In the next to the last equation we shall change the order of integration, write

$$\cos R\lambda k = \cos R\lambda\nu \cos R\lambda(k-\nu) - \sin R\lambda\nu \sin R\lambda(k-\nu)$$

and use the following theorem from the theory of Fourier integrals[1]: If $f(x)$ is a differentiable function in $[a,\infty]$, if $f''(x_0)$, $x_0 > a$, exists, and if $f(x)/x$ and $f'(x)/x$ are both absolutely integrable in $[a,\infty]$, then, as $R\to\infty$,

$$\int_a^\infty f(x)\,\frac{\sin R(x-x_0)}{x-x_0}\,dx = \pi f(x_0) + O\left(\frac{1}{R}\right), \quad \mathrm{PV}\int_a^\infty f(x)\,\frac{\cos R(x-x_0)}{x-x_0}\,dx = O\left(\frac{1}{R}\right). \tag{13.16}$$

Remembering that both $r^{-1}$ and $r_1^{-1}$ are $O(R^{-1})$, one finds

$$\varphi_1(x,y,z) = -4\nu\, e^{\nu(y+b)}\int_0^1 (1-\lambda^2)^{-\frac{1}{2}} \sin R\lambda\nu\, d\lambda + O(R^{-1}).$$

The asymptotic expansion of this integral is well known[2] and we may write

$$\varphi_1(x,y,z) = -2\pi\nu\, e^{\nu(y+b)}\sqrt{\frac{2}{\pi R\nu}}\sin\left(R\nu - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right).$$

If we can find a harmonic function $\varphi_2$ satisfying 1., 2. and 4. and having the asymptotic behavior

$$\varphi_2(x,y,z) = 2\pi\nu\, e^{\nu(y+b)}\sqrt{\frac{2}{\pi R\nu}}\cos\left(R\nu - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right),$$

then

$$\varphi_1\cos\sigma t + \varphi_2\sin\sigma t = -2\pi\nu\, e^{\nu(y+b)}\sqrt{\frac{2}{\pi R\nu}}\sin\left(R\nu - \sigma t - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right)$$

will be a solution. The following function fulfils the requirements[3]:

$$\varphi_2(x,y,z) = 2\pi\nu\, e^{\nu(y+b)} J_0(R\nu).$$

We note in passing that $\varphi_1$ has the same asymptotic behavior as

$$-2\pi\nu\, e^{\nu(y+b)} Y_0(R\nu).$$

The final result is

$$\left.\begin{aligned}\Phi(x,y,z,t) = {} & \left[\frac{1}{r} + \mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\, e^{k(y+b)} J_0(kR)\,dk\right]\cos\sigma t + \\ & + 2\pi\nu\, e^{\nu(y+b)} J_0(\nu R)\sin\sigma t, \qquad \nu = \sigma^2/g.\end{aligned}\right\} \tag{13.17}$$

Haskind (1954), using a derivation having some similarity to that used below for the two-dimensional case, has found the following form for $\Phi$:

$$\left.\begin{aligned}\Phi(x,y,z,t) = {} & \left[\frac{1}{r} + \frac{1}{r_1} + 2\nu\, e^{\nu y}\int_\infty^y \frac{e^{-\nu y}}{r_1}\,dy - 2\pi\nu\, e^{\nu(y+b)} Y_0(\nu R)\right]\cos\sigma t + \\ & + 2\pi\nu\, e^{\nu(y+b)} J_0(\nu R)\sin\sigma t.\end{aligned}\right\} \tag{13.17'}$$

It is sometimes convenient to use the complex form for the potential, $\varphi\, e^{-i\sigma t}$, with

$$\varphi(x,y,z) = \frac{1}{r} + \mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\, e^{k(y+b)} J_0(kR)\,dk + i\,2\pi\nu\, e^{\nu(y+b)} J_0(\nu R), \tag{13.17''}$$

---

[1] See, e.g.: S. Bochner, Vorlesungen über Fouriersche Integrale, Leipzig, 1932, ch. I and § 8.
[2] See, e.g., A. Erdélyi: Asymptotic expansions, p. 43. Dover, New York 1956.
[3] See the first Eq. (13.7) and G. N. Watson: Bessel functions, p. 199. Cambridge 1949.

for then $\operatorname{Re} \varphi\, e^{-i\sigma t}$ gives (13.17) and $\operatorname{Im} \varphi\, e^{-i\sigma t}$ the source potential for an outgoing wave with singularity of the form $r^{-1} \sin \sigma t$. Eq. (13.17′) may be written analogously. By deforming the path of integration in a familiar way one may also express $\varphi(x, y, z)$ in the form [cf. Havelock (1942, 1955)]:

$$\left.\begin{aligned}\varphi(x, y, z) = \frac{1}{r} + \frac{1}{r_1} - \frac{4\nu}{\pi}\int_0^\infty [\nu \cos k(y+b) - k \sin k(y+b)] \frac{K_0(kR)}{k^2+\nu^2}\, dk - \\ - 2\pi\nu\, e^{\nu(y+b)}\, Y_0(\nu R) + i\, 2\pi\nu\, e^{\nu(y+b)}\, J_0(\nu R).\end{aligned}\right\} \quad (13.17''')$$

In the analogous problem for finite depth $h$ one replaces 4. by 4′. $\varphi_y(x, -h, z) = 0$ and proceeds somewhat similarly. However, in order to satisfy 4′. it is convenient to look for a solution in the form

$$\Phi(x, y, z, t) = [r^{-1} + r_2^{-1} + \varphi_0(x, y, z)] \cos \sigma t + \varphi_2(x, y, z) \sin \sigma t,$$

where

$$r_2^2 = (x-a)^2 + (y+2h+b)^2 + (z-c)^2.$$

Eq. (13.11) then becomes

$$\tilde{\varphi}_0 = A_0(k, \vartheta) \cosh k(y+h)$$

and (13.14), now more complicated because of $r^{-1}$ and $r_2^{-1}$, becomes

$$A_0(k, \vartheta) = \frac{2(k+\nu)\, e^{-kh} \cosh k(b+h)}{k \sinh kh - \nu \cosh kh}\, e^{-ik(a\cos\vartheta + c \sin\vartheta)}.$$

The final formula for the velocity potential is

$$\left.\begin{aligned}&\Phi(x, y, z, t) \\ &= \left[\frac{1}{r} + \frac{1}{r_2} + \mathrm{PV}\int_0^\infty \frac{2(k+\nu)\, e^{-kh} \cosh k(b+h) \cosh k(y+h)}{k \sinh kh - \nu \cosh kh} J_0(kR)\, dk\right] \cos \sigma t + \\ &+ \frac{2\pi(m_0+\nu)\, e^{-m_0 h} \sinh m_0 h \cosh m_0(b+h) \cosh m_0(y+h)}{\nu h + \sinh^2 m_0 h} J_0(m_0 R) \sin \sigma t,\end{aligned}\right\} \quad (13.18)$$

where $m_0 \tanh m_0 h - \nu = 0$, $\nu = \sigma^2/g$. The form of the last term of (13.18) may be altered by using the identities

$$\frac{e^{-m_0 h} \sinh m_0 h}{\nu h + \sinh^2 m_0 h} = \frac{2 e^{-m_0 h} \cosh m_0 h}{2 m_0 h + \sinh 2 m_0 h} = \frac{m_0 - \nu}{m_0^2 h - \nu^2 h + \nu}.$$

John (1950, p. 95) has derived the following series for $\Phi$, the analogue of (13.17‴)

$$\left.\begin{aligned}\Phi(x, y, z, t) = 2\pi \frac{\nu^2 - m_0^2}{h m_0^2 - h\nu^2 + \nu} \cosh m_0(y+h) \cosh m_0(b+h) \times \\ \times [Y_0(m_0 R) \cos \sigma t - J_0(m_0 R) \sin \sigma t] + \\ + 4 \sum_{k=1}^\infty \frac{m_k^2 + \nu^2}{h m_k^2 + h\nu^2 - \nu} \cos m_k(y+h) \cos m_k(b+h)\, K_0(m_k R) \cos \sigma t,\end{aligned}\right\} \quad (13.19)$$

where $m_k$, $k > 0$, are the positive real roots of $m \tan mh + \nu = 0$. Either expression may also be given in complex form as in (13.17″).

Potential functions satisfying the condition (13.9), but with $r^{-1} \cos \sigma t$ in 3. replaced by a higher-order singularity have been given by Thorne (1953) and Havelock (1955). In fact, Thorne gives a rather complete census of the possible singular solutions for both two and three dimensions and for finite and infinite depth. Included are series expansions as well as integrals. For infinite depth the general expression which includes (13.17) is

$$\left.\begin{aligned}\Phi(x,y,z,t) = \left[\frac{P_n^m(\cos\Theta)}{r^{n+1}} + \frac{(-1)^m}{(n-m)!}\,\mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\,k^n\,e^{k(y+b)}\,J_m(kR)\,dk\right]\times\\ \times\cos m\alpha\cos\sigma t + \frac{(-1)^m}{(n-m)!}\,2\pi\nu^{n+1}\,e^{\nu(y+b)}\,J_m(\nu R)\cos m\alpha\sin\sigma t,\end{aligned}\right\}\quad(13.20)$$

where $\cos\Theta=(y-b)/r$, $x=R\cos\alpha$, $z=R\sin\alpha$. Here $P_n^m$ are the associated Legendre polynomials defined by

$$P_n^m(\mu) = (1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_n(\mu),\qquad m\leqq n.$$

The asymptotic behavior of (13.20) is given by

$$\Phi(x,y,z,t) = \frac{(-1)^{m+1}}{(n-m)!}\,2\pi\nu^{n+1}e^{\nu(y+b)}\sqrt{\frac{2}{\pi\nu R}}\sin\left(\nu R-\sigma t-\frac{2m+1}{4}\pi\right)+O\left(\frac{1}{R}\right).$$

It has been pointed out by both HAVELOCK (1955) and MACCAMY (1954) that solutions can be constructed which vanish much faster than this at infinity. Let the function of (13.20) be denoted by $\Phi_n$. Then $\Phi_{n+1}-\nu(n-m+1)^{-1}\Phi_n$ is the following function:

see errata

$$\left.\begin{aligned}\left[\frac{P_{n+1}^m(\cos\Theta)}{r^{n+2}} - \frac{\nu}{n-m+1}\frac{P_n^m(\cos\Theta)}{r^{n+1}} + (-1)^m\frac{P_{n+1}^m(-\cos\Theta_1)}{r_1^{n+1}} + \right.\\ \left. + (-1)^m\frac{\nu}{n-m+1}\frac{P_n^m(-\cos\Theta_1)}{r_1^{n+1}}\right]\cos m\alpha\cos\sigma t,\end{aligned}\right\}\quad(13.21)$$

where $\cos\Theta_1=(y+b)/r_1$, $r_1^2=(x-a)^2+(y+b)^2+(z-c)^2$. For $y=0$ and large $R$ these solutions are $O(R^{-n-1})$ if $m$ and $n$ are both odd, $O(R^{-n-2})$ if one is even and one odd, and $O(R^{-n-3})$ if both are even. Although they have the form of standing waves, they satisfy the radiation condition because they decrease so rapidly with large $R$.

In addition to the papers cited above, one can find treatments of the submerged source of pulsating strength in KOCHIN (1940), HAVELOCK (1942), JOHN (1950, p. 92ff.), where a detailed discussion is given for the case of finite depth, HASKIND (1944), and LIU (1952). The definition of the improper integral in (13.15) and following is not always the same in these different treatments. In some cases the variable $k$ is treated as complex and the path of integration deflected around the singularity $k=\nu$ by following a small semi-circle in the lower half of the $k$-plane. The radiation condition is then automatically satisfied if one writes $\Phi$ in the complex form $\varphi\,e^{-i\sigma t}$, $\varphi=\varphi_1+i\varphi_2$. Other treatments achieve the same end by introducing a "fictitious viscosity" $i\mu$ which has the effect of replacing the singularity at $k=\nu$ by one at $k=\nu+i\mu$ and thus placing the path of integration below the singularity. In the end one must find the limit of the solution as $\mu\to 0$. The fictitious viscosity has no relation to real viscosity and may be considered a mathematical device to enable one to interpret an improper integral in a suitable way (for the purpose it seems to be infallible).

S o u r c e  a n d  v o r t e x  o f  p u l s a t i n g  s t r e n g t h  i n  t w o  d i m e n s i o n s. The two-dimensional problem can be formulated analogously to (13.9), and solutions found in a similar manner. The fundamental singularities will now be of the form $\log r\cos\sigma t$, $r^{-n}\cos n\Theta\cos\sigma t$ and $r^{-n}\sin n\Theta\cos\sigma t$, $n=1,2,\ldots$. The results are given in the paper of THORNE (1953) cited earlier. We shall follow a different method here in order to illustrate the use of complex variables to solve such problems.

We shall consider simultaneously a source of strength $Q$ and a vortex of intensity $\Gamma$ at the point $c = a + ib$, $b < 0$. In the notation used at the end of Sect. 11, we shall be looking for a function $f(z, t)$ analytic in $z$ and of the form

$$\left.\begin{aligned} f(z,t) &= \left[\frac{\Gamma + iQ}{2\pi i}\log(z-c) + f_0(z)\right]\cos\sigma t + f_2(z)\sin\sigma t, \qquad \operatorname{Im} c < 0, \\ &= f_1(z)\cos\sigma t + f_2(z)\sin\sigma t, \end{aligned}\right\} \tag{13.22}$$

where $f_0$ and $f_2$ have no singularities in the lower half-plane. In addition, $f_1$ and $f_2$ must each satisfy the free-surface condiction (11.7) which we write

$$\operatorname{Im}\{f_k'(x - i0) + i\nu f_k(x - i0)\} = 0, \qquad \nu = \sigma^2/g, \qquad k = 1, 2. \tag{13.23}$$

Condition 4 of (13.9) will be taken in the somewhat stronger form,

$$|f_k'| \leqq M \quad \text{for} \quad |z| \geqq m \quad \text{and} \quad \lim_{y\to-\infty} |f_k'| = 0, \tag{13.24}$$

where $m$ and $M$ are given constants. The radiation condition becomes:

$$\lim_{x\to\pm\infty} \operatorname{Re}\{f_1' \pm \nu f_2\} = 0, \qquad \lim_{x\to\pm\infty} \operatorname{Re}\{f_2' \mp \nu f_1\} = 0. \tag{13.25}$$

Following a method apparently originally due to Levi-Civita (see Tonolo, 1913), but used frequently by Keldysh (1935), Kochin (e.g., 1939), Stoker (1947), Lewy (1946) and others, we introduce the functions

$$A_k(z) = f_k'(z) + i\nu f_k(z). \tag{13.26}$$

Then (13.23) becomes

$$\operatorname{Im} A_k(x - i0) = 0, \qquad k = 1, 2, \tag{13.27}$$

and (13.22) becomes: the two functions

$$A_0(z) = A_1(z) - \frac{\Gamma + iQ}{2\pi i}\,\frac{1}{z-c} - \nu\,\frac{\Gamma + iQ}{2\pi}\log(z-c)$$

and $A_2(z)$ are both regular everywhere in the lower half-plane. A function $A(z)$ with $\operatorname{Im} A(x - i0) = 0$ may be continued into the upper half-plane by defining $A(x + iy) = \overline{A(x - iy)}$, $y > 0$, the bar indicating complex conjugate. Since $A_2$ is regular in the lower half-plane, the extended function will be regular in the whole plane. In addition, one may derive easily from (13.24) that $|A_2(z)| < C|z| + D$ for sufficiently large $|z|$: then, from the regularity of $A_2$, such an inequality holds in the whole half-plane and hence in the whole plane after reflection. It then follows from a known generalization of Liouville's Theorem[1] that $A_2(z) = az + b$, where $a$ and $b$ are constants. It follows from (13.27) that $a$ and $b$ are real. The differential equation

$$f_2'(z) + i\nu f_2(z) = az + b$$

has the solution

$$f_2(z) = C\,e^{-i\nu z} - \frac{ia}{\nu}z - \frac{ib}{\nu} + \frac{a}{\nu^2}.$$

The condition $\lim_{y\to-\infty} |f_2'| = 0$ requires $a = 0$. Thus, finally

$$f_2(z) = C_2\,e^{-i\nu z} + i B_2, \qquad B_2 \text{ real}.$$

One may set $B_2 = 0$ without loss of generality. Incidentally, this provides a proof of the theorem of Stoker and Keldysh mentioned earlier [shortly after Eq. (13.6)].

[1] See C. Carathéodory: Theory of functions of a complex variable, Vol. I, § 168. Chelsea, New York 1954.

The function $A_1(z)$, after extension into the upper half-plane, will consist of four singular terms plus a function regular in the whole complex plane, say $A_3(z)$:

$$A_1(z) = \frac{\Gamma + iQ}{2\pi i}\frac{1}{z-c} + \nu\frac{\Gamma+iQ}{2\pi}\log(z-c) - \frac{\Gamma - iQ}{2\pi i}\frac{1}{z-\bar{c}} + \\ + \nu\frac{\Gamma - iQ}{2\pi}\log(z-\bar{c}) + A_3(z).$$

Since $A_1$ satisfies (13.27), and the four singular terms taken together also have vanishing imaginary part for $y=0$, the same must hold for $A_3$. Hence $A_3$ must have the same form as $A_2$. Substituting the resulting expression for $A_1$ in (13.26), one has a differential equation for $f_1(z)$. The solution is

$$f_1(z) = \frac{\Gamma+iQ}{2\pi i}\log(z-c) + \frac{\Gamma - iQ}{2\pi i}\log(z-\bar{c}) - \\ - \frac{\Gamma - iQ}{\pi i}e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\bar{c}}\,du + C_1 e^{-i\nu z} + i\,B_1,$$

where $B_1$ is real and the path of integration is in the lower half-plane. As in the case $f_2$, we may set $B_1 = 0$. $C_1$ and $C_2$ must now be chosen to satisfy (13.25). Making use of

$$\int_{-\infty}^{\infty}\frac{e^{i\nu u}}{u-\bar{c}}\,du = 2\pi i\, e^{i\nu\bar{c}},$$

one can show that

$$f_1' + \nu f_2 = -i\nu C_1 e^{-i\nu z} + \nu C_2 e^{-i\nu z} + O(z^{-1}) \quad \text{as} \quad x\to+\infty,$$
$$f_1' - \nu f_2 = -2i(\Gamma - iQ)e^{-i\nu(z-\bar{c})} - i\nu C_1 e^{-i\nu z} - \nu C_2 e^{-i\nu z} + O(z^{-1}) \quad \text{as} \quad x\to-\infty.$$

This gives

$$C_1 = -(\Gamma - iQ)e^{i\nu\bar{c}}, \qquad C_2 = -i(\Gamma - iQ)e^{i\nu\bar{c}}.$$

One may easily verify that this choice of $C_1$ and $C_2$ does produce outgoing waves.

If one makes the change of variable $\nu(u-z) = -k(z-\bar{c})$ in the integral term in $f_1$ and deforms the resulting path to $Ox$, one finds

$$-e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\bar{c}}\,du = \mathrm{PV}\int_0^{\infty}\frac{e^{-ik(z-\bar{c})}}{k-\nu}\,dk + \pi i\, e^{-i\nu(z-\bar{c})}.$$

Substituting this in the expression for $f_1$, one finally obtains

$$\left.\begin{aligned} f(z,t) = \Bigg[&\frac{\Gamma+iQ}{2\pi i}\log(z-c) + \frac{\Gamma - iQ}{2\pi i}\log(z-\bar{c}) + \\ &+ \frac{\Gamma - iQ}{\pi i}\,\mathrm{PV}\int_0^{\infty}\frac{e^{-ik(z-\bar{c})}}{k-\nu}\,dk\Bigg]\cos\sigma t - i(\Gamma - iQ)\,e^{-i\nu(z-\bar{c})}\sin\sigma t. \end{aligned}\right\} \tag{13.28}$$

Singularities of higher order may be found by differentiating (13.28) with respect to $z$. The expression for $f'(z,t)$ may be put into a somewhat different form by using

see errata

$$\frac{\Gamma - iQ}{2\pi i}\frac{1}{z-\bar{c}} = \frac{\Gamma - iQ}{2\pi}\int_0^{\infty}e^{-ik(z-c)}\,dk.$$

Then

$$f'(z,t) = \left[\frac{\Gamma + iQ}{2\pi i}\,\frac{1}{z-c} - \frac{\Gamma - iQ}{2\pi}\,\mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\,e^{-ik(z-\bar{c})}\,dk\right]\cos\sigma t - \left.\begin{matrix} \\ - \nu(\Gamma - iQ)\,e^{-i\nu(z-\bar{c})}\sin\sigma t. \end{matrix}\right\} \tag{13.29}$$

One may continue differentiating, using either form for $f'(z,t)$. Thus, from (13.29)

$$\left.\begin{aligned} f^{(n)}(z,t) = \Bigg[&\frac{\Gamma + iQ}{2\pi i}\,\frac{(-1)^{n-1}(n-1)!}{(z-c)^n} - \\ &- \frac{\Gamma - iQ}{2\pi}(-i)^{n-1}\,\mathrm{PV}\int_0^\infty k^{n-1}\frac{k+\nu}{k-\nu}\,e^{-ik(z-\bar{c})}\,dk\Bigg]\cos\sigma t - \\ &- \nu^n(-i)^{n-1}(\Gamma - iQ)\,e^{-i\nu(z-\bar{c})}\sin\sigma t. \end{aligned}\right\} \tag{13.30}$$

By setting $\Gamma = 0$, $z - c = r\,e^{i(\frac{1}{2}\pi - \Theta)} = ir\,e^{-i\Theta}$ (rather than the conventional $r\,e^{i\Theta}$ in order to distinguish easily symmetrical from unsymmetrical solutions) and taking the appropriate real or imaginary part, one finds the following formulas for $\Phi(x,y,t)$:

$$\left.\begin{aligned} \Phi(x,y,t) &= \left[\frac{Q}{2\pi}\log\frac{r}{r_1} - \frac{Q}{\pi}\,\mathrm{PV}\int_0^\infty \frac{e^{k(y+b)}\cos k(x-a)}{k-\nu}\,dk\right]\cos\sigma t - \\ &\quad - Q\,e^{\nu(y+b)}\cos\nu(x-a)\sin\sigma t, \\ \Phi(x,y,t) &= \Bigg[\frac{Q}{2\pi}\,\frac{\cos n\Theta}{r^n} - \\ &\quad - \frac{(-1)^{n-1}}{(n-1)!}\,\frac{Q}{2\pi}\,\mathrm{PV}\int_0^\infty k^{n-1}\frac{k+\nu}{k-\nu}\,e^{k(y+b)}\cos k(x-a)\,dk\Bigg]\cos\sigma t - \\ &\quad - \frac{(-1)^{n-1}}{(n-1)!}\,Q\,\nu^n\,e^{\nu(y+b)}\cos\nu(x-a)\sin\sigma t, \\ \Phi(x,y,t) &= \Bigg[\frac{Q}{2\pi}\,\frac{\sin n\Theta}{r^n} + \\ &\quad + \frac{(-1)^{n-1}}{(n-1)!}\,\frac{Q}{2\pi}\,\mathrm{PV}\int_0^\infty k^{n-1}\frac{k+\nu}{k-\nu}\,e^{k(y+b)}\sin k(x-a)\,dk\Bigg]\cos\sigma t + \\ &\quad + \frac{(-1)^{n-1}}{(n-1)!}\,Q\,\nu^n\,e^{\nu(y+b)}\sin\nu(x-a)\sin\sigma t. \end{aligned}\right\} \tag{13.31}$$

In the formula for the logarithmic singularity $r_1$ may be eliminated and the coefficient of $\cos\sigma t$ written as [see JOHN (1950, p. 100)]:

$$\frac{Q}{2\pi}\log r + \frac{Q}{2\pi}\,\mathrm{PV}\int_0^\infty\left[\frac{k+\nu}{(k-\nu)k}\,e^{k(y+b)}\cos k(x-a) + \frac{1}{k}\,e^{-k}\right]dk.$$

For water of finite depth the method used above does not work as conveniently because of the difficulty of formulating the boundary condition on the bottom, $\mathrm{Im}\,f'(x - ih) = 0$, in terms of the function $A(z)$. However, it can be done, yielding a differential-difference equation for $f(z)$ which can be solved by use of

Laplace transforms[1]. The method used for the three-dimensional problem can also be carried through [see HASKIND (1942b), JOHN (1950), and THORNE (1953)].

It is convenient to separate the vortex from the source. The resulting functions are as follows:

vortex:

$$\left.\begin{aligned} f(z,t) = &\left[\frac{\Gamma}{2\pi i}\log(z-c) - \frac{\Gamma}{2\pi i}\log(z-c_2) - \right.\\ &\left. - \frac{\Gamma}{\pi}\,\mathrm{PV}\int_0^\infty \frac{k+\nu}{k}\,\frac{e^{-kh}\sinh k(h+b)\sin k(z-a+ih)}{k\sinh kh - \nu\cosh kh}\,dk\right]\cos\sigma t - \\ &- \Gamma\frac{\nu+m_0}{m_0}\cdot\frac{e^{-m_0 h}\sinh m_0 h\sinh m_0(h+b)\sin m_0(z-a+ih)}{\nu h+\sinh^2 m_0 h}\sin\sigma t; \end{aligned}\right\} \tag{13.32}$$

source:

$$\left.\begin{aligned} f(z,t) = &\left[\frac{Q}{2\pi}\log(z-c) + \frac{Q}{2\pi}\log(z-c_2) - \frac{Q}{\pi}\log ih - \right.\\ &\left. - \frac{Q}{\pi}\,\mathrm{PV}\int_0^\infty \left\{\frac{k+\nu}{k}\,\frac{e^{-kh}\cosh k(h+b)\cos k(z-a+ih)}{k\sinh kh - \nu\cosh kh} + \frac{e^{-kh}}{k}\right\}dk\right]\cos\sigma t - \\ &- Q\frac{\nu+m_0}{m_0}\,\frac{e^{-m_0 h}\sinh m_0 h\cosh m_0(h+b)\cos m_0(z-a+ih)}{\nu h+\sinh^2 m_0 h}\sin\sigma t. \end{aligned}\right\} \tag{13.33}$$

Here $c_2 = a - ib - 2ih$. The remark following (13.18) concerning the form of the last term of that formula applies also here. The real part of either of these gives the corresponding potential function.

For the source, the integral representation and the series representation analogous to (13.19) are:

see errata

$$\left.\begin{aligned} \Phi(x,y,t) = &\left[\frac{Q}{2\pi}\log\frac{r}{h} + \frac{Q}{2\pi}\log\frac{r_2}{h} - \right.\\ &\left. - \frac{Q}{\pi}\,\mathrm{PV}\int_0^\infty \left\{\frac{k+\nu}{k}\,\frac{e^{-kh}\cosh k(h+b)\cosh k(y+h)\cos k(x-a)}{k\sinh kh-\nu\cosh kh} - \frac{e^{-kh}}{k}\right\}dk\right]\cos\sigma t - \\ &- Q\frac{\nu+m_0}{m_0}\,\frac{e^{-m_0 h}\sinh m_0 h\cosh m_0(h+b)\cosh m_0(y+h)\cos m_0(x-a)}{\nu h+\sinh^2 m_0 h}\sin\sigma t, \\ = &\; Q\frac{1}{m_0}\,\frac{m_0^2-\nu^2}{h m_0^2 - h\nu^2+\nu}\cosh m_0(y+h)\cosh m_0(b+h)\sin[m_0|x-a|-\sigma t] - \\ &- Q\sum_{k=1}^\infty \frac{1}{m_k}\,\frac{m_k^2+\nu^2}{h m_k^2+h\nu^2-\nu}\cos m_k(y+h)\cos m_k(b+h)\,e^{-m_k|x-a|}\sin\sigma t. \end{aligned}\right\} \tag{13.34}$$

THORNE (1953) gives the potential functions for the higher-order singularities and the function for the logarithmic singularity in a form involving $r$ and $r_1$ and hence more analogous to the one in (13.31). VOITSENYA (1958) has derived the complex potential for a source-vortex situated in an infinitely deep fluid of density $\varrho_1$ lying beneath another of density $\varrho_2 < \varrho_1$ and of thickness $d$.

Source of constant strength in uniform motion: three dimensions. We shall assume the source moving in the direction $Ox$ with constant velocity $u_0$. Let $(x, y, z)$ be coordinates in a system moving with velocity $u_0$ in direc-

[1] Cf. S. BOCHNER: Vorlesungen über Fouriersche Integrale, pp. 167–168. Leipzig 1932.

tion $Ox$ and let the source be at $(a, b, c)$, $b<0$. Then, from Sect. 11, we wish to find a function $\varphi(x, y, z)$ satisfying

$$\left.\begin{aligned}
&1.\quad \Delta\varphi = 0 \quad \text{except at } (a, b, c),\\
&2.\quad \varphi_{xx}(x, 0, z) + \nu\,\varphi_y(x, 0, z) = 0, \qquad \nu = g/u_0^2,\\
&3.\quad \varphi(x, y, z) = r^{-1} + \varphi_0(x, y, z),\\
&\qquad\qquad \text{where } \varphi_0 \text{ is harmonic in the region } y<0,\\
&4.\quad \lim_{y\to-\infty} \operatorname{grad}\varphi = 0,\\
&5.\quad \lim_{x\to+\infty} \operatorname{grad}\varphi = 0.
\end{aligned}\right\} \tag{13.35}$$

see errata

For fluid of finite depth $h$, 4. is replaced by 4'. $\varphi_y(x, -h, z) = a$. Without condition 5, demanding vanishing of the motion far ahead of the source, the solution would not be unique. The profile of the free surface is obtained from $\eta(x, z) = u_0 g^{-1}\varphi_x(x, 0, z)$. Strictly speaking, the solution of (13.35) will represent a sink, i.e. a source of strength $-1$. However, we shall continue to call such solutions sources.

A solution to this problem may be obtained by methods very similar to those used for the source of pulsating strength. The details will not be repeated, but can be found in HAVELOCK (1932), SRETENSKII (1937), KOCHIN (1937), LUNDE (1951), PETERS and STOKER (1957), TIMMAN and VOSSERS (1955) and elsewhere. The result is

$$\left.\begin{aligned}
\varphi(x,y,z) = {} & \frac{1}{r} - \frac{1}{r_1} - \frac{4\nu}{\pi}\int_0^{\frac{1}{2}\pi} d\vartheta\,\mathrm{PV}\!\int_0^\infty \frac{e^{k(y+b)}\cos[k(x-a)\cos\vartheta]\cos[k(z-c)\sin\vartheta]}{k\cos^2\vartheta-\nu}\,dk - \\
& - 4\nu\int_0^{\frac{1}{2}\pi} e^{\nu(y+b)\sec^2\vartheta}\sin[\nu(x-a)\sec\vartheta]\cos[\nu(z-c)\sin\vartheta\sec^2\vartheta]\sec^2\vartheta\,d\vartheta,
\end{aligned}\right\} \tag{13.36}$$

where

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2, \quad r_1^2 = (x-a)^2 + (y+b)^2 + (z-c)^2, \quad \nu = g/u_0^2.$$

The potential functions for higher-order singularities are unwieldy and will not be given. The one corresponding to $r^{-n-1}P_n(\cos\Theta)$ can be easily obtained by $n$-fold differentiation with respect to $y$, if one remembers that

$$\frac{P_n(\cos\Theta)}{r^{n+1}} = \frac{(-1)^n}{n!}\,\frac{\partial^n}{\partial y^n}\left(\frac{1}{r}\right).$$

The dipole with axis in the direction $Ox$ is obtained by differentiating (13.36) with respect to $x$ and will be used later.

The velocity potential for a source moving in fluid of finite depth has been calculated by SRETENSKII (1937) and by HASKIND (1945b). The form given below is essentially that given by LUNDE (1951):

$$\left.\begin{aligned}
\varphi(x, y, z) = {} & \frac{1}{r} + \frac{1}{r_2} - \\
& - \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} d\vartheta\,\mathrm{PV}\!\int_0^\infty \frac{e^{-kh}\cosh k(y+h)\,[\cosh k(b+h)\,(k\cos^2\vartheta+\nu)-\nu]}{k\cos^2\vartheta\cosh kh - \nu\sinh kh}\times\\
&\qquad \times\cos[k(x-a)\cos\vartheta]\cos[k(z-c)\sin\vartheta]\,dk - \\
& - 4\int_{\vartheta_0}^{\frac{1}{2}\pi} \frac{e^{-k_0 h}\operatorname{sech}k_0 h\cosh k_0(y+h)\,[\cosh k_0(b+h)\,(k_0\cos^2\vartheta+\nu)-\nu]}{\cos^2\vartheta - \nu h\operatorname{sech}^2 k_0 h}\times\\
&\qquad \times\sin[k_0(x-a)\cos\vartheta]\cos[k_0(z-c)\sin\vartheta]\,d\vartheta,
\end{aligned}\right\} \tag{13.37}$$

see errata

where $k_0 = k_0(\vartheta)$ is the real positive root of

$$k_0 - \nu \sec^2\vartheta \tanh k_0 h = 0, \qquad \vartheta_0 < \vartheta < \tfrac{1}{2}\pi,$$

where $\vartheta_0 = \arccos\sqrt{\nu h}$ if $\nu h = gh/u_0^2 < 1$, $\vartheta_0 = 0$ if $\nu h \geqq 1$. As before, $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ and $r_2^2 = (x-a)^2 + (y+2h+b)^2 + (z-c)^2$. We note that $k_0(\vartheta) < \nu\sec^2\vartheta$, $k_0(\vartheta) \to 0$ as $\vartheta \to \vartheta_0$, $k_0(\vartheta)/\nu\sec^2\vartheta \to 1$ as $\vartheta \to \frac{1}{2}\pi$ and $k_0(\vartheta) \to \nu\sec^2\vartheta$ as $h \to \infty$. In the double integral the principal value is necessary only for $\vartheta_0 < \vartheta < \frac{1}{2}\pi$, for the singularity does not occur in the denominator for $0 \leqq \vartheta < \vartheta_0$. The part of the double integral with $0 \leqq \vartheta < \vartheta_0$ approaches zero as $x \to \pm\infty$, so that no correction is necessary in order to satisfy condition 5. This is the explanation of the lower limit $\vartheta_0$ in the second integral. In this integral the denominator vanishes only at $\vartheta = \vartheta_0$. One may verify that the integral is convergent by noting that

$$k_0'(\vartheta) = \frac{k_0 \sin 2\vartheta}{\cos^2\vartheta - \nu h\,\mathrm{sech}^2 k_0 h}$$

and rewriting it as an integral with respect to $k_0$. When $h \to \infty$, (13.37) reduces to a form of (13.36) in which $r_1$ is absorbed into the double integral.

For the stationary pulsating source the asymptotic form of the velocity potential for large $R$ was found in the course of deriving the potential function. For the moving source of constant strength the asymptotic form is more difficult to compute. Since the form of the free surface, $\eta = g u_0^{-1}\varphi_x(x, 0, z)$ is of principal physical interest, we shall discuss the asymptotic form of $\varphi_x$ instead of $\varphi$.

Introduce cylindrical coordinates $x - a = R\cos\alpha$, $z - c = R\sin\alpha$ into the $x$ derivative of (13.36):

$$\left.\begin{aligned}\varphi_x(R,\alpha,y) &= \frac{-R\cos\alpha}{[R^2+(y-b)^2]^{\frac{3}{2}}} + \frac{R\cos\alpha}{[R^2+(y+b)^2]^{\frac{3}{2}}} + \\ &+ \frac{2\nu}{\pi}\int_0^{\frac{1}{2}\pi}\sec\vartheta\,d\vartheta\,\mathrm{PV}\int_0^\infty e^{k(y+b)}\frac{\sin[kR\cos(\vartheta-\alpha)] + \sin[kR\cos(\vartheta+\alpha)]}{k - \nu\sec^2\vartheta}\,k\,dk - \\ &- 2\nu^2\int_0^{\frac{1}{2}\pi} e^{\nu(y+b)\sec^2\vartheta}\{\cos[\nu R\sec^2\vartheta\cos(\vartheta-\alpha)] + \\ &\qquad + \cos[\nu R\sec^2\vartheta\cos(\vartheta+\alpha)]\}\sec^3\vartheta\,d\vartheta.\end{aligned}\right\} \quad (13.38)$$

see errata

For large $R$ the first two terms taken together are $O(R^{-3})$. Apply the theorem (13.16) to the integral with respect to $k$. This gives, after combining with the second integral,

$$\left.\begin{aligned}\varphi_x(R,\alpha,y) &= 2\nu^2\int_0^{\frac{1}{2}\pi}\sec^3\vartheta\, e^{\nu(y+b)\sec^2\vartheta}\times \\ &\times\{\cos[\nu R\sec^2\vartheta\cos(\vartheta-\alpha)][-1+\operatorname{sgn}\cos(\vartheta-\alpha)] + \\ &+ \cos[\nu R\sec^2\vartheta\cos(\vartheta+\alpha)][-1+\operatorname{sgn}\cos(\vartheta+\alpha)]\}\,d\vartheta + O(R^{-1}).\end{aligned}\right\} \quad (13.39)$$

Since $\varphi_x$ is symmetric in $\alpha$, we consider only $0 \leqq \alpha \leqq \pi$. We have for $0 \leqq \alpha \leqq \frac{1}{2}\pi$

$$\varphi_x(R,\alpha,y) = -4\nu^2\int_{\frac{1}{2}\pi-\alpha}^{\frac{1}{2}\pi}\sec^3\vartheta\, e^{\nu(y+b)\sec^2\vartheta}\times \cos[\nu R\sec^2\vartheta\cos(\vartheta+\alpha)]\,d\vartheta + O(R^{-1}),$$

and for $\frac{1}{2}\pi < \alpha \leqq \pi$ (13.40)

$$\begin{aligned}\varphi_x(R,\alpha,y) &= -4\nu^2\int_0^{\frac{1}{2}\pi}\sec^3\vartheta\, e^{\nu(y+b)\sec^2\vartheta}\cos[\nu R\sec^2\vartheta\cos(\vartheta+\alpha)]\,d\vartheta - \\ &- 4\nu^2\int_0^{\alpha-\frac{1}{2}\pi}\sec^3\vartheta\, e^{\nu(y+b)\sec^2\vartheta}\cos[\nu R\sec^2\vartheta\cos(\vartheta-\alpha)]\,d\vartheta + O(R^{-1}).\end{aligned}$$

Consider the two integrals containing $\cos(\vartheta+\alpha)$ and let

$$\lambda = \sec^2\vartheta \cos(\vartheta+\alpha).$$

Then for $0 \leqq \alpha \leqq \frac{1}{2}\pi$ the integral becomes

$$-4\nu^2 \int_0^{-\infty} \frac{2\, e^{\nu(y+b)\sec^2\vartheta}}{\sin(2\vartheta+\alpha) - 3\sin\alpha} \cos\nu R\lambda\, d\lambda.$$

If $\frac{1}{2}\pi < \alpha < \pi$, the lower limit is $\cos\alpha$. In either case one may show that the coefficient of $\cos\nu R\lambda$ is single-valued, continuous, absolutely integrable and monotonically decreasing as a function of $\lambda$. By integration by parts one may then establish the following estimates as $R\to\infty$ (cf. S. Bochner, Vorlesungen über Fouriersche Integrale, Leipzig, 1932, § 3):

for $0 \leqq \alpha \leqq \frac{1}{2}\pi$

$$O(R^{-2});$$

for $\frac{1}{2}\pi < \alpha < \pi$

$$-4\nu \frac{e^{\nu(y+b)}}{R\sin\alpha} \sin(\nu R\cos\alpha) + O(R^{-2}).$$

If $\alpha = \pi$, the two integrals in (13.40) combine to give

$$8\nu^2 \int_1^\infty e^{\nu(y+b)\lambda^2} \frac{\lambda^2}{\sqrt{\lambda^2-1}} \cos\nu R\lambda\, d\lambda$$
$$= 4\nu^2 \sqrt{2\pi} \frac{e^{\nu(y+b)}}{\sqrt{\nu R}} \cos\left(\nu R - \frac{3}{4}\pi\right) + O(R^{-1}).$$

Consider now the remaining integral in (13.40), and let

$$\mu(\vartheta) = \sec^2\vartheta\cos(\vartheta-\alpha).$$

The integral takes the form

$$-8\nu^2 \int_{\cos\alpha}^0 e^{\nu(y+b)\sec^2\vartheta} \frac{\cos\nu R\mu}{\sin(2\vartheta-\alpha)+3\sin\alpha}\, d\mu.$$

The denominator now becomes zero when

$$\tan\vartheta = -\tfrac{1}{4}\cot\alpha\left[1 \pm \sqrt{1-8\tan^2\alpha}\right], \tag{13.41}$$

an equation which has real roots when $\tan^2\alpha \leqq \frac{1}{8}$, i.e. when

$$180° - 19°28' \ldots < \alpha < 180°.$$

When $\frac{1}{2}\pi < \alpha < \pi - \arcsin\frac{1}{3} = \alpha_c$, the Fourier-integral estimate used for the other integral may be applied to give

$$4\nu\, e^{\nu(y+b)} \frac{\sin(\nu R\cos\alpha)}{R\sin\alpha} + O(R^{-2}).$$

When $\alpha_2 < \alpha < \pi$, $\vartheta$ is a two-branched function of $\mu$ and the resulting two integrals each have singularities at one of the limits. Thus the elementary method of analysis used above can no longer be applied. However, a modification of the method above can be carried through[1]; the classical treatment is by the method of stationary phase which is well discussed in Stoker (1957, Chap. 8).

[1] See, e.g., A. Erdélyi: Asymptotic expansions, pp. 46—56. Dover, New York 1956.

The estimates already derived are of the same order as the remainder term in (13.39). Analysis of this term produces terms which cancel the terms in $1/R \sin\alpha$ already derived, thus removing an apparent singular behavior near the $x$-axis

The asymptotic form for the surface $\eta(R, \alpha)$ above a source of strength $m$ (i.e. $-m/r$) is as follows:

for $0 \leqq \alpha < \pi - \arcsin \frac{1}{3} = \alpha_c$

$$\eta(R, \alpha) = O\big((\nu R)^{-2}\big); \tag{13.42}$$

for $\alpha = \alpha_c$

$$\eta(R, \alpha_c) = 4 \cdot 3^{\frac{5}{6}} \Gamma\left(\frac{1}{3}\right) \frac{m\nu}{u_0} (\nu R)^{-\frac{1}{3}} e^{\frac{3}{2}\nu b} \cos\left(\frac{\sqrt{3}}{2} \nu R\right) + O\big((\nu R)^{-\frac{2}{3}}\big);$$

for $\alpha_c < \alpha < \pi$

$$\eta(R, \alpha) = 4\sqrt{2\pi}\, \frac{m\nu}{u_0} \frac{(\nu R)^{-\frac{1}{2}}}{[1 - 9\sin^2\alpha]^{\frac{1}{4}}} \times$$

$$\times \{\sec^{\frac{3}{2}}\vartheta_1 e^{\nu b \sec^2\vartheta_1} \cos(\nu R \mu_1 - \tfrac{1}{4}\pi) + \sec^{\frac{3}{2}}\vartheta_2 e^{\nu b \sec^2\vartheta_2} \cos(\nu R \mu_2 + \tfrac{1}{4}\pi)\} + O\big((\nu R)^{-1}\big);$$

for $\alpha = \pi$

$$\eta(R, \pi) = -4\sqrt{2\pi}\, \frac{m\nu}{u_0} (\nu R)^{-\frac{1}{2}} e^{\nu b} \cos(\nu R - \tfrac{3}{4}\pi) + O\big((\nu R)^{-1}\big).$$

Here $\vartheta_1$ and $\vartheta_2 > \vartheta_1$ are the two roots (13.41) and $\mu_1 (<0)$ and $\mu_2 < \mu_1$ the corresponding values of $\sec^2\vartheta \cos(\vartheta - \alpha)$. As $\alpha \to \alpha_c$, $\vartheta_i \to \arctan \frac{1}{2}\sqrt{2}$, $\mu_i \to -\frac{1}{2}\sqrt{3}$; as $\alpha \to \pi$, $\vartheta_1 \to 0$, $\mu_1 \to -1$, $\vartheta_2 \to \frac{1}{2}\pi$, $\mu_2 \to -\infty$. In order to have some idea of the form of the free surface far behind the source, one may graph the curves

$$\nu R \mu_1(\alpha) - \tfrac{1}{4}\pi = -2n\pi, \quad \nu R \mu_2(\alpha) + \tfrac{1}{4}\pi = -2n\pi, \quad n > 0,$$

showing the traces of the wave crests in the region

$$\pi - \arcsin \tfrac{1}{3} < \alpha < \pi + \arcsin \tfrac{1}{3}.$$

This gives the well known pattern shown in Fig. 1a. The first equation gives the transverse waves, the second one the diverging waves. The wavelength along $\alpha = \pi$ is $2\pi/\nu$ and along the boundary lines $4\pi\sqrt{3}/3\nu$. The expansion is not suitable in the region near the boundary lines $\alpha = \alpha_c$. As $\alpha \to \alpha_c$, $\alpha > \alpha_c$, the term $[1 - 9\sin^2\alpha]^{\frac{1}{4}} \to 0$ and the amplitudes become infinite. A special investigation of the region near $\alpha = \alpha_c$ is necessary and shows $(\nu R)^{-\frac{1}{3}}$ as leading term; $\eta$ may be expressed in terms of Airy functions [cf. URSELL (1960)].

Essentially the same pattern is produced by a moving concentrated pressure on the free surface; it was first analysed by KELVIN (1906 = Papers, Vol. IV, pp. 407—413). The asymptotic behavior for moving pressure distributions has been extensively studied [e.g., HOGNER (1923), TETURÔ INUI (1936), PETERS (1949), BARTELS and DOWNING (1955)]. LAMB (1926) has given the asymptotic form of the surface over a moving submerged dipole. The form of the surface near the moving dipole has been investigated by HAVELOCK (1928), who gives traces of the profile on planes $\alpha$ = const for several values of $\alpha$ between $\frac{1}{2}\pi$ and $\pi$ (the radial lines of Figs. 1b and c) and for $|b\nu| = \frac{1}{2}$ and 4. HAVELOCK's computations were later used by WIGLEY (1930) to produce the contour curves shown in Figs. 1b and c.

A similar analysis may be made for (13.37), a source moving in fluid of finite depth. For a moving pressure distribution the problem has been treated by HAVELOCK (1908) and TETURÔ INUI (1936). The pattern is modified as follows.

If $\nu h > 1$, the pattern is qualitatively like that for $h = \infty$. However, the wedge within which the disturbance is chiefly contained has a wider aperture and as $\nu h \to 1$ the aperture approaches $\frac{1}{2}\pi$ radians on each side of the line of motion.

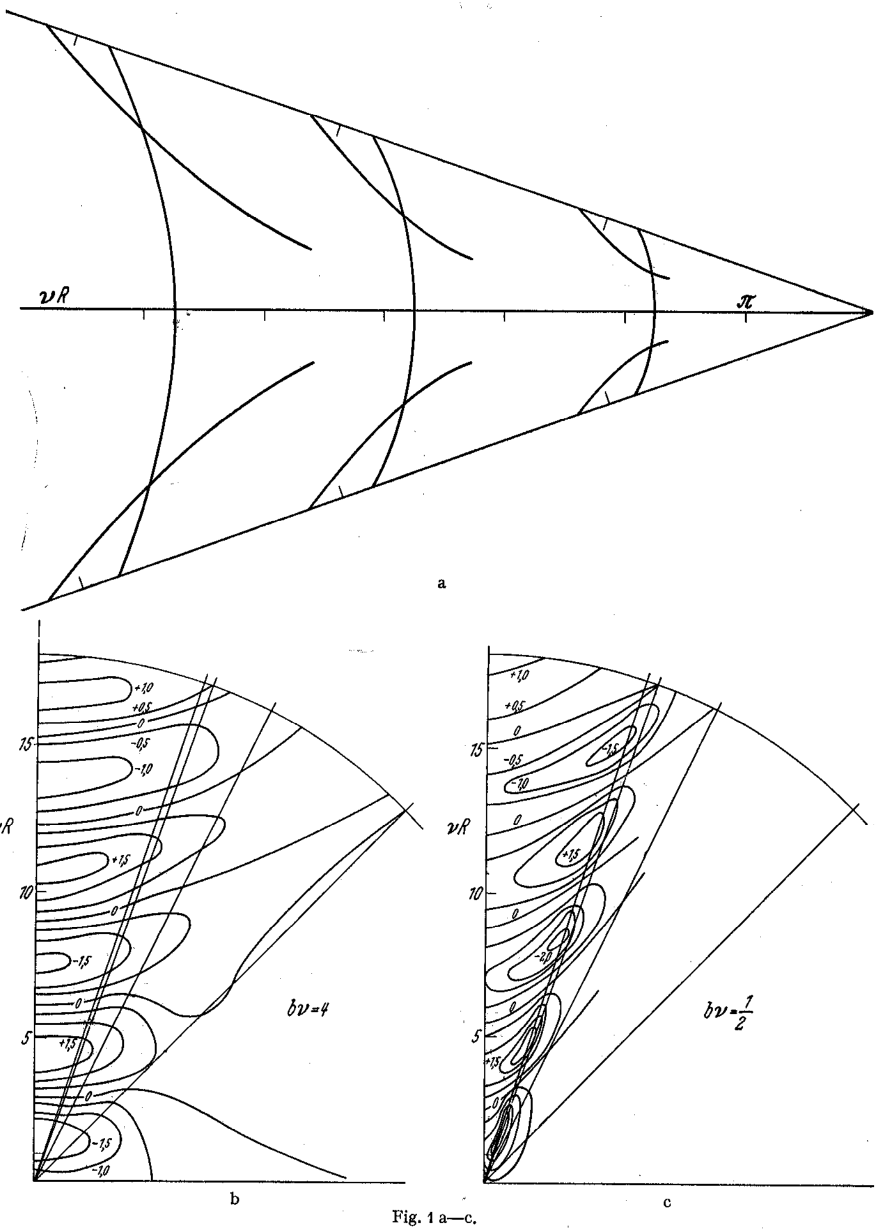



Fig. 1 a—c.

In addition, the wave length of the transverse wave system increases and approaches infinity as $\nu h \to 1$. When $\nu h \geqq 1$, the transverse wave system is missing completely, but diverging waves still occur in a wedge of aperture varying from $\pi$ to 0 as $\nu h \to 0$. [See also EKMAN (1906), who has considered the free surface over a dipole on a flat bottom.]

Fig. 2 from HAVELOCK (1908) shows the half-angle of the aperture.

KOCHIN (1938c) has gone further in this type of problem. He has derived the potential for a source situated in a fluid of density $\varrho_1$ and depth $h$, bounded below by a plane, over which is lying another fluid of density $\varrho_2 < \varrho_1$, extending infinitely far upwards. The lower fluid moves with velocity $c_1$, the upper with velocity $c_2$ in the same direction. He also finds the asymptotic behavior of the solution.

Singularities of constant strength in uniform motion: two dimensions. For submerged sources and vortices in two-dimensional motion the complex-variable method used for the pulsating source may again be applied. For infinitely deep fluid, the computation has been carried out in this way by KELDYSH and LAVRENT'EV (1937) and KOCHIN (1937); a detailed exposition is given in the textbook of KOCHIN, KIBEL' and ROZE (1948, Chap. VIII, § 19). HAVELOCK (1927) and SRETENSKII (1938) have treated the problem by different methods. The complex velocity potential for a combined source of strength $Q$ and vortex of intensity $\Gamma$ at $c = a + ib$, $b < 0$, is given by

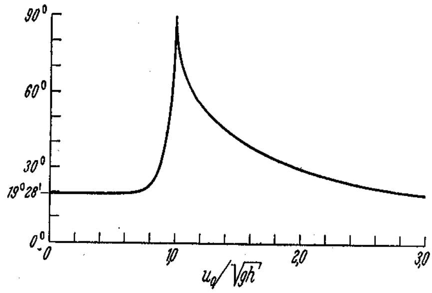


Fig. 2.

see errata

see errata

$$\left.\begin{aligned} f(z) &= \frac{\Gamma + iQ}{2\pi i}\log(z - c) - \frac{\Gamma - iQ}{2\pi i}\log(z - \bar{c}) + \frac{\Gamma - iQ}{\pi i}\, e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u - \bar{c}}\, du, \\ &= \frac{\Gamma + iQ}{2\pi i}\log(z - c) - \frac{\Gamma - iQ}{2\pi i}\log(z - \bar{c}) - 2(\Gamma - iQ)\, e^{-i\nu(z - \bar{c})} + \\ &\qquad + \frac{\Gamma - iQ}{\pi i}\, e^{-i\nu z}\int_{-\infty}^{z}\frac{e^{i\nu u}}{u - \bar{c}}\, du, \\ &= \frac{\Gamma + iQ}{2\pi i}\log(z - c) - \frac{\Gamma - iQ}{2\pi i}\log(z - \bar{c}) - \frac{\Gamma - iQ}{\pi i}\,\mathrm{PV}\int_{0}^{\infty}\frac{e^{-ik(z - \bar{c})}}{k - \nu}\, dk - \\ &\qquad - (\Gamma - iQ)\, e^{-i(z - \bar{c})}. \end{aligned}\right\} \qquad (13.43)$$

The real velocity potential for, say, a submerged source can be obtained from any of these equations. The last one gives a form analogous to (13.36):

$$\left.\begin{aligned} \varphi(x, y) &= \frac{Q}{2\pi}\log r + \frac{Q}{2\pi}\log r_1 + \\ &\quad + \frac{Q}{\pi}\,\mathrm{PV}\int_{0}^{\infty}\frac{e^{k(y+b)}\cos k(x - a)}{k - \nu}\, dk + Q\, e^{\nu(y+b)}\sin\nu(x - a). \end{aligned}\right\} \qquad (13.44)$$

Higher-order singularities can be obtained by differentiating (13.43). The complex velocity potential for a dipole of moment $M$ and axis in the direction $e^{i\alpha}$ is given by

$$\left.\begin{aligned} f(z) &= -\frac{M}{2\pi}\frac{e^{i\alpha}}{z - c} + \frac{M}{2\pi}\frac{e^{-i\alpha}}{z - \bar{c}} - \frac{iM\nu}{\pi}\, e^{-i\alpha}\, e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u - \bar{c}}\, du, \\ &= -\frac{M}{2\pi}\frac{e^{i\alpha}}{z - c} + \frac{M}{2\pi}\frac{e^{-i\alpha}}{z - \bar{c}} + \frac{iM\nu}{\pi}\, e^{-i\alpha}\,\mathrm{PV}\int_{0}^{\infty}\frac{e^{-ik(z - \bar{c})}}{k - \nu}\, dk - \\ &\qquad - M\nu\, e^{-i\alpha}\, e^{-i\nu(z - \bar{c})}. \end{aligned}\right\} \qquad (13.45)$$

If in the last term of the first equation of either (13.43) or (13.45), one makes use of the identity

$$e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\bar{c}}\,du=\int_{-\infty}^{0}\frac{-e^{-i\nu u}}{z-u-\bar{c}}\,du,$$

it is not difficult to see that this last term is equivalent to a distribution of dipoles on the ray from $\bar{c}$ to $-\infty$ parallel to the $x$-axis. The moment density and axis can be determined for the three cases, source, vortex and dipole, by comparison of the integrand with the first term of (13.45).

For the case of finite depth the complex velocity potential has been calculated by Tikhonov (1940) and is also given by Haskind (1945a) for both source and vortex. We give separately source, vortex and dipole:

see errata

$$\left.\begin{aligned}&\text{source:}\\ &f(z)=\frac{Q}{2\pi}\log(z-c)+\frac{Q}{2\pi}\log(z-c_2)+\\ &\quad+\frac{2Q}{\pi}\,\mathrm{PV}\!\int_0^\infty\frac{k+\nu}{k}\,e^{-kh}\frac{\cosh k(b+h)}{\nu\sinh kh-k\cosh kh}\sin^2\tfrac{1}{2}k(z-a+ih)\,dk-\\ &\quad-\frac{Q\nu}{k_0}\,\frac{\cosh k_0(b+h)}{\nu h-\cosh^2 k_0 h}\sin k_0(z-a+ih);\end{aligned}\right\}\quad(13.46)$$

see errata

$$\left.\begin{aligned}&\text{vortex:}\\ &f(z)=\frac{\Gamma}{2\pi i}\log(z-c)-\frac{\Gamma}{2\pi i}\log(z-c_2)-\\ &\quad-\frac{\Gamma}{\pi}\,\mathrm{PV}\!\int_0^\infty\frac{k+\nu}{k}\,e^{-kh}\frac{\sinh k(b+h)}{\nu\sinh kh-k\cosh kh}\sin k(z-a+ih)\,dk+\\ &\quad+\frac{\Gamma\nu}{k_0}\,\frac{\sinh k_0(b+h)}{\nu h-\cosh^2 k_0 h}\cos k_0(z-a+ih);\end{aligned}\right\}\quad(13.47)$$

see errata

$$\left.\begin{aligned}&\text{dipole:}\\ &f(z)=-\frac{M}{2\pi}\,\frac{e^{i\alpha}}{z-c}-\frac{M}{2\pi}\,\frac{e^{-i\alpha}}{z-c_2}-\\ &\quad-\frac{M}{2\pi}\,\mathrm{PV}\!\int_0^\infty(k+\nu)\,e^{-kh}\frac{e^{i\alpha}\sin k(z-c)+e^{-i\alpha}\sin k(z-c_2)}{\nu\sinh kh-k\cosh kh}\,dk+\\ &\quad+\frac{\nu M}{2}\,\frac{e^{i\alpha}\cos k_0(z-c)+e^{-i\alpha}\cos k_0(z-c_2)}{\nu h-\cosh^2 k_0 h}.\end{aligned}\right\}\quad(13.48)$$

Here $c_2=a-ib+2ih$ and the last summand in each of (13.46) to (13.48) is to be deleted if $\nu h=gh/c^2\leqq 1$; $k_0$ is the positive real root of $\nu\sinh kh-k\cosh kh=0$, which exists only if $\nu h>1$.

Asymptotic form of these functions as $x\to-\infty$ is easily seen to be given by double the last term in each expression. When $\nu h<1$, the disturbance is only local, a fact which corresponds to the absence of transverse waves behind the three-dimensional source for $\nu h<1$.

Kochin (1937a, b) has derived the complex velocity potential when fluid of density $\varrho_2$ overlies the fluid of density $\varrho_1>\varrho_2$ containing the singularity. The lower fluid may be of infinite or finite depth; the upper one is taken infinitely deep. Their velocities may be different.

Source of variable strength, starting from rest and following an arbitrary path. Consider now a source whose position and strength at time

$t \geqq 0$ are given by $(a(t), b(t), c(t))$ and $m(t)$, where $b(t) < 0$. Let $m(t) = 0$ for $t < 0$. The conditions to be satisfied by the velocity potential $\Phi(x, y, z, t)$ are

1. $\Delta \Phi = 0, \quad y < 0, \quad (x, y, z) \neq (a(t), b(t), c(t))$,
2. $\Phi_{tt}(x, 0, z, t) + g\, \Phi_y(x, 0, z, t) = 0$,
3. $\Phi(x, y, z, t) = m(t)\, r^{-1} + \Phi_0(x, y, z, t)$, $\Phi_0$ harmonic everywhere in $y < 0$,
4. $\lim\limits_{y \to -\infty} \operatorname{grad} \Phi = 0$ for all $x, z$ and $t$,
5. $\lim\limits_{R \to \infty} \operatorname{grad} \Phi = 0$ for all $t$,
6. $\Phi(x, 0, z, 0) = \Phi_t(x, 0, z, 0) = 0$.

Here $r^2 = (x - a(t))^2 + (y - b(t))^2 + (z - c(t))^2$, $R^2 = (x - a(t))^2 + (z - c(t))^2$.

If one assumes a solution in the form

see errata

$$\Phi = m\, r^{-1} - m_1^{-1}\, r + \Phi_1$$

where $r_1^2 = (x - a)^2 + (y + b)^2 + (z - c)^2$, then $\Phi_1$ must be harmonic in $y < 0$ and satisfy 4., 5., 6. and

$$\Phi_{1tt}(x, 0, z, t) + g\, \Phi_{1y}(x, 0, z, t) = -2g\, m(t)\, b(t)\, [(x-a)^2 + b^2 + (z-c)^2]^{-\frac{3}{2}}, \quad t \geqq 0.$$

It follows from the conditions that, for $t < 0$, $\Phi_1 = \text{const}$, which we may take as zero. Let $\overline{\Phi}_1$ be the Laplace transform of $\Phi_1$:

$$\overline{\Phi}_1(x, y, z, s) = \int_0^\infty e^{-st}\, \Phi_1(x, y, z, t)\, dt.$$

Then $\overline{\Phi}_1$ is a harmonic function in $y < 0$ satisfying 4. and 5. for each $s$ and also, after making use of 6., the condition

$$s^2\, \overline{\Phi}_1(x, 0, z, s) + g\, \overline{\Phi}_{1y}(x, 0, z, s) = -2g \int_0^\infty e^{-st}\, m(t)\, b(t)\, [(x-a)^2 + b^2 + (z-c)^2]^{-\frac{3}{2}}\, dt.$$

Since

$$s^2\, \overline{\Phi}_1(x, y, z, s) + g\, \overline{\Phi}_{1y}(x, y, z, s) + \\ + 2g \int_0^\infty e^{-st}\, m(t)\, (y + b(t))\, [(x-a)^2 + (y+b)^2 + (z-c)^2]^{-\frac{3}{2}}\, dt$$

is a harmonic function in $y < 0$ vanishing on $y = 0$ and at infinity, it is identically zero. Making use of (13.12) differentiated with respect to $y$ and changing the order of integration, one obtains

$$\begin{aligned} s^2\, \overline{\Phi}_1(x, y, z, s) &+ g\, \overline{\Phi}_{1y}(x, y, z, s) \\ &= \frac{g}{\pi} \int_0^\infty k\, dk \int_0^\infty dt\, e^{-st} m(t)\, e^{k(y+b)} \int_{-\pi}^{\pi} d\vartheta\, e^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]} \\ &= 2g \int_0^\infty k\, dk \int_0^\infty dt\, e^{-st}\, m(t)\, e^{k(y+b)}\, J_0\!\left(k\,[(x-a)^2 + (z-c)^2]^{\frac{1}{2}}\right). \end{aligned}$$

The solution for $\overline{\Phi}_1$ is

$$\overline{\Phi}_1(x, y, z, s) = 2g \int_0^\infty dk\, \frac{k}{s^2 + g\,k} \int_0^\infty dt\, e^{-st}\, m(t)\, e^{k(y+b)}\, J_0(k\, R(t)).$$

Making use of the convolution theorem and the fact that $(s^2+gk)^{-1}$ is the transform of $(gk)^{-\frac{1}{2}} \sin (gk)^{\frac{1}{2}} t$, one may find the original function $\Phi_1(x, y, z, t)$:

$$\Phi_1(x, y, z, t) = 2\int_0^\infty dk\,(g\,k)^{\frac{1}{2}} \int_0^t d\tau \sin\left[(g\,k)^{\frac{1}{2}}(t-\tau)\right] m(\tau)\, e^{k(y+b(\tau))} J_0\big(k\,R(\tau)\big).$$

For fixed $t$ one may easily verify, using known properties of the Fourier-Bessel transform[1], that $\Phi_1$ is $o(R^{-\frac{1}{2}})$ and hence that 5. is satisfied. One has then the result

$$\left.\begin{aligned}\Phi(x, y, z, t) &= \frac{m(t)}{r(t)} - \frac{m(t)}{r_1(t)} + 2\int_0^\infty dk\,(g\,k)^{\frac{1}{2}} \int_0^t d\tau\, m(\tau) \sin\left[(g\,k)^{\frac{1}{2}}(t-\tau)\right] \times \\ &\qquad \times e^{k(y+b)} J_0\big(k\,R(\tau)\big) \\ &= \frac{m(t)}{r(t)} - \frac{m(t)}{r_1(t)} + \frac{1}{\pi}\int_{-\pi}^{\pi} d\vartheta \int_0^\infty dk (g\,k)^{\frac{1}{2}} \int_0^t d\tau\, m(\tau) \sin\left[(g\,k)^{\frac{1}{2}}(t-\tau)\right] \times \\ &\qquad \times e^{k[y+b(\tau)+i(x-a(\tau))\cos\vartheta + i(z-c(\tau))\sin\vartheta]}.\end{aligned}\right\} \quad (13.49)$$

see errata

By a more refined analysis of the behavior for large $R$ [cf. Stoker (1957, pp. 190 to 191)] one may establish that $\Phi$ is $O(R^{-2})$ and $\Phi_R$ and $\Phi_y$ are $O(R^{-3})$ as $R \to \infty$.

For some time $t > t_0 \geqq 0$, one may write $\Phi$ in the form

$$\begin{aligned}\Phi(x, y, z, t) &= 2\int_0^\infty dk\,(g\,k)^{\frac{1}{2}} \int_0^{t_0} d\tau\, m(\tau) \sin\left[(g\,k)^{\frac{1}{2}}(t-\tau)\right] e^{k[y+b(\tau)]} J_0\big(k\,R(\tau)\big) + \\ &\quad + \frac{m(t)}{r(t)} - \frac{m(t)}{r_1(t)} + 2\int_0^\infty dk\,(g\,k)^{\frac{1}{2}} \int_0^{t-t_0} d\tau\, m(\tau+t_0) \sin\left[(g\,k)^{\frac{1}{2}}(t-t_0-\tau)\right] \times \\ &\quad \times e^{k[y+b(\tau+t_0)]} J_0\big(k\,R(\tau+t_0)\big) = \Phi_2(x, y, z, t) + \Phi_3(x, y, z, t).\end{aligned}$$

Here the first summand $\Phi_2$ represents the effect at time $t > t_0$ of the action of the source from $t=0$ to $t=t_0$. The remaining terms, $\Phi_3$, are the same as (13.49) with $t$ measured from $t_0$ $\big(m(t) = m(t-t_0+t_0)$, etc.$\big)$, and show the effect at time $t$ of the action of the source from $t=t_0$ to $t=t$. (This is, of course, what one would expect from the linearity of the problem and the fact that the choice of $t=0$ is arbitrary.) When $t=t_0$, $\Phi_3$ reduces to

$$\frac{m(t_0)}{r(t_0)} - \frac{m(t_0)}{r_1(t_0)}.$$

Thus

$$\Phi_3(x, 0, z, t_0) = 0.$$

This fact provides a basis for Havelock's procedure in similar problems, a procedure originating with Kelvin in the treatment of moving pressure distributions. The idea is roughly as follows. Divide the path of the source into small segments of time span $\Delta t$. If $\Delta t$ is small enough, the effect of gravity upon the fluid motion produced by the source during this time interval will be negligible, and one may take the boundary condition at the free surface to be $\Phi = 0$. The distortion of the surface resulting from the action of the source during this short interval is found and the future behavior of the distortion computed while taking account of gravity. Summing over all $\Delta t$ and taking the limit leads to the potential function.

[1] Cf. G. N. Watson: Bessel functions, § 14.41. Cambridge 1944.

The expression (13.49) has been essentially given by HASKIND (1946b) and BRARD (1948a). Special choices of $m(t)$ and of the motion of the source lead to cases similar to those treated earlier. Thus, if $m(t)=m\cos\sigma t$ and $(a, b, c)$ is fixed, one has the potential function for a stationary source of oscillating strength, starting to oscillate at $t=0$. Carrying out the $t$ integration and taking a limit by using, say, the Fourier Integral Theorem (13.16) allow one to derive (13.17). The radiation condition is automatically satisfied. The velocity potential for finite values of $t$ may be written in the form

$$\left.\begin{aligned}\Phi(x, y, z, t) = {} & \frac{m\cos\sigma t}{r} - \frac{m\cos\sigma t}{r_1} + \\ & + 2m\cos\sigma t\,\mathrm{PV}\int_0^\infty \frac{k}{k-\sigma^2/g}\, e^{k(y+b)} J_0(kR)\,dk - \\ & - 2m\,\mathrm{PV}\int_0^\infty \frac{k}{k-\sigma^2/g}\cos(gk)^{\frac{1}{2}}t\, e^{k(y+b)} J_0(kR)\,dk.\end{aligned}\right\} \tag{13.50}$$

The leading term in the asymptotic expansion of the last summand gives the last summand of (13.17).

If one takes $m(t)=m=$ a constant, $a(t)=a_0+u_0 t$, $b(t)=b_0$, $c(t)=c_0$, one obtains the velocity potential for a source suddenly brought into existence at $t=0$ and moving with constant velocity in the direction $Ox$ [cf. LUNDE (1951, p. 18)]. A limit as $t\to\infty$ will give (13.36), the proper boundary conditions at infinity being again automatically satisfied. For finite $t$ the velocity potential in a coordinate system moving with velocity $u_0$ in direction $Ox$ $\left(\bar{x}=x-u_0 t\right.$, so that $\left.\Phi(x, y, z, t)=\varphi(\bar{x}, y, z, t)\right)$ is given by

$$\left.\begin{aligned}\varphi(\bar{x}, y, z, t) = {} & \frac{m}{r} - \frac{m}{r_1} + \\ & + \frac{m}{\pi}\int_{-\pi}^{\pi} d\vartheta \int_0^\infty dk\,(gk)^{\frac{1}{2}}\, e^{k[y+b_0+i\omega(\vartheta)]}\int_0^t d\tau\,\sin\tau(gk)^{\frac{1}{2}}\, e^{iku_0\tau\cos\vartheta}, \\ & \omega(\vartheta) = (\bar{x}-a_0)\cos\vartheta + (z-c_0)\sin\vartheta.\end{aligned}\right\} \tag{13.51}$$

The two cases just discussed may be combined by choosing $m(t)=m\cos\sigma t$ and $a(t)=a_0+u_0 t$, $b(t)=b_0$, $c(t)=c_0$. The modification of (13.51) is simple: a factor $\cos\sigma t$ must be put with the first two terms and a factor $\cos\sigma(t-\tau)$ put at the end of the integral. The asymptotic form as $t\to\infty$ can again be found by use of the Fourier Integral Theorem (13.16) or simple modifications. However, if the resulting formula is written out as principal-value integrals plus other terms, the expression is very unwieldy; it may be found in HAVELOCK (1958). Use of complex integrals allows one to compress the formula. Let

$$\varphi(\bar{x}, y, z, t) = m\cos\sigma t\left(\frac{1}{r}-\frac{1}{r_1}\right) + m\,\mathrm{Re}\, e^{-i\sigma t}\varphi_0, \qquad \varphi_0=\varphi_1+i\varphi_2.$$

Then

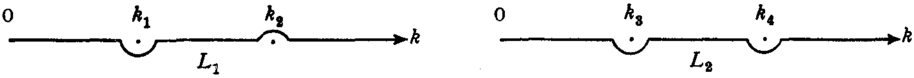
$$\varphi_0 = \frac{2g}{\pi}\int_0^\gamma d\vartheta\int_0^\infty dk\, F(\vartheta, k) + \frac{2g}{\pi}\int_\gamma^{\frac{1}{2}\pi} d\vartheta\int_{L_1} dk\, F(\vartheta, k) + \frac{2g}{\pi}\int_{\frac{1}{2}\pi}^{\pi} d\vartheta\int_{L_2} dk\, F(\vartheta, k),$$

$$F(\vartheta, k) = \frac{k\, e^{k[y+b_0+i(\bar{x}-a_0)\cos\vartheta]}\cos[k(z-c_0)\sin\vartheta]}{gk-(\sigma+ku_0\cos\vartheta)^2}, \tag{13.52}$$

where

$$\tau = u_0 \sigma / g ,$$

$$\gamma = \begin{cases} 0 & \text{if } \tau < \frac{1}{4} \\ \arccos \frac{1}{4\tau} & \text{if } \tau \geqq \frac{1}{4} , \end{cases}$$




$$\sqrt{g\,k_1},\ \sqrt{g\,k_3} = \frac{1 - \sqrt{1 - 4\tau \cos\vartheta}}{2\tau\cos\vartheta}\,\sigma ,$$

$$\sqrt{g\,k_2},\ -\sqrt{g\,k_4} = \frac{1 + \sqrt{1 - 4\tau \cos\vartheta}}{2\tau\cos\vartheta}\,\sigma .$$

This potential has been derived by HASKIND (1946a), BRARD (1948a, b), HANAOKA (1953), SRETENSKII (1954), the last with an unfortunate mistake in sign

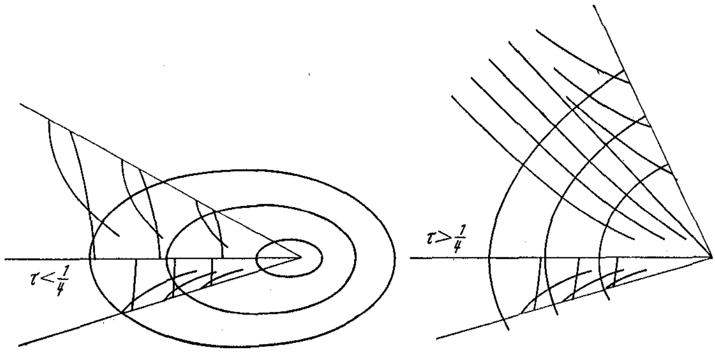



Fig. 3.

in one term, EGGERS (1957), and HAVELOCK (1958). HANAOKA, BRARD, EGGERS, and SRETENSKII have each considered the asymptotic form of the surface for large $R$. Fig. 3 shows qualitatively (cf. BECKER 1958) the curves of equal phase, say the crests, for the various systems of waves formed. The patterns must be completed by reflection in the $x$-axis.

Motion of a source on a circular path of radius $D$ may be treated by taking $a = D\cos\sigma t$, $c = D\sin\sigma t$ in (13.49). For constant $m$ this problem has been considered by SRETENSKII (1946a, b, 1957), HAVELOCK (1950), and STOKER (1957).

One may derive a formula analogous to (13.49) when the source moves in the presence of both a horizontal bottom at $y = -h$ and a free surface. The derivation may be carried out along lines similar to those used in deriving (13.49). The resulting velocity potential is [cf. LUNDE (1951, p. 32)]

$$\left.\begin{aligned} \Phi(x,y,z,t) &= \frac{m(t)}{r(t)} + \frac{m(t)}{r_2(t)} - 2m(t)\int_0^\infty e^{-kh}\,\frac{\cosh k(h+b(t))}{\cosh k h} \times \\ &\times \cosh k(y+h)\, J_0\big(k R(t)\big)\, dk + 2\int_0^\infty dk\,\sqrt{g k}\,\frac{\cosh k(y+h)}{\cosh^2 k h\,\sqrt{\tanh k h}} \times \\ &\times \int_0^t d\tau \sin\big[(t-\tau)\sqrt{g\,k\tanh k h}\big]\, m(\tau)\cosh k\big(h+b(\tau)\big)\, J_0\big(k R(\tau)\big), \end{aligned}\right\} \quad (13.53)$$

where
$$r_2^2 = (x - a(t))^2 + (y + 2h + b(t))^2 + (z - c(t))^2.$$

Two-dimensional formulas corresponding to (13.49) and (13.53) may also be derived. They are as follows, with the source and vortex separated for finite depth:

infinite depth:

see errata

$$\left.\begin{aligned} f(z,t) &= \frac{\Gamma(t) + i\,Q(t)}{2\pi i}\log\left(z - c(t)\right) + \frac{\Gamma(t) + i\,Q(t)}{2\pi i}\log\left(z - \bar{c}(t)\right) + \\ &+ \frac{g}{\pi i}\int_0^t [\Gamma(\tau) - i\,Q(\tau)]\,d\tau \int_0^\infty \frac{1}{\sqrt{g\,k}}\,e^{-i\,k\,(z - \bar{c}(\tau))}\sin\left[\sqrt{g\,k}\,(t-\tau)\right]dk; \end{aligned}\right\} \quad (13.54)$$

depth $h$, source:

$$\left.\begin{aligned} f(z,t) &= \frac{Q(t)}{2\pi}\log\left(z - c(t)\right) + \frac{Q(t)}{2\pi}\log\left(z - \bar{c}(t) + 2i\,h\right) + \\ &+ \frac{Q(t)}{\pi}\int_0^\infty \frac{e^{-k\,h}}{k\cosh k\,h}\cosh k\left(b(t) + h\right)\cos k\left(z - a(t) + i\,h\right)dk - \\ &- \frac{g}{\pi}\int_0^\infty \frac{\operatorname{sech}^2 k\,h}{\sqrt{g\,k\tanh k\,h}}\,dk\int_0^t Q(\tau)\cosh k\left(b(\tau) + h\right)\cos k\left(z - a(\tau) + i\,h\right)\times \\ &\qquad\times\sin\left[\sqrt{g\,k\tanh k\,h}\,(t-\tau)\right]d\tau; \end{aligned}\right\} \quad (13.55)$$

depth $h$, vortex:

$$\left.\begin{aligned} f(z,t) &= \frac{\Gamma(t)}{2\pi i}\log\left(z - c(t)\right) - \frac{\Gamma(t)}{2\pi i}\log\left(z - \bar{c}(t) + 2i\,h\right) + \\ &+ \frac{\Gamma(t)}{\pi}\int_0^\infty \frac{e^{-k\,h}}{k\cosh k\,h}\sinh k\left(b(t) + h\right)\sin k\left(z - a(t) + i\,h\right) - \\ &- \frac{g}{\pi}\int_0^\infty \frac{\operatorname{sech}^2 k\,h}{\sqrt{g\,k\tanh k\,h}}\,dk\int_0^t \Gamma(\tau)\sinh k\left(b(\tau) + h\right)\sin k\left(z - a(\tau) + i\,h\right)\times \\ &\qquad\times\sin\left[\sqrt{g\,k\tanh k\,h}\,(t-\tau)\right]dk. \end{aligned}\right\} \quad (13.56)$$

Higher-order singularities may be generated by taking derivatives with respect to $z$. One may transfer to moving coordinates, etc., just as in the three-dimensional case [see HAVELOCK (1949) for (13.54) in moving coordinates]. The velocity potential for a steadily moving source of pulsing strength in two dimensions has been given by HASKIND (1954, p. 23 ff.), who also gives the asymptotic expressions for large values of $\pm x$. When $\tau < \frac{1}{4}$, there exist one wave far ahead of the moving source propagating in the same direction and three far behind, one propagating in the same direction and two in the opposite direction; when $\tau > \frac{1}{4}$, there exist two waves far behind propagating in the opposite direction. The analysis for finite depth has been given by BECKER (1956).

**14. Some simple physical solutions.** In this section we consider periodic waves in an ocean of infinite horizontal extent, either infinitely deep or with a horizontal bottom, in canals, and at an interface. The linearizing parameter $\varepsilon$ of Sect. 10α may be taken to be the ratio of amplitude to wave length.

α) *Standing waves in an infinite ocean.* It is appropriate to the physical problem to require that the motion remain bounded everywhere.

Consider first two-dimensional motion. Then, from Sect. 13β, the only solutions of the form $\Phi = \varphi\cos(\sigma t + \tau)$ are given by

$$\Phi(x, y, t) = a\,e^{\nu y}\cos(\nu x + \alpha)\cos(\sigma t + \tau), \qquad \nu = \sigma^2/g \tag{14.1}$$

for infinite depth, and

$$\left.\begin{aligned}\Phi(x, y, t) &= a \cosh m_0(y+h) \cos(m_0 x + \alpha) \cos(\sigma t + \tau),\\ m_0 \tanh m_0 h - \nu &= 0,\end{aligned}\right\} \tag{14.2}$$

for finite depth.

The corresponding forms of the free surface are given by

see errata

$$\eta(x, t) = A \cos(\nu x + \alpha) \sin(\sigma t + \tau)$$

and

$$\eta(x, t) = A \cos(m_0 x + \alpha) \sin(\sigma t + \tau),$$

respectively. These represent standing waves according to our definition in Sect. 7. We recall that $m_0 > \nu$.

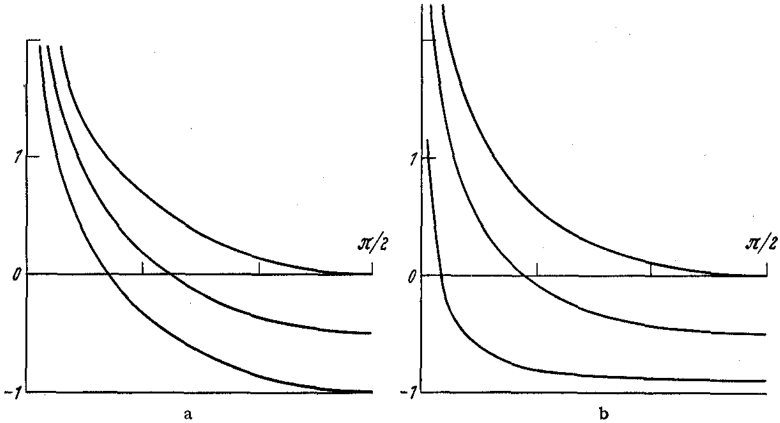



Fig. 4 a and b.

It is of interest to examine the streamlines and the paths of the individual fluid particles. The streamlines of the motion can be easily found from

$$\frac{dy}{dx} = \frac{\Phi_y}{\Phi_x} = \cot(\nu x + \alpha)$$

and

$$\frac{dy}{dx} = \frac{\Phi_y}{\Phi_x} = -\tanh m_0(y+h) \cot(m_0 x + \alpha),$$

respectively. The streamlines are then

$$e^{\nu(y-y_m)} |\sin(\nu x + \alpha)| = 1$$

and

$$\sinh m_0(y+h) |\sin(m_0 x + \alpha)| = \sinh m_0(y_m + h), \qquad 0 \geqq y_m \geqq h, \tag{14.3}$$

for infinite and finite depth respectively; here $y_m$ is the lowest point of the streamline. If the fluid is infinitely deep, the streamlines are all congruent. Fig. 4a shows three of them for a quarter wave length and $\alpha = 0$, $\nu = 1$. The vertical line $x = 0$ is also a streamline. If the fluid is of finite depth, the streamlines vary with depth. Fig. 4b shows streamlines corresponding to $y_m = 0, -0.5, -0.9$ for $\alpha = 0$, $h = 1$, $m_0 = 1$. The horizontal line $y = -1$ and the vertical line $x = 0$ are also streamlines.

Since the streamlines are time-independent, they also contain the curves for the trajectories of individual particles. However, the trajectory of an individual particle will be an oscillating motion of small amplitude along a segment of the streamline passing through the point. Thus, in Fig. 4b the particles on the bottom simply oscillate back and forth about an equilibrium position, those directly beneath a crest, i.e. at $x=0$, oscillate vertically, etc. In view of the infinitesimal-wave approximation used in this chapter the streamlines have physical significance for only a small distance above the equilibrium free surface, $y=0$.

In order to investigate, at least approximately, the behavior of the trajectories more fully, we may replace the actual trajectory by its tangent at an average position, say $(x_0, y_0)$, an approximation consistent with the assumptions made in linearizing. Then the equations describing the trajectory become (setting $\alpha=\tau=0$)

$$\frac{dx}{dt} = -a\nu\, e^{\nu y_0} \sin \nu x_0 \cos \sigma t, \qquad \frac{dy}{dt} = a\nu\, e^{\nu y_0} \cos \nu x_0 \cos \sigma t$$

for infinite depth, and

$$\frac{dx}{dt} = -a m_0 \cosh m_0 (y_0 + h) \sin m_0 x_0 \cos \sigma t,$$
$$\frac{dy}{dt} = a m_0 \sinh m_0 (y_0 + h) \cos m_0 x \cos \sigma t$$

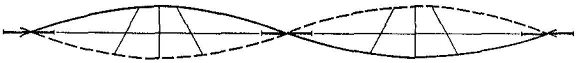


Fig. 5.

for finite depth. The approximate trajectories are then

$$x = x_0 - a\sigma^{-1}\nu\, e^{\nu y_0} \sin \nu x_0 \sin \sigma t, \qquad y = y_0 + a\sigma^{-1}\nu\, e^{\nu y_0} \cos \nu x_0 \sin \sigma t \tag{14.4}$$

for infinite depth, and

$$\left.\begin{aligned} x &= x_0 - a\sigma^{-1} m_0 \cosh m_0 (y_0 + h) \sin m_0 x_0 \sin \sigma t, \\ y &= y_0 + a\sigma^{-1} m_0 \sinh m_0 (y_0 + h) \cos m_0 x_0 \sin \sigma t \end{aligned}\right\} \tag{14.5}$$

for finite depth. For infinite depth, the amplitude of oscillation drops off very rapidly as depth of the equilibrium position increases, the ratio of the amplitude at depth $y_0$ to the amplitude at the free surface being $e^{\nu y_0}$. The same ratio for the case of finite depth is

$$\frac{\sinh^2 m_0 (y_0 + h) + \sin^2 m_0 x_0}{\sinh^2 m_0 h + \sin^2 m_0 x_0}.$$

Thus, on the bottom, when $y_0 = -h$, the amplitude is zero under the crests and maximum under the nodes. As is evident from the equations of the trajectories, the path lines of particles on the free surface are approximately as in Fig. 5. see errata In order to explain the apparently inconsistent behavior at the nodes one mus. go to a higher approximation than the linearized theory used in this chaptert

Let us now consider three-dimensional solutions. The standing-wave solutions are of the form

see errata

$$\Phi(x, y, z, t) = e^{\nu y} \chi(x, y) \cos(\sigma t + \tau) \qquad \text{for finite depth,}$$

or

$$\Phi(x, y, z, t) = \cosh m_0 (y + h)\, \chi(x, z) \cos(\sigma t + \tau) \qquad \text{for finite depth,}$$

where $\chi(x, z)$ is a solution of

$$\Delta_2 \chi + \nu^2 \chi = 0 \quad \text{or} \quad \Delta_2 \chi + m_0 \chi = 0,$$

respectively, regular everywhere in $y \leqq 0$.

Two particular cases are of especial interest. The first corresponds to separation of variables in rectangular coordinates [see (13.5) and (13.6)]. The solutions are

see errata

$$\left.\begin{aligned}\Phi(x, y, z, t) &= a\, e^{\nu y} \cos(k_1 x + \alpha) \cos(k_2 z + \gamma) \cos(\sigma t + \tau),\\ k_1^2 + k_2^2 &= \nu^2 = \sigma^2/g,\end{aligned}\right\} \tag{14.6}$$

for infinite depth, and

$$\left.\begin{aligned}\Phi(x, y, z, t) &= a \cosh m_0(y+h) \cos(k_1 x+\alpha) \cos(k_2 z+\gamma) \cos(\sigma t+\tau),\\ k_1^2 + k_2^2 &= m_0^2, \qquad m_0 \tanh m_0 h - \nu = 0,\end{aligned}\right\} \tag{14.7}$$

for finite depth. The other solutions result from separating variables in polar coordinates [see (13.7) and (13.8)]. They are

$$\Phi(R, \alpha, y, t) = a\, e^{\nu y} J_n(\nu R) \cos(n\alpha + \delta) \cos(\sigma t + \tau), \qquad n = 0, 1, \ldots, \tag{14.8}$$

for infinite depth, and

$$\left.\begin{aligned}\Phi(R, \alpha, y, t) &= a \cosh m_0(y + h)\, J_n(m_0 R) \cos(n\alpha + \delta) \cos(\sigma t + \tau),\\ n &= 0, 1, \ldots,\end{aligned}\right\} \tag{14.9}$$

for finite depth. The form of the free surface may be found immediately from $\eta(x, z, t) = -\Phi_t(x, 0, z, t)/g$. These are all standing waves.

The streamlines and path lines may be found for these two cases with no special difficulty. For the first case for finite depth the streamlines are the intersections of the surfaces

$$\left.\begin{aligned}|\sin k_1 x|^{k_2^2} &= C_1 |\sin k_2 z|^{k_1^2},\\ |\sin k_1 x \sin k_2 z| \sinh m_0(y + h) &= C_2.\end{aligned}\right\} \tag{14.10}$$

The vertical lines, $x = p\pi/k_1$, $z = q\pi/k_2$, passing through the maxima and minima are streamlines. The points on the vertical lines $x = (p + \frac{1}{2})\,\pi/k_1$, $z = (q + \frac{1}{2})\,\pi/k_2$ passing through the sattlepoints are all stagnation points. The projection on $y = 0$ of the field of streamlines is indicated qualitatively by Fig. 6. The behavior in a projection on a vertical plane is similar to that for two-dimensional motion.

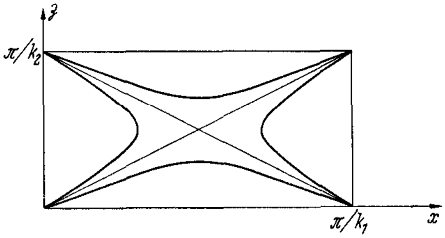



Fig. 6.

see errata

In the second case above one may easily visualize the streamlines for the case of pure ring waves, $n = 0$. For finite depht they are given in a plane $\alpha = \text{const}$ by

$$m_0 R\, J_1(m_0 R) \sinh m_0(y+h) = C_1 \tag{14.11}$$

together with the vertical lines at the zeros of $J_1(m_0 R)$. The behavior of the curves is qualitatively similar to that of the two-dimensional case.

In both cases approximations to the path lines can be found as in the two-dimensional case.

*β) Progressive waves in an infinite ocean.* By taking the proper linear combinations of the standing-wave solutions one may obtain progressive waves. Thus, adding

$$\Phi_1 = a\, e^{\nu y} \cos \nu x \cos \sigma t \quad \text{and} \quad \Phi_2 = a\, e^{\nu y} \sin \nu x \sin \sigma t,$$

one obtains

$$\Phi = a\, e^{\nu y} \cos(\nu x - \sigma t) \tag{14.12}$$

which represents a progressive wave moving to the right with velocity

$$c = \frac{\sigma}{\nu} = \frac{g}{\sigma} = \sqrt{\frac{g}{\nu}} = \sqrt{\frac{g\lambda}{2\pi}}, \tag{14.13}$$

where $\lambda = 2\pi/\nu$ is the wavelength. Subtracting yields a progressive wave moving to the left. If one takes the coefficient in $\Phi_1$ as $a_1$ and in $\Phi_2$ as $a_2$, the sum may be written

$$\Phi = \tfrac{1}{2} e^{\nu y} [(a_1 + a_2) \cos(\nu x - \sigma t) + (a_1 - a_2) \cos(\nu x + \sigma t)].$$

This is a superposition of two progressive waves of different amplitudes, one moving to the left and one to the right. If $a_1 = a_2$, a pure progressive wave is obtained; if $a_2 = 0$, one obtains again a standing wave, as a superposition of two progressive waves moving in opposite directions.

For water of finite depth $h$ the corresponding expressions for $\Phi$ may be obtained by replacing $e^{\nu y}$ by $\cosh m_0(y+h)$ and $\nu$ by $m_0$. The phase velocity is given by

$$c = \frac{\sigma}{m_0} = \sqrt{\frac{g \tanh m_0 h}{m_0}} = \sqrt{\frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}}. \tag{14.14}$$

As $h \to \infty$, the velocity approaches that obtained above for deep water. In fact, if $h/\lambda > 0.2$, the velocity is already within 0.1 of the value for deep water. $c$ is an increasing function of $\lambda$, but cannot increase indefinitely as in the case of infinitely deep water, for (14.14) implies

$$c < \sqrt{g h}. \tag{14.15}$$

The streamlines for the progressive wave moving to the right are given by

$$e^{\nu y} |\sin(\nu x - \sigma t)| = C \quad \text{and} \quad \sinh m_0(y+h) |\sin(m_0 x - \sigma t)| = C \tag{14.16}$$

for infinite and finite depth, respectively. At a given instant $t$ these have the same shape relative to a crest as the streamlines for a standing wave. However, since they are time-dependent, the path lines for particles do not lie on the streamlines. The path lines may be found approximately for a particle with equilibrium position $(x_0, y_0)$ from the equations

$$\frac{dx}{dt} = \Phi_x(x_0, y_0, t), \qquad \frac{dy}{dt} = \Phi_y(x_0, y_0, t).$$

This approximation is consistent with the assumptions made in linearizing the boundary condition, as can be seen by assuming a solution in the form

$$x(t) = x_0 + \varepsilon x_1(t) + \cdots, \qquad y(t) = y_0 + \varepsilon y_1(t) + \cdots,$$

where $\varepsilon = a\sigma\nu/2\pi g$ for infinite depth and $\varepsilon = a\sigma m_0/2\pi g$ for finite depth, substituting in the exact path equations, and retaining only first-order terms.

For infinite depth the particle trajectories are given by

$$x = x_0 - a\nu\sigma^{-1} e^{\nu y_0} \cos(\nu x_0 - \sigma t), \quad y = y_0 - a\nu\sigma^{-1} e^{\nu y_0} \sin(\nu x_0 - \sigma t). \tag{14.17}$$

The particles follow circular orbits of radius $a\nu\sigma^{-1} e^{\nu y_0}$ about the equilibrium position $(x_0, y_0)$; at the top of the orbit they are moving in the same direction as the wave. The orbital velocity is $a\nu\, e^{\nu y_0}$, so that the motion dies out quickly as $|y_0|$ increases; for example, at a depth of one wave length the velocity and orbit radius are only $\frac{1}{535}$ the value at the free surface. Although the particles at the crest of a wave are moving in the same direction as the wave, their velocity is not necessarily the same and is, in fact, much smaller in view of the assumed smallness of $\varepsilon = (a\nu/c)(\nu/2\pi)$.

For finite depth the orbits are elliptical with the major axis horizontal:

$$\left.\begin{aligned} x &= x_0 - a\, m_0\, \sigma^{-1} \cosh m_0 (y_0 + h) \cos (m_0\, x - \sigma t)\,, \\ y &= y_0 - a\, m_0\, \sigma^{-1} \sinh m_0 (y_0 + h) \sin (m_0\, x - \sigma t)\,. \end{aligned}\right\} \tag{14.18}$$

The particles again trace the orbit in a clockwise direction except that on the bottom they simply oscillate along a horizontal segment. Fig. 7 from RUELLAN and WALLET (1950) shows the path lines for a variety of cases of superposed waves. The topmost picture shows the orbits for a pure progressive wave moving to the right. The bottom picture is a superposition of progressive waves of equal amplitudes moving in opposite directions, i.e. a pure standing wave. The intermediate cases show superpositions with varying ratios of the amplitudes. The intermediate cases are instructive in that not only path lines, but also streamlines are visible.

Since the progressive-wave solutions are steady with respect to a coordinate system moving with the wave, it is clear that we could have obtained a steady-state solution as a small motion superposed upon a uniform flow. If we take a complex velocity potential in the form

$$F(z) = -c\, z + f(z)\,. \tag{14.19}$$

Then [see Eq. (11.6)] $f$ must satisfy

$$\mathrm{Re}\{i\, g\, f + c^2 f'\} = 0 \qquad \text{for } y = 0$$

and either $|f'| \to 0$ as $y \to -\infty$ or $\mathrm{Im}\, f' = 0$ for $y = -h$. The solution for the first case, infinite depth, is given by

$$f = a\, \mathrm{e}^{-i\nu z} = a\, \mathrm{e}^{\nu y} [\cos \nu\, x - i \sin \nu\, x]\,, \qquad \nu = g/c^2. \tag{14.20}$$

The solution for the finite-depth case is given by

$$\left.\begin{aligned} f &= a \cos m_0 (z + i\, h) \\ &= a\, [\cos m_0\, x \cosh m_0\, (y + h) - i \sin m_0\, x \sinh m_0\, (y + h)]\,, \end{aligned}\right\} \tag{14.21}$$

where $m_0$ must satisfy

$$c^2\, m_0 - g \tanh m_0\, h = 0\,.$$

The same relation is found in (14.14). A real solution does not exist if $c^2/g\, h > 1$ and in this case there is no wave-like motion consistent with the linearized theory. The streamlines, identical here with the path lines, are obtained from

$$-c\, y + \psi(x, y) = 0\,.$$

One may replace this equation, consistently with the linearization assumptions [cf. (10.18)], by

$$-c\, y + \psi(x, y_0) = 0\,,$$

where $y_0$ is the mean height of the streamline. Thus, for finite depth, they are given by

$$y = -\frac{a}{c} \sinh m_0\, (y_0 + h) \sin m_0\, x\,, \tag{14.22}$$

an easily constructed family of curves. In the foregoing we have tacitly taken $a$ to be real. However, it may be complex and thus include waves of different phase.

We note finally that (14.8) or (14.9) allow one to construct waves progressing like the spokes of a wheel. However, outwardly progressing waves can be constructed only when the solution involving $Y_n$ is used, and this has a singularity at the origin.

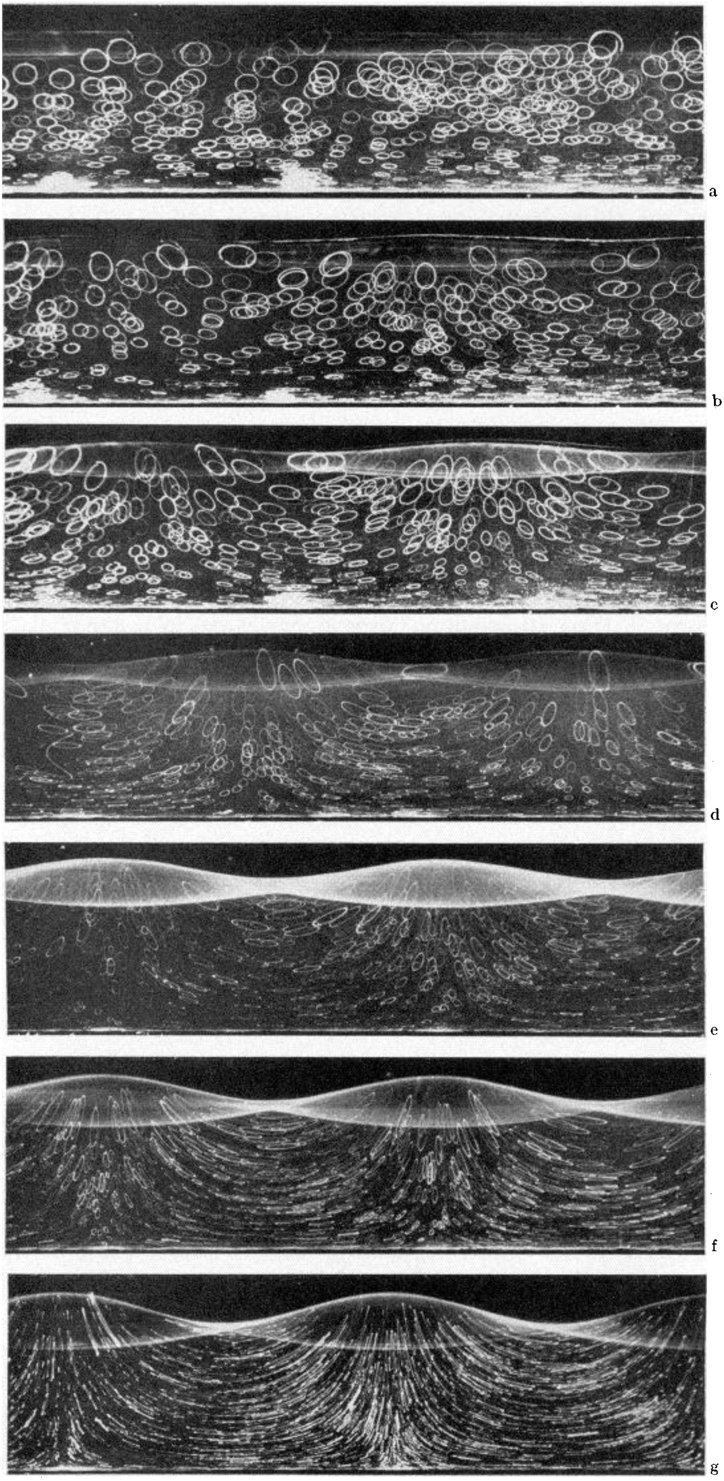



Fig. 7 a—g. Particle trajectories in progressive and standing waves.

*γ) Periodic waves in rectangular canals.* Let us suppose that the fluid is contained between the planes $z=0$ and $z=d$. Then the velocity potential must satisfy the additional conditions

$$\Phi_z(x, y, 0, t) = \Phi_z(x, y, d, t) = 0. \tag{14.23}$$

This condition is automatically satisfied by the two-dimensional waves discussed in 14α, so that they present no special interest here. However, condition (14.4) does restrict the three-dimensional solutions (14.6) and (14.7), for $k_2$ must now satisfy (taking $\gamma=0$).

$$k_2 = \frac{n\pi}{d}, \quad n = 1, 2, \ldots .$$

Since $k_1^2+k_2^2=\nu^2$ or $m_0^2$, there can be no solution periodic in $x$ unless

$$n < \frac{\nu d}{\pi} \quad \text{or} \quad n < \frac{m_0 d}{\pi}, \tag{14.24}$$

respectively. Hence, for frequencies below a certain critical frequency $\sigma_1$, where

$$\sigma_1 = \sqrt{\frac{\pi g}{d}} \quad \text{or} \quad \sigma_1 = \sqrt{\frac{\pi g}{d}\tanh\frac{\pi h}{d}} \tag{14.25}$$

for infinite or finite depth respectively, there can exist no three-dimensional standing waves in a canal.

Let us form a three-dimensional progressive wave in a canal of finite depth by adding standing-wave solutions:

$$\Phi(x, y, z, t) = a\cosh m_0(y+h)\cos k_2 z\cos(k_1 x - \sigma t), \qquad k_2 = n\pi/d.$$

The velocity of the progressive wave is given by

$$c^2 = \frac{\sigma^2}{k_1^2} = g h\left(1 - \frac{n^2\pi^2}{m_0^2 d^2}\right)^{-1}\frac{\tanh m_0 h}{m_0 h} < g h\left(1 - \frac{n^2\pi^2}{m_0^2 d^2}\right)^{-1}. \tag{14.26}$$

As in the case treated above, there can exist no three-dimensional progressive waves unless $\sigma > \sigma_1$. However, if they exist, their velocity is higher than the velocity of two-dimensional waves of the same frequency.

One may define similarly a sequence of critical frequencies $\sigma_1, \sigma_2, \ldots$, where

$$\sigma_k = \sqrt{\frac{k\pi g}{d}} \quad \text{or} \quad \sigma_k = \sqrt{\frac{k\pi g}{d}\tanh\frac{k\pi h}{d}};$$

when $\sigma_k < \sigma < \sigma_{k+1}$, $k$ types of three-dimensional waves are possible with $n = 1, 2, \ldots, k$.

*δ) Waves at an interface.* Let us now suppose that two fluids are present, one lying over the other. Variables referring to the upper and lower fluids have subscripts 2 and 1 respectively. From (10.7) and (10.8) the linearized boundary conditions for a small disturbance are

$$\left.\begin{aligned} \Phi_{1y} &= \Phi_{2y}, \\ \varrho_1[\Phi_{1tt} + g\Phi_{1y}] &= \varrho_2[\Phi_{2tt} + g\Phi_{2y}], \end{aligned}\right\} \tag{14.27}$$

both equations to be satisfied at the equilibrium position of the interface. We shall consider several typical problems.

Let the upper fluid fill the region $y>0$, and the lower fluid the region $y<0$. We require of a solution that

$$|\mathrm{grad}\,\Phi_1| \to 0 \quad \text{as} \quad y \to -\infty \quad \text{and} \quad |\mathrm{grad}\,\Phi_2| \to 0 \quad \text{as} \quad y \to +\infty.$$

In looking for a standing-wave solution, one may, following Sect. 14α, take

$$\Phi_1 = a_1 e^{my} \varphi(x, z) \cos(\sigma t + \tau), \qquad \Phi_2 = a_2 e^{-my} \varphi(x, z) \cos(\sigma t + \tau),$$

where the relation between $a_1$ and $a_2$ and $m$ and $\sigma$ is to be determined by (14.27), and $\varphi$ satisfies

$$\Delta_2 \varphi + m^2 \varphi = 0.$$

The first Eq. (14.27) gives immediately that

$$a_1 + a_2 = 0.$$

The second one gives the relation

$$\sigma^2 = \frac{\varrho_1 - \varrho_2}{\varrho_1 + \varrho_2} m g. \tag{14.28}$$

The equation of the interface may be obtained from (10.8):

$$\eta(x, z, t) = \frac{a_1 m}{\sigma} \varphi(x, z) \sin(\sigma t + \tau).$$

Since $a_1 = -a_2$, there is a discontinuity in $u$ (and $w$ if the motion is three-dimensional) as one crosses the interface.

The special choices of $\varphi(x, z)$ made in Sect. 14β may, of course, also be made here. In particular, one may make progressive and standing waves. If one forms two-dimensional progressive waves at the interface, one finds for the velocity

$$c^2 = \frac{\varrho_1 - \varrho_2}{\varrho_1 + \varrho_2} \frac{g}{m}. \tag{14.29}$$

If one assumes the fluids bounded above and below by planes $y = h_2$ and $y = -h_1$, respectively, a similar calculation shows

$$\sigma^2 = \frac{\varrho_1 - \varrho_2}{\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2} g m. \tag{14.30}$$

It is clear from (14.28) and (14.30) that these solutions exist only if $\varrho_2 < \varrho_1$. The case $\varrho_2 > \varrho_1$ will be discussed later.

A more complicated problem of this type is the following [cf. LAMB (1932, § 231), GREENHILL (1887)]. Suppose there is a solid horizontal bottom at $y = -h$, an interface at $y = -d$ and a free surface at $y = 0$. Then, in addition to (14.27) at $y = -d$, $\Phi_1$ and $\Phi_2$ must satisfy

$$\Phi_{2tt} + g \Phi_{2y} = 0 \quad \text{at} \quad y = 0, \qquad \Phi_{1y} = 0 \quad \text{at} \quad y = -h.$$

If one seeks solutions of the form

$$\Phi_2 = (a_2 \cosh m y + b_2 \sinh m y) \varphi(x, z) \cos(\sigma t + \tau),$$
$$\Phi_1 = a_1 \cosh m(y + h) \varphi(x, z) \cos(\sigma t + \tau),$$

substitution in the various boundary conditions yields the following relation between $\sigma$ and $m$:

$$\left.\begin{aligned} &\left(\frac{\sigma^2}{g m}\right)^2 [\varrho_1 \coth m d \coth m(h - d) + \varrho_2] - \\ &\qquad - \frac{\sigma^2}{g m} \varrho_1 [\coth m d + \coth m(h - d)] + (\varrho_1 - \varrho_2) = 0. \end{aligned}\right\} \tag{14.31}$$

If $\varrho_2 < \varrho_1$, one may establish that there exist two positive solutions for $\sigma^2$ for a given $m$, so that two possible frequencies are possible for a given wave pattern.

If the bottom fluid is taken infinitely deep, one replaces $\coth m(h-d)$ by 1 in (14.31) and the two solutions simplify to

$$\sigma_1^2 = g\,m, \qquad \sigma_2^2 = g\,m \frac{\varrho_1 - \varrho_2}{\varrho_1 \coth m\,d + \varrho_2} < \sigma_1^2. \tag{14.32}$$

The first solution, $\sigma_1$, is the same as would be obtained if the two fluids were identical (and there is no discontinuity in $u$ and $w$ at the interface); the second, $\sigma_2$, is interpreted below. The inequality $\sigma_2^2 < \sigma_1^2$ holds in general, and one may establish

$$\frac{\sigma_2^2}{g\,m} < \{\tanh m\,d,\ \tanh m\,(h-d)\} \leqq \frac{\sigma_1^2}{g\,m} \leqq \min\left\{1, \frac{\varrho_1}{\varrho_2} \tanh m\,h\right\}. \tag{14.33}$$

If one computes the ratio of the amplitude of the disturbance at the interface to that at the free surface, one finds, no matter whether $h$ is finite or not,

$$\cosh m\,d - \frac{g\,m}{\sigma^2} \sinh m\,d. \tag{14.34}$$

An examination of the roots of (14.31) shows that the ratio (14.34) is negative for the smaller of the two roots and positive for the larger. Thus, in the solution associated with the smaller root, a maximum of the disturbance at the interface is associated with a minimum of that at the free surface, and vice versa. On the other hand, with the larger root the maxima and minima go together. For the values given in (14.32), the ratio becomes

$$e^{-m\,d} \quad \text{and} \quad -\frac{\varrho_1}{\varrho_1 - \varrho_2} e^{m\,d}, \tag{14.35}$$

respectively. We note that, although the first ratio is $<1$, the second is in absolute value $>1$ if $\varrho_2(1+e^{m\,d}) > \varrho_1 > \varrho_2$, a condition satisfied if $\varrho_1$ is only slightly greater than $\varrho_2$. In fact, the ratio may become very large.

For a given wave length and amplitude of the wave at the free surface one may also compare the amplitudes of the two different modes of motion at the interface. If $A_i$ is the amplitude associated with the frequency $\sigma_i$, then for the case $h=\infty$ one finds

$$\left|\frac{A_2}{A_1}\right| = \frac{\varrho_2}{\varrho_1 - \varrho_2} \frac{1 + \tanh m\,d}{1 - \tanh m\,d},$$

which may be either less than or greater than 1.

It is of some interest to examine somewhat further the solution associated with the smaller root $\sigma_2$ of (14.31). Then, since $a_2/b_2 = g\,m/\sigma^2$, the inequality (14.39) implies that there exists an $h_0$ with $0<h_0<d$ such that

see errata

$$\frac{\sigma_2^2}{g\,m} = \frac{b_2}{a_2} = \tanh m\,h_0 < \tanh m\,d < 1$$

and that

$$\Phi_2 = \sqrt{a_2^2 - b_2^2} \cosh m\,(y+h_0)\, \varphi(x,z) \cos(\sigma\,t + \tau).$$

Thus the part of the top fluid between $y=0$ and $y=-h_0$ behaves as if there were a solid boundary at $y=-h_0$; and, of course, the fluid between $y=-h_0$ and $y=-h$ as if it were between solid boundaries. If one has selected solutions for $\varphi$ which can be combined to form a progressive plane wave, then one may conclude that the velocity $c_2 = \sigma_2/m$ associated with this mode of motion has an upper bound:

$$c_2 = \sqrt{\frac{g}{m} \tanh m\,h_0} < \sqrt{g\,d}.$$

In fact, when $h=\infty$, one may verify immediately from (14.32) that

$$c_2 = \sqrt{\frac{g}{m}\tanh m d \frac{\varrho_1-\varrho_2}{\varrho_1+\varrho_2\tanh m d}} \leqq \sqrt{g d \frac{\varrho_1-\varrho_2}{\varrho_1}} = c_{2\max}.$$

Thus for $h=\infty$ a progressive wave travelling faster than $c_{2\max}$ will consist of only the one mode of motion, i.e. the one associated with $\sigma_1$. If $c<c_{2\max}$, there may be two modes of motion excited. This fact is associated with the phenomenon of "dead-water" resistance of ships [see LAMB (1916a), EKMAN (1904), SRETENSKII (1934)].

For superposed fluids one may also find solutions analogous to (14.20) and (14.21). Let us suppose that the first (upper) fluid flows to the left with mean velocity $c_2$ and the second with mean velocity $c_1$. We wish to find the possible steady periodic profiles of the interface, assuming as usual that the disturbance is small. The complex velocity potential for each fluid is taken in the form

$$F_1(z) = -c_1 z + f_1(z), \qquad F_2(z) = -c_2 z + f_2(z). \tag{14.36}$$

The conditions to be satisfied at the mean common boundary, $y=0$, are:

$$\left.\begin{aligned} c_1^{-1}\operatorname{Im} f_1 &= c_2^{-1}\operatorname{Im} f_2,\\ \varrho_1 c_1^{-1}\operatorname{Re}\{i g f_1 + c_1^2 f_1'\} &= \varrho_2 c_2^{-1}\operatorname{Re}\{i g f_2 + c_2^2 f_2'\}.\end{aligned}\right\} \tag{14.37}$$

If each fluid extends infinitely far vertically, then

$$f_1 = a_1 e^{-imz}, \qquad f_2 = a_2 e^{imz}$$

give a steady-state solution if

$$\frac{a_1}{c_1} = -\frac{\bar a_2}{c_2}$$

and

$$m = \frac{g(\varrho_1-\varrho_2)}{\varrho_1 c_1^2 + \varrho_2 c_2^2} > 0, \tag{14.38}$$

where $\bar a_2$ is the complex conjugate of $a_2$. If the upper fluid is bounded by $y=h_2$ and the lower by $y=-h_1$, then the solution is

$$f_1 = a_1 \cos m(z + i h_1), \qquad f_2 = a_2 \cos m(z - i h_2),$$

where, letting $a_k = \alpha_k + i\beta_k$, $k=1, 2$,

$$\frac{\alpha_1}{c_1}\sinh m h_1 = -\frac{\alpha_2}{c_2}\sinh m h_2, \qquad \frac{\beta_1}{c_1}\cosh m h_1 = \frac{\beta_2}{c_2}\cosh m h_2$$

and

$$m = \frac{g(\varrho_1-\varrho_2)}{\varrho_1 c_1^2 \coth m h_1 + \varrho_2 c_2^2 \coth m h_2}. \tag{14.39}$$

In either case the equation of the interface is given by

$$y = \frac{1}{c_k}\psi_k(x, 0).$$

SRETENSKII (1952b) has considered a three-dimensional analogue of the above problem in which the direction of flow of one of the fluids makes an angle $\vartheta$ with that of the other. Thus, take velocity potentials of the following form:

$$\left.\begin{aligned} \Phi_2(x,y,z) &= -c_2(x\cos\vartheta + z\sin\vartheta) + \varphi_2(x,y,z),\\ \Phi_1(x,y,z) &= -c_1 x + \varphi_1(x,y,z).\end{aligned}\right\} \tag{14.40}$$

The following are the boundary conditions at the interface $\eta(x, z)$ for small disturbance:

$$\left.\begin{aligned}&\varphi_{2y}(x,0,z)+c_2(\eta_x\cos\vartheta+\eta_z\sin\vartheta)=0, \qquad \varphi_{1y}(x,0,z)+c_1\eta_x=0,\\ &g(\varrho_1-\varrho_2)\,\eta=\varrho_1 c_1\varphi_{1x}(x,0,z)-\varrho_2 c_2[\varphi_{2x}(x,0,z)\cos\vartheta+\varphi_{2z}(x,0,z)\sin\vartheta].\end{aligned}\right\}\tag{14.41}$$

For a solution in the form

$$\varphi_1=A_1\,\mathrm{e}^{my}\cos(k_1x+k_2z),\qquad \varphi_2=A_2\,\mathrm{e}^{-my}\cos(k_1x+k_2z),\qquad k_1^2+k_2^2=m^2,$$

the following relations must hold

$$\left.\begin{aligned}&\frac{A_2}{A_1}=-\frac{c_2}{c_1}\,\frac{k_1\cos\vartheta+k_2\sin\vartheta}{k_1}\\ \text{and}\qquad &\varrho_1c_1^2k_1^2+\varrho_2c_2^2(k_1\cos\vartheta+k_2\sin\vartheta)^2=g\,m(\varrho_1-\varrho_2).\end{aligned}\right\}\tag{14.42}$$

These reduce to (14.38) for $\vartheta=0$, $k_1=m_1$ as they should. The equation for the interface is

$$y=-A_1\frac{m}{k_1c_1}\sin(k_1x+k_2z).\tag{14.43}$$

SRETENSKII studies the properties of the solution in more detail.

As a further extension of the preceding cases one may consider a time-dependent disturbance at the interface between two fluids flowing at different velocities. This will be treated in the section on stability of motion.

A natural generalization of the two-fluid system is the $n$-fluid system [see GREENHILL (1887)] and then the heterogeneous fluid with density given as a series

$$\varrho(x,y,z,t)=\varrho_0(y)+\varepsilon\,\varrho^{(1)}(x,y,z,t)+\varepsilon^2\varrho^{(2)}+\cdots.$$

If one assumes a similar expansion for $p$ and expansions for $u$, $v$, $w$, and $\eta$ starting with $\varepsilon$, one may derive easily the linearized equations. These, discussion of some periodic solutions, and references to the literature may be found in LAMB (1932, § 235). GROEN (1948) has shown that the period for simple harmonic motion in the linearized problem is a monotonic increasing function of the wave length starting with the minimum $2\pi\sqrt{-\varrho_0(y)/g\,\varrho_0'(y)}$ for $\lambda=+0$. This theorem has been generalized by HEYNA and GROEN (1958) to allow a free upper surface. GROEN (1950) discusses properties of internal waves in an expository way and gives further references to the more recent literature. For some pertinent theorems about waves in heterogeneous fluids see Sect. $32\beta$.

**15. Group velocity and the propagation of disturbances and of energy.** In the last section we considered periodic waves at a free surface or interface. In this section we wish to consider waves of a given but fairly general initial form and study the way in which they propagate. Although this will entail writing down the solution to a particular initial-value problem, this is of only incidental interest, the chief interest being in the history of the form of the free surface or interface. Initial-value problems as such will be treated in more detail later on. In fact, the remarks below apply equally well to other initial-value problems, for example, an initial distribution of velocity on the surface. What is essential is the resolution of the subsequent motion into a set of waves moving to the right and of ones moving to the left, as in (15.2).

The property of the fluid and its boundaries which is most important for this investigation is the functional relation between the frequency $\sigma$ and the wave number $k$. The earlier parts of this chapter have shown that considerable variation is possible in the form of $\sigma(k)$. The two-fluid example with both free surface and interface gave a doubly valued function. A multiply valued function could have been obtained with more layers. However, each branch, or the branch, is a decreasing function of $k$, approaching zero as $k \to \infty$. When surface tension is taken into account (see Sect. 24), the form of $\sigma(k)$ for large $k$ changes; it then becomes an increasing function, behaving like $k^{\frac{3}{2}}$. If $h$ is large enough, $\sigma(k)$ decreases initially, i.e. for $k < k_m$, reaches a minimum at $k_m$ and then increases; if $h$ is small enough $\sigma(k)$ is everywhere increasing. It will be convenient to extend the definition of $\sigma(k)$ to negative $k$ by setting $\sigma(-k) = -\sigma(k)$.

α) *The propagation of an initial elevation.* Let us suppose that at time $t=0$ the free surface is given by $y=\eta(x,0)$ and that the fluid is at rest. How does the free surface behave subsequently? One may conveniently think of this as an initial humping up of the fluid near one point, but this is not essential. We shall also suppose that $\eta(x,0)$ is sufficiently restricted to allow a Fourier-integral representation. In part of what follows we shall also assume it to be square integrable, i.e. the total available energy is finite, and on occasion that $x\eta$ is square integrable. Let

$$\left.\begin{aligned}\eta(x,0) &= \int_0^\infty [C(k)\cos kx + S(k)\sin kx]\,dk\\ &= \int_{-\infty}^\infty e^{-ikx} E(k)\,dk = 2\,\mathrm{Re}\int_0^\infty e^{-ikx} E(k)\,dk,\end{aligned}\right\} \tag{15.1}$$

where

$$C(k) = \frac{1}{\pi}\int_{-\infty}^\infty \eta(x,0)\cos kx\,dx, \qquad S(k) = \frac{1}{\pi}\int_{-\infty}^\infty \eta(x,0)\sin kx\,dx,$$

$$E(k) = \frac{1}{2\pi}\int_{-\infty}^\infty \eta(x,0)\,e^{ikx}\,dx = \frac{1}{2}[C(k) + i\,S(k)].$$

We shall call $E(k)$ the *spectrum* of $\eta(x,0)$. Note that $E(-k) = E^*(k)$, the complex conjugate of $E(k)$ (we change notation temporarily in order to avoid conflict with the notation for averages introduced below).

A formal solution for $\Phi$ and $\eta(x,t)$ may be written down immediately:

$$\left.\begin{aligned}\Phi(x,y,t) &= -\int_0^\infty \frac{\sigma(k)}{k} Y(y)\,[C(k)\cos kx + S(k)\sin kx]\sin\sigma t\,dk\\ &= -\int_{-\infty}^\infty \frac{\sigma(k)}{k} Y(y)\,E(k)\,e^{-ikx}\sin\sigma t\,dk\\ &= \frac{1}{2}i\int_{-\infty}^\infty \frac{\sigma(k)}{k} Y(y)\,E(k)\,[e^{-i(kx-\sigma t)} - e^{-i(kx+\sigma t)}]\,dk,\\ \eta(x,t) &= \int_0^\infty [C(k)\cos kx + S(k)\sin kx]\cos\sigma t\,dk\\ &= \int_{-\infty}^\infty e^{-ikx} E(k)\cos\sigma t\,dk = \frac{1}{2}\int_{-\infty}^\infty E(k)\,[e^{-i(kx-\sigma t)} + e^{-i(kx+\sigma t)}]\,dk.\end{aligned}\right\} \tag{15.2}$$

Here $Y(y) = \cosh k(y+h)/\sinh kh$ for a single fluid of depth $h$, $Y(y) = e^{|k|y}\,\mathrm{sgn}\,k$ for infinite depth (the peculiar modification of $Y$ for $h=\infty$ is necessary for

$k<0$). However, more general situations are allowable in which, for example, $\eta(x,t)$ describes an interface. The choice of an expression for $\Phi$ has been based upon the kinematic boundary condition $\Phi_y(x,0,t)=\eta_t(x,t)$ in order not to exclude the possibility of surface tension. For simplicity we also restrict ourselves to single-valued $\sigma$'s. For more complicated problems, such as the two-fluid problem with both free surface and interface discussed in Sect. 14 $\delta$, the freedom to fix both $\eta_1(x,0)$ and $\eta_2(x,0)$ independently requires the determination of two spectra for each surface with relations between them set by (14.34). The remarks below will still apply to motion resulting from each spectrum separately. Finally, we note that a statement concerning specific conditions to be satisfied by $\eta(x,0)$ for the case of a single free surface may be found in a paper by KAMPÉ DE FÉRIET and KOTIK (1953).

It is clear from (15.2) that one may express $\eta(x,t)$ as a sum of two functions, one, say $\eta_R(x,t)$, representing a superposition of waves moving to the right, the other, $\eta_L$, waves moving to the left. We consider only $\eta_R$ since similar remarks apply to $\eta_L$ with $x$ replaced by $-x$. The spectrum of $\eta_R$ is given by $\frac{1}{2}E(k)\,e^{i\sigma(k)t}$, so that clearly $\sigma(k)$ plays an important role in the change of shape of $\eta_R$. Since each harmonic component in $\eta_R$ is moving to the right with velocity $\sigma(k)/k$, and since this is not a constant in the cases we have been considering, the different components will move with different velocities and we shall expect $\eta_R$ to change its shape with time, even though moving as a whole to the right.

In order to get some idea of the overall motion it is reasonable to try to compute an average position of $\eta_R(x,t)$ and find how this moves. One must first decide how to define the average position. One possibility, which, as we shall see presently, is unsatisfactory is to use $\eta_R$ itself as the weighting function, i.e. to define

$$\bar{x}_R(t)=\int_{-\infty}^{\infty}x\,\eta_R(x,t)\,dx\Big/\int_{-\infty}^{\infty}\eta_R(x,t)\,dx$$

when this exists. An easy computation shows that

$$\bar{x}_R(t)=\bar{x}_R(0)+\sigma'(0)\,t,$$

i.e. the average motion is, on this definition, independent of the form of $\sigma(k)$ except near $k=0$. For deep-water gravity waves $\sigma'(0)=\infty$; for depth $h$, $\sigma'(0)=\sqrt{gh}$, the maximum velocity [see Eq. (14.15)]. In conformity with the above one may define the "spread" of the hump to be

$$\int_{-\infty}^{\infty}[x-\bar{x}_R(t)]^2\,\eta_R(x,t)\,dx\Big/\int_{-\infty}^{\infty}\eta_R(x,t)\,dx.$$

A computation shows that this remains constant in time, when it exists. This definition of average is unsatisfactory, as could have been expected inasmuch as the weighting function can become negative. We note in passing that

$$\int_{-\infty}^{\infty}\eta_R(x,t)\,dx=\int_{-\infty}^{\infty}\eta_R(x,0)\,dx,$$

an expression of conservation of mass.

Another possible weighting function without this shortcoming, but still allowing ease of computation, is $\eta_R^2(x,t)$. We note first that

$$\int_{-\infty}^{\infty}\eta_R^2(x,t)\,dx=\int_{-\infty}^{\infty}\eta_R^2(x,0)\,dx=\tfrac{1}{2}\pi\int_{-\infty}^{\infty}E(k)\,E^*(k)\,dk.$$