arguments need to be supported in terms of analytic continuation by the reflection principle, but the essential idea is the same as that used more recently (cf., e.g., LEHMAN, 1954, § 3, or PETERS, 1950, § 3).

KIRCHHOFF proceeds to solve this equation in the special case $\gamma=m\pi/n$, m and n relatively prime integers, by assuming

$$f(z)=\sum_{k=0}^{n-1}A_k\exp(i\lambda\nu z\beta^k),\qquad \beta=e^{-i\frac{2m\pi}{n}}. \tag{17.33}$$

Substitution in (17.32) gives

$$A_k(\beta^k\lambda+1)=A_{k-1}(\beta^k\lambda-1),\qquad k=0,\dots,n-1, \tag{17.34}$$

with $A_{-1}\equiv A_{n-1}$. Multiplying all equations together and remembering that 1, $\beta,\dots,\beta^{n-1}$ are all n-th roots of unity, one finds

$$\lambda^n-(-1)^n=\lambda^n-1,$$

which can hold only if n is even, say $n=2q$ (hence m is odd). With $\lambda=-1=\beta^q$, the above equations determine successively $A_1, \ldots, A_{q-1}$ in terms of A_0, and $A_q=\cdots=A_{n-1}=0$:

$$A_k = i A_{k-1} \cot k\gamma = i^k A_0 \cot\gamma \cot 2\gamma \ldots \cot k\gamma. \tag{17.35}$$

Then

$$f(z) = \sum_{k=0}^{q-1} A_k \exp(-i\nu\beta^k z). \tag{17.36}$$

A_0 is still an arbitrary complex constant. The differential-difference equation is a necessary condition for $f(z)$, but not sufficient to ensure that all boundary conditions are satisfied. If one substitutes the above expression for $f(z)$ in (17.31), one finds after some computation that one must take

$$A_0 = B_0 e^{-i\pi(q-1)/4}, \tag{17.37}$$

where B_0 is pure imaginary (say $i B_0'$) if both $\frac{1}{2}(m+1)$ and q are even and otherwise is real. With this choice of A_0 one has

$$A_{q-k} = \overline{A}_{k-1}. \tag{17.38}$$

As Kirchhoff points out, the solution is physically acceptable for the problem at hand only if $m=1$; otherwise, φ does not remain bounded as $x\to+\infty$. If $m=1$, then for $y=0$, the dominant term as $x\to\infty$ is given by

$$\left.\begin{aligned} f(x) &\sim B_0 \exp\left(-i\nu x - i\pi\frac{q-1}{4}\right) \\ \text{or}\qquad \varphi(x,0) &\sim B_0 \cos\left(\nu x + \pi\frac{q-1}{4}\pi\right). \end{aligned}\right\} \tag{17.39}$$

Here are several easily computable special cases of (17.36):
$\gamma=90°$ $(m=1, q=1, \beta=-1)$:

$$f(z) = B_0 e^{-i\nu z} = B_0 e^{\nu y}(\cos\nu x - i\sin\nu x); \tag{17.40}$$

$\gamma=45°$ $(m=1, q=2, \beta=-i)$:

$$\left.\begin{aligned} f(z) &= B_0 e^{-i\frac{\pi}{4}}\left[e^{-i\nu z} + i e^{-\nu z}\right] \\ &= B_0\left[e^{\nu y}\cos\left(\nu x + \frac{\pi}{4}\right) + e^{-\nu x}\cos\left(\nu y - \frac{\pi}{4}\right)\right] - \\ &\quad - i B_0\left[e^{\nu y}\sin\left(\nu x + \frac{\pi}{4}\right) + e^{-\nu x}\sin\left(\nu y - \frac{\pi}{4}\right)\right]; \end{aligned}\right\} \tag{17.41}$$

$\gamma=30°$ $(m=1, q=3, \beta=\frac{1}{2}(\sqrt{3}-i))$:

$$\begin{aligned} f(z) &= B_0 e^{-i\frac{\pi}{2}}\left[e^{-i\nu z} + i\sqrt{3}\, e^{-\frac{1}{2}(\sqrt{3}+i)\nu z} - e^{-\frac{1}{2}(\sqrt{3}-i)\nu z}\right] \\ &= B_0\{-e^{\nu y}\sin\nu x - e^{-\frac{1}{2}\nu(x\sqrt{3}+y)}\sin\tfrac{1}{2}\nu(x-y\sqrt{3}) + \\ &\quad + \sqrt{3}\, e^{-\frac{1}{2}\nu(x\sqrt{3}-y)}\cos\tfrac{1}{2}\nu(x+y\sqrt{3})\} + \\ &\quad + i B_0\{-e^{\nu y}\cos\nu x + e^{-\frac{1}{2}\nu(x\sqrt{3}+y)}\cos\tfrac{1}{2}\nu(x-y\sqrt{3}) - \\ &\quad - \sqrt{3}\, e^{-\frac{1}{2}\nu(x\sqrt{3}-y)}\sin\tfrac{1}{2}\nu(x+y\sqrt{3})\}. \end{aligned}$$

Numerical computations for $\varphi(x,y)$ for $\gamma=6°$ $(q=15)$ as well as for the above cases were carried out by STOKER (1947) and are presented graphically in his paper.

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KIRCHHOFF'S solution is limited to the special choice of angle noted above and furthermore presents only solutions which are bounded at the origin. The solution of the differential-difference equation (17.32) for arbitrary γ, $0<\gamma\leqq\pi$, has been given by both PETERS (1950), ISAACSON (1950), and ROSEAU (1952, Chap. V). All use Laplace transforms. However, the method cannot be expounded briefly and we refer to either the original papers or STOKER'S *Water waves* for the details.

The special case $\gamma=\pi/2q$ can be treated fairly simply by the reduction method used in the problem of the vertical barrier.

From (17.32) we have

$$f^{(k+1)}(z)+i\nu f^{(k)}(z)=\beta^{k+1}f^{(k+1)}(\beta z)-i\nu\beta^k f^{(k)}(\beta z),\quad k=0,1,\ldots. \tag{17.42}$$

The free surface condition [cf. (11.7)] implies

$$\operatorname{Im}\{f^{(k+1)}(x)+i\nu f^{(k)}(x)\}=0,\quad x>0. \tag{17.43}$$

Hence also

$$\operatorname{Im}\{\beta^{k+1}f^{(k+1)}(x\beta)-i\nu\beta^k f^{(k)}(x\beta)\}=0,\quad x>0.$$

This last equation can also be written

$$\operatorname{Im}\{\beta^{k+1}f^{(k+1)}(z)-i\nu\beta^k f^{(k)}(z)\}=0\quad\text{for}\quad z=r\,\mathrm{e}^{-2i\gamma}. \tag{17.44}$$

If the numbers a_k and a'_k are real, (17.43) and (17.44) imply

$$\left.\begin{aligned}&\operatorname{Im}\left\{\sum_{k=0}^{s}a_k[f^{(k+1)}(x)+i\nu f^{(k)}(x)]\right\}=0,\\&\operatorname{Im}\left\{\sum_{k=0}^{s}a'_k\beta^k[\beta f^{(k+1)}(r\,\mathrm{e}^{-2i\gamma})-i\nu f^{(k)}(r\,\mathrm{e}^{-2i\gamma})]\right\}=0.\end{aligned}\right\} \tag{17.45}$$

We wish to find numbers $\{a_k\}$ and $\{a'_k\}$ such that

$$\sum_{k=0}^{s}a_k[f^{(k+1)}(z)+i\nu f^{(k)}(z)]\equiv\sum_{k=0}^{s}a'_k\beta^k[\beta f^{(k+1)}(z)-i\nu f^{(k)}(z)]. \tag{17.46}$$

Comparing coefficients of derivatives of the same order, one finds

$$\left.\begin{aligned}a_0&=-a'_0,\\a_{k-1}+i\nu a_k&=\beta^k(a'_{k-1}-i\nu a'_k),\quad k=1,\ldots,s,\\a_s&=\beta^{s+1}a'_s.\end{aligned}\right\} \tag{17.47}$$

These relations will be satisfied if one takes $s=q-1$ (for $\beta^q=-1$) and

$$\left.\begin{aligned}a_k&=-a'_k=a_{k-1}\frac{1}{i\nu}\,\frac{\beta^k+1}{\beta^k-1}=a_{k-1}\frac{1}{\nu}\cot k\gamma,\\&=\frac{a_0}{\nu^k}\cot\gamma\cot 2\gamma\ldots\cot k\gamma,\quad k=1,\ldots,s.\end{aligned}\right\} \tag{17.48}$$

We note that $\nu^{q-k}a_{q-k}=\nu^{k-1}a_{k-1}$. With this choice of the coefficients $\{a_k\}$, define

$$\left.\begin{aligned}g(z)&=\sum_{k=0}^{q-1}a_k\{f^{(k+1)}(z)+i\nu f^{(k)}(z)\}=P\left(\frac{d}{dz}\right)\left(\frac{d}{dz}+i\nu\right)f(z),\\&=-\sum_{k=0}^{q-1}a_k\beta^k\{\beta f^{(k+1)}(z)-i\nu f^{(k)}(z)\}=-P\left(\beta\frac{d}{dz}\right)\left(\beta\frac{d}{dz}-i\nu\right)f(z)\\&=\sum_{k=0}^{q-1}a_k\{f^{(k+1)}(\beta z)-i\nu f^{(k)}(\beta z)\}=P\left(\frac{d}{dz}\right)\left(\frac{d}{dz}-i\nu\right)f(\beta z)\end{aligned}\right\} \tag{17.49}$$