Surface Waves.

By

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With 56 Figures.

A. Introduction.

The various problems of fluid motion treated in this article have in common the property that the fluid is subject to a gravitational force. In addition, in almost all cases they also have in common the presence of surfaces separating two fluids of different densities or, if only one fluid is present, of so-called free surfaces. However, not all fluid flows falling into this category are treated here: tidal motion is treated in Vol. XLVIII in the article by A. DEFANT. The observed properties of ocean waves and their generation by wind are treated in the article by H. U. ROLL, also in Vol. XLVIII. Closely related problems concerning flows with free surfaces are treated in the article by D. GILBARG in this volume.

The subject of water waves engaged many of the mathematicians and mathematical physicists of the last century. Moreover, the last several years have brought a renewed interest in the theory of water waves. In addition to this extensive literature on theoretical aspects of the subject, there have also been many experimental investigations, usually carried out by hydraulic engineers. Hydraulic engineers have also produced an extensive literature, both theoretical and experimental, on open channel flow, flow over weirs and through sluice-gates, etc.; included is a considerable literature on numerical and graphical methods of solving the equations involved. Oceanographers have produced their own literature, usually emphasizing different aspects of the subject. The theory of ship waves has produced its own literature.

All this material is pertinent to this article. Clearly some selection has to be made. We have followed roughly the following rules: Fundamental results are derived in full. The treatments of various special problems are selected so as to exemplify particular methods, other methods being mentioned only by literature citation. Experimental results are not usually reproduced, but references are given. Numerical methods of solving equations are not treated at all. The more special problems of hydraulic engineering are also not treated. Geophysical aspects which are omitted have already been mentioned.

Several excellent expositions of the theory of waves or of various parts of it already exist. We mention the following[2]: LAMB [1932, Chaps. VIII (pp. 250 to 362) and IX (pp. 363–475)]; BASSET [1888, Chap. XVII (pp. 144–187)]; WIEN [1900, Chap. V (pp. 166–224)]; KOCHIN, KIBEL', and ROZE [1948, Chap. 8 (pp. 394–526)]; MILNE-THOMSON [1956, Chap. XIV (pp. 374–431)]; AIRY (1845); BOUASSE (1924); AUERBACH (1931); THORADE (1931); SRETENSKII (1936); KHRISTIANOVICH (1938); KEULEGAN (1950); ECKART (1951); and STOKER (1957). The last cited book by STOKER gives an up-to-date account of much of the

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[2] References are collected at the end and identified in the text by author and date.

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fundamental theory. For observation of waves of many kinds, CORNISH (1910, 1934) und MICHE (1954) should be consulted. SHULEIKIN [1953, part 3 (pp. 213 to 292)] contains a general discussion of topics of interest in oceanography. RUSSELL and MACMILLAN (1952) give a rather nontechnical discussion of ocean waves. A volume published by the Society of Naval Architects of Japan (Zôsen Kyôkai) contains expository papers on various aspects of water-wave theory related to ships [see MARUO (1957), JINNAKA (1957), NISHIYAMA (1957), BESSHO (1957), and INUI (1957)].

For extensive bibliographies one should consult THORADE (1931, pp. 195 to 211); SRETENSKII (1936, pp. 294—303); KAMPÉ DE FÉRIET (1932, pp. 225—229); and STOKER (1957, pp. 545—560). SRETENSKII (1950, 1951) in a survey of the accomplishments of the USSR during the years 1917—1947 has given a rather complete bibliography of Russian papers during those years. TAKAO INUI (1954) has included a valuable bibliography of Japanese papers in a survey of Japanese contributions to the theory of ship waves. An interesting early history of the subject may be found in a paper by ST. VENANT and FLAMANT (1887). The treatise by the WEBER brothers (1825) is still of interest for its content, and especially for its many references to and summaries of the early papers on water waves. The section on waves in the article on hydrodynamics by LOVE (1914), as modified by APPELL, BEGHIN and VILLAT, in the *Encyclopédie des sciences mathématiques* gives brief indications of the contents of many of the papers published up to about 1912.

B. Mathematical formulation.

1. Coordinate systems and conventions. In the mathematical description of waves one may, as in fluid mechanics in general, describe the motion by describing either the paths of individual fluid particles ("Lagrangian" description) or the velocity (and acceleration) field in the region occupied by fluid at a given moment ("Eulerian" description). Generally, but not always, the Eulerian description will be used.

Rectangular coordinates may be used conveniently for almost all problems. The y-axis will be taken directed oppositely to the force of gravity, the x-axis and z-axis so as to form a right-handed system (i.e., if the y-axis is toward the top of the page and the x-axis is toward the right, the z-axis will point toward the reader). This is a somewhat unconventional choice for the z-axis, but has the obvious advantage that in two-dimensional problems one can delete z-dependent terms from the equations, have conventional (x, y) coordinates, and set $z = x + iy$ without ambiguity when complex-variable methods are convenient.

It seems hardly worth while to try to formulate rules concerning when a moving coordinate system is preferable to a fixed one. However, use of a moving coordinate system is clearly convenient in those cases where it allows one to formulate a problem in a time-independent manner.

The following well-established convention with regard to use of certain letters will be adhered to. The components of the velocity vector $\boldsymbol{v}$ will be denoted by u, v, w the pressure by p and the density by ϱ. The coefficient of viscosity of the fluid will be denoted by μ, the coefficient of kinematic viscosity, μ/ϱ, by ν. The acceleration resulting from gravity is denoted by g.

In the Eulerian formulation one seeks $\boldsymbol{v}, p$ and ϱ as functions of x, y, z, t i.e., at any instant t one seeks a vector function and two scalar functions defined on the region occupied by fluid at that instant. In the Lagrangian system one focuses attention on the trajectories of individual particles in the fluid: if a, b, c

are the coordinates of a particle at time $t=0$, then one seeks the position $x(a,b,c,t)$, $y(a,b,c,t)$, $z(a,b,c,t)$ of this point at a later time t. One may pass from one system to the other by means of the equations

$$\frac{dx}{dt}=u(x,y,z,t),\quad \frac{dy}{dt}=v(x,y,z,t),\quad \frac{dz}{dt}=w(x,y,z,t) \tag{1.1}$$

with $x=a$, $y=b$, $z=c$ at $t=0$ as initial conditions.

2. Equations of motion. Derivations of the fundamental equations describing fluid motion are available in many places (e.g., Vol. VIII, Part 1 of this Encyclopedia). The equations are reproduced here for convenience of reference.

The equation of continuity in Eulerian coordinates is

$$\frac{\partial \varrho}{\partial t}+\frac{\partial(\varrho u)}{\partial x}+\frac{\partial(\varrho v)}{\partial y}+\frac{\partial(\varrho w)}{\partial z}=0. \tag{2.1}$$

If the fluid is incompressible, but not necessarily homogeneous, $d\varrho/dt=0$ (but not necessarily $\partial\varrho/\partial t=0$) and Eq. (2.1) becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{2.2}$$

In Lagrangian coordinates this may be written

$$\varrho(x,y,z,t)\,D=\varrho(a,b,c,0) \tag{2.3}$$

where

$$D=\begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c}\\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c}\\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial z}\end{vmatrix}.$$

For an incompressible fluid $\varrho(x,y,z,t)=\varrho(a,b,c,0)$ and (2.3) becomes

$$D=1. \tag{2.4}$$

The dynamical equations take different forms according as one does or does not try to take account of viscosity. The Navier-Stokes equations for the motion of an incompressible viscous fluid, when the only external force is that of gravity, are as follows in Eulerian coordinates:

$$\left.\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&= -\frac{1}{\varrho}\frac{\partial p}{\partial x}+\frac{\mu}{\varrho}\Delta u,\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}&=-g-\frac{1}{\varrho}\frac{\partial p}{\partial y}+\frac{\mu}{\varrho}\Delta v,\\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}&= -\frac{1}{\varrho}\frac{\partial p}{\partial z}+\frac{\mu}{\varrho}\Delta w. \end{aligned}\right\} \tag{2.5}$$

If viscosity is neglected, the last two terms on the right side of the equations are to be deleted and one obtains the equations for an "ideal" fluid:

$$\left.\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&= -\frac{1}{\varrho}\frac{\partial p}{\partial x},\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}&=-g-\frac{1}{\varrho}\frac{\partial p}{\partial y},\\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}&= -\frac{1}{\varrho}\frac{\partial p}{\partial z}. \end{aligned}\right\} \tag{2.6}$$

In Lagrangian coordinates the latter equations become:

$$\left.\begin{aligned}\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a}+\left(g+\frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial a}+\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a}&=-\frac{1}{\varrho}\frac{\partial p}{\partial a},\\ \frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b}+\left(g+\frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial b}+\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b}&=-\frac{1}{\varrho}\frac{\partial p}{\partial b},\\ \frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c}+\left(g+\frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial c}+\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c}&=-\frac{1}{\varrho}\frac{\partial p}{\partial c}.\end{aligned}\right\}\tag{2.7}$$

The equations of two-dimensional motion result if one deletes all terms containing z, w, and c.

The motion is called irrotational if it satisfies the additional equations

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=0,\qquad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0,\qquad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0,\tag{2.8}$$

or, in two-dimensional motion,

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0.\tag{2.8'}$$

In the case of irrotational motion there exists a potential function $\Phi(x, y, z, t)$ such that

$$u=\frac{\partial\Phi}{\partial x},\qquad v=\frac{\partial\Phi}{\partial y},\qquad w=\frac{\partial\Phi}{\partial z}.\tag{2.9}$$

It is a classical theorem of hydrodynamics [cf. LAMB (1932, §§ 17, 33)] that, if the motion of an inviscid fluid with $\varrho=\varrho(p)$ is irrotational at any instant, it is so thereafter. In particular, a motion started from rest is irrotational.

If $\varrho=\varrho(p)$ is the equation of state, the following integral of the equations of motion exists for irrotational motion:

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}(u^2+v^2+w^2)+g y+P=A(t)\tag{2.10}$$

where

$$P=\int_{p_0}^{p}\varrho^{-1}\,dp$$

and $A(t)$ is an arbitrary function of t. If the fluid is incompressible, the usual case in this article, ϱ is independent of p and the integral becomes:

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$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}+(u^2+v^2+w^2)+g y+\frac{p-p_0}{\varrho}=A(t).\tag{2.10'}$$

In this case one obtains also from (2.2) and (2.9)

$$\Delta\Phi\equiv\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0.\tag{2.11}$$

Even if the motion is not irrotational, there still exists an integral like (2.10) if the motion is steady, the so-called Bernoulli integral:

$$\tfrac{1}{2}(u^2+v^2+w^2)+g y+P=C.\tag{2.10''}$$

Here C is constant along a single streamline:

$$\frac{dx}{dt}=u(x, y, z),\qquad \frac{dy}{dt}=v(x, y, z),\qquad \frac{dz}{dt}=w(x, y, z),$$

but may vary from one streamline to another.

There will be occasion in the following to treat problems in moving coordinate systems. Let $Oxyz$ be a fixed coordinate system and $\bar{O}\bar{x}\bar{y}\bar{z}$ be a system moving with respect to $Oxyz$ but without rotation. Let $\boldsymbol{v}_0$ be the vector $\frac{d}{dt}O\bar{O}$, the velocity of a particle referred to $Oxyz$ be $\boldsymbol{v}$ and to $\bar{O}\bar{x}\bar{y}\bar{z}$ be $\bar{\boldsymbol{v}}$. Then $\boldsymbol{v} = \bar{\boldsymbol{v}} + \boldsymbol{v}_0$. We shall generally want either to describe the absolute motion $\boldsymbol{v}$ with respect to the moving coordinate system $\bar{O}\bar{x}\bar{y}\bar{z}$ or the relative motion $\bar{\boldsymbol{v}}$ with respect to this coordinate system. In either case the continuity equation remains the same in form

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho u)}{\partial \bar{x}} + \frac{\partial(\varrho v)}{\partial \bar{y}} + \frac{\partial(\varrho w)}{\partial \bar{z}} = 0, \qquad \varrho = \varrho(\bar{x}, \bar{y}, \bar{z}, t) \tag{2.12}$$

or

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho \bar{u})}{\partial \bar{x}} + \frac{\partial(\varrho \bar{v})}{\partial \bar{y}} + \frac{\partial(\varrho \bar{w})}{\partial \bar{z}} = 0, \qquad \varrho = \varrho(\bar{x}, \bar{y}, \bar{z}, t). \tag{2.13}$$

The dynamical equations for an ideal fluid for the absolute motion described in the moving coordinate system are:

$$\left.\begin{aligned}
\frac{\partial u}{\partial t} + (u - u_0)\frac{\partial u}{\partial \bar{x}} + (v - v_0)\frac{\partial u}{\partial \bar{y}} + (w - w_0)\frac{\partial u}{\partial \bar{z}} &= \qquad -\frac{1}{\varrho}\frac{\partial p}{\partial \bar{x}},\\
\frac{\partial v}{\partial t} + (u - u_0)\frac{\partial v}{\partial \bar{x}} + (v - v_0)\frac{\partial v}{\partial \bar{y}} + (w - w_0)\frac{\partial v}{\partial \bar{z}} &= -g - \frac{1}{\varrho}\frac{\partial p}{\partial \bar{y}},\\
\frac{\partial w}{\partial t} + (u - u_0)\frac{\partial w}{\partial \bar{x}} + (v - v_0)\frac{\partial w}{\partial \bar{y}} + (w - w_0)\frac{\partial w}{\partial \bar{z}} &= \qquad -\frac{1}{\varrho}\frac{\partial p}{\partial \bar{z}}.
\end{aligned}\right\} \tag{2.14}$$

The dynamical equations for the relative motion are:

$$\left.\begin{aligned}
\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}} &= \qquad -\frac{1}{\varrho}\frac{\partial p}{\partial \bar{x}} - \dot{u}_0,\\
\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{v}}{\partial \bar{z}} &= -g - \frac{1}{\varrho}\frac{\partial p}{\partial \bar{y}} - \dot{v}_0,\\
\frac{\partial \bar{w}}{\partial t} + \bar{u}\frac{\partial \bar{w}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{w}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}} &= \qquad -\frac{1}{\varrho}\frac{\partial p}{\partial \bar{z}} - \dot{w}_0.
\end{aligned}\right\} \tag{2.15}$$

Let us suppose that the motion is irrotational and let $\Phi(x, y, z, t)$ be the velocity potential for the absolute motion in the fixed coordinate system. Let

$$\Phi(x, y, z, t) = \Phi\left(\bar{x} + \int^t u_0\,dt, \bar{y} + \int^t v_0\,dt, \bar{z} + \int^t w_0\,dt, t\right) = \bar{\Phi}(\bar{x}, \bar{y}, \bar{z}, t).$$

Then $\bar{\Phi}$ is the velocity potential for the absolute motion in the moving coordinate system:

$$\frac{\partial \bar{\Phi}}{\partial \bar{x}} = u, \qquad \frac{\partial \bar{\Phi}}{\partial \bar{y}} = v, \qquad \frac{\partial \bar{\Phi}}{\partial \bar{z}} = w.$$

The integral (2.10) becomes:

$$\frac{\partial \bar{\Phi}}{\partial t} + \frac{1}{2}\left[(u - u_0)^2 + (v - v_0)^2 + (w - w_0)^2\right] + g\bar{y} + P = \bar{A}(t), \tag{2.16}$$

where $\bar{A}(t) = A(t) + \frac{1}{2}(u_0^2 + v_0^2 + w_0^2) - g\int^t v_0\,dt$. If one defines $\bar{\bar{\Phi}}$ by

$$\bar{\bar{\Phi}}(\bar{x}, \bar{y}, \bar{z}, t) = \bar{\Phi}(\bar{x}, \bar{y}, \bar{z}, t) - u_0\bar{x} - v_0\bar{y} - w_0\bar{z},$$

then $\bar{\bar{\Phi}}$ is the velocity potential for the relative motion:

$$\frac{\partial \bar{\bar{\Phi}}}{\partial \bar{x}} = \bar{u}, \qquad \frac{\partial \bar{\bar{\Phi}}}{\partial \bar{y}} = \bar{v}, \qquad \frac{\partial \bar{\bar{\Phi}}}{\partial \bar{z}} = \bar{w},$$

and the integral (2.10) may be written:

$$\frac{\partial \bar{\bar{\Phi}}}{\partial t} + \dot{u}_0 \bar{x} + \dot{v}_0 \bar{y} + \dot{w}_0 \bar{z} + \frac{1}{2}(\bar{u}^2 + \bar{v}^2 + \bar{w}^2) + g\bar{y} + P = \bar{A}(t). \tag{2.17}$$

The more general equations when the system $\bar{O}\bar{x}\bar{y}\bar{z}$ is also rotating will not be necessary for this article.

3. Boundary conditions at an interface. Let us now suppose that we are given two immiscible fluids with a common boundary surface, $S(t)$. The one fluid, with density ϱ_1 and viscosity μ_1, will occupy region $R_1(t)$; the other, with density ϱ_2 and viscosity μ_2, the region $R_2(t)$. Let $F(x, y, z, t) = 0$ describe the surface $S(t)$; we assume $F_x^2 + F_y^2 + F_z^2 > 0$ (where $F_x = \partial F/\partial x$, etc.).

The first condition which the surface $S(t)$ must satisfy is a kinematic one. As the surface moves, the velocity of a point (x, y, z) on the surface in the direction of the normal to the surface is given by $-F_t/\sqrt{F_x^2 + F_y^2 + F_z^2}$. Here one takes the normal in the direction (F_x, F_y, F_z). A particle of fluid at the same point of the surface at that instant will have a velocity component in the direction of the surface normal given by $\frac{u F_x + v F_y + w F_z}{\sqrt{F_x^2 + F_y^2 + F_z^2}} = v_n$. For $S(t)$ to be a bounding surface means, of course, that there can be no transfer of matter across the surface. Consequently the following equation must be satisfied:

$$u F_x + v F_y + w F_z = -F_t, \tag{3.1}$$

where we have used the assumption $F_x^2 + F_y^2 + F_z^2 > 0$ in dropping the denominators. If one defines the "material derivative" by the equation

$$\frac{DF}{Dt} = u F_x + v F_y + w F_z + F_t,$$

then (3.1) is the same as

$$\frac{DF}{Dt} = 0. \tag{3.1'}$$

This condition must be satisfied by any bounding surface, whether an interface or a rigid boundary[1].

There are further dynamical conditions to be satisfied at an interface. Let us first consider the general case of viscous fluids with surface tension at the interface. The following assumptions are made:

1. The effect of surface tension as one passes through the interface is to produce a discontinuity in the normal stress proportional to the mean curvature of the boundary surface.

2. For viscous fluids the tangential stress must be continuous as one passes through the interface.

3. For viscous fluids the tangential component of the velocity must be continuous as one passes through the interface.

In order to formulate these statements in mathematical language, we introduce the following notation. Let $g(x, y, z)$ be some function defined in both R_1 and R_2 and let (x_0, y_0, z_0) be a point of the interface S. Assuming that the following limit exists, we shall write

$$g_i(x_0, y_0, z_0) = \lim g(x, y, z) \quad \text{as} \quad (x, y, z) \to (x_0, y_0, z_0), (x, y, z) \text{ in } R_i,$$

[1] For further discussion of this condition see C. TRUESDELL: Bull. Tech. Univ. Istanbul **3** (1950), No. 1, 71–78 (1951); L. LICHTENSTEIN: Grundlagen der Hydromechanik, pp. 159 to 170, 234ff. Berlin: Springer 1929.

and

$$[g(x_0, y_0, z_0)] = g_2(x_0, y_0, z_0) - g_1(x_0, y_0, z_0).$$

Let the components of the stress tensor be denoted by

$$\begin{matrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}. \end{matrix}$$

Consider an element of area of the surface S at a point (x, y, z) of S. Let the unit normal vector to S at (x, y, z) be (l, m, n). Then the surface element will have associated with it the stress vector with components:

$$\sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n, \quad \sigma_{yx} l + \sigma_{yy} m + \sigma_{yz} n, \quad \sigma_{zx} l + \sigma_{zy} m + \sigma_{zz} n.$$

Let R_1 and R_2 be the principal radii of curvature of S at (x, y, z). Then 1. and 2. are combined in the one equation

$$\left.\begin{aligned} [\sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n] &= T(R_1^{-1} + R_2^{-1})\, l, \\ [\sigma_{yx} l + \sigma_{yy} m + \sigma_{yz} n] &= T(R_1^{-1} + R_2^{-1})\, m, \\ [\sigma_{zx} l + \sigma_{zy} m + \sigma_{zz} n] &= T(R_1^{-1} + R_2^{-1})\, n, \end{aligned}\right\} \tag{3.2}$$

where T is a constant of proportionality depending upon the two fluids (and their temperatures, but this will not be considered here). T is called the coefficient of surface tension[1].

The kinematic condition imposed in (3.1) is clearly equivalent to continuity of the normal component of the velocity as one passes through S. Consequently, the condition 3. above may be combined with this to give

$$u_1 = u_2, \quad v_1 = v_2, \quad w_1 = w_2. \tag{3.3}$$

In the linearized theory of viscosity the stress tensor for an incompressible fluid is given by

$$\left.\begin{matrix} p - 2\mu u_x & -\mu(u_y + v_x) & -\mu(u_z + w_x) \\ -\mu(v_x + u_y) & p - 2\mu v_y & -\mu(v_z + w_y) \\ -\mu(w_x + u_z) & \mu(w_y + v_z) & p - 2\mu w_z. \end{matrix}\right\} \tag{3.4}$$

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The geometric quantity $R_1^{-1} + R_2^{-1}$ is given by the formula[2]

$$\left.\begin{aligned} \frac{1}{R_1} + \frac{1}{R_2} &= -\frac{\partial}{\partial x}\frac{F_x}{\sqrt{F_x^2 + F_y^2 + F_z^2}} - \frac{\partial}{\partial y}\frac{F_y}{\sqrt{F_x^2 + F_y^2 + F_z^2}} - \frac{\partial}{\partial z}\frac{F_z}{\sqrt{F_x^2 + F_y^2 + F_z^2}} \\ &= -\frac{F_{xx}(F_y^2 + F_z^2) + F_{yy}(F_x^2 + F_z^2) + F_{zz}(F_x^2 + F_y^2) - 2(F_{xy} F_x F_y + F_{yz} F_y F_z + F_{zx} F_z F_x)}{[F_x^2 + F_y^2 + F_z^2]^{\frac{3}{2}}}. \end{aligned}\right\} \tag{3.5}$$

The sign is so selected that, if it is positive, the direction of increase of the normal component of the stress vector at the interface is in the direction

$$(l, m, n) = \left(\frac{F_x}{\sqrt{F_x^2 + F_y^2 + F_z^2}}, \frac{F_y}{\sqrt{F_x^2 + F_y^2 + F_z^2}}, \frac{F_z}{\sqrt{F_x^2 + F_y^2 + F_z^2}}\right). \tag{3.6}$$

[1] For an air-water interface $T = 72.8$ dynes/cm at 20° C, for mercury-air interface $T = 485$ dynes/cm at 20° C, for a mercury-water interface $T = 412$ dynes/cm, for benzene-air $T = 28.9$ dynes/cm at 20° C, for liquid helium-helium vapor $T = 0.24$ dynes/cm at −270° C.

[2] See, e.g., A. Duschek and W. Mayer: Lehrbuch der Differentialgeometrie, Vol. I, pp. 150–152. Leipzig u. Berlin: Teubner 1930.

In the case of a surface given by $y=\eta(x, z)$ Eq. (3.5) becomes

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{\eta_{xx}(1+\eta_z^2)+\eta_{zz}(1+\eta_x^2)-2\eta_{xz}\eta_x\eta_z}{(1+\eta_x^2+\eta_z^2)^{\frac{3}{2}}}, \tag{3.5'}$$

where the direction of increase is upwards. In the case of two-dimensional motion this simplifies further to the well-known formula

$$\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}. \tag{3.5''}$$

If one now substitutes (3.4) to (3.6) in (3.2), one obtains the general boundary condition at the interface. The result is unwieldy in its general form[1].

If the interface is given by $y=\eta(x, z)$, the boundary condition becomes

$$\left.\begin{aligned}
[p]\,\eta_x-\{2[\mu u_x]\,\eta_x-[\mu(u_y+v_x)]+[\mu(u_z+w_x)]\,\eta_z\} &= T(R_1^{-1}+R_2^{-1})\,\eta_x,\\
[p]+\{[\mu(v_x+u_y)]\,\eta_x-2[\mu v_y)+[\mu(v_z+w_y)]\,\eta_z\} &= T(R_1^{-1}+R_2^{-1}),\\
[p]\,\eta_z-\{[\mu(w_x+u_z)]\,\eta_x-[\mu(w_y+v_z)]+2[\mu w_z]\,\eta_z\} &= T(R_1^{-1}+R_2^{-1})\,\eta_z
\end{aligned}\right\} \tag{3.7}$$

with $R_1^{-1}+R_2^{-1}$ given by (3.5'). Here fluid$_1$ is the lower and fluid$_2$ the upper fluid. For two-dimensional motion the equations take the following form:

$$\left.\begin{aligned}
[p]\,\eta'(x)-\{2[\mu u_x]\,\eta'(x)-[\mu(u_y+v_x)]\} &= T\frac{\eta''(x)}{(1+\eta'(x)^2)^{\frac{3}{2}}}\,\eta'(x),\\
[p]+\{[\mu(u_y+v_x)]\,\eta'(x)-2[\mu v_y]\} &= T\frac{\eta''(x)}{(1+\eta'(x)^2)^{\frac{3}{2}}}.
\end{aligned}\right\} \tag{3.8}$$

One may also write this condition in terms of the components of the stress vector normal and tangential to the interface:

see errata

$$\left.\begin{aligned}
[p]-2\frac{[\mu u_x]\,\eta'^2-[\mu(u_y+v_x)]\,\eta'+[\mu v_y]}{(1+\eta'^2)^{\frac{1}{2}}} &= T\frac{\eta''(x)}{(1+\eta'^2)^{\frac{3}{2}}},\\
\frac{2[\mu(u_x-v_y)]\,\eta'+[\mu(u_y+v_x)]\,(\eta'^2-1)}{(1+\eta'^2)^{\frac{1}{2}}} &= 0.
\end{aligned}\right\} \tag{3.8'}$$

If surface tension is to be neglected, one obtains the resulting boundary condition by setting $T=0$ in the various equations above. In this case, Eq. (3.2) simply states the continuity of the stress vector as one passes through the interface.

If viscosity is neglected, but not necessarily surface tension, the condition on the stress vector becomes simply

$$[p]=T(R_1^{-1}+R_2^{-1}), \tag{3.9}$$

where, of course, the mean curvature is still given by (3.5). The other boundary condition (3.3) changes more drastically upon neglecting viscosity: Condition 3. stating the continuity of the tangential component of velocity is abandoned. The continuity of the normal component, i.e. (3.1), is still retained, of course.

[1] In tensor notation the condition is somewhat more perspicuous:

$$\{[p]\,\delta_{ij}-[\mu(u_{i,j}+u_{j,i})]\}\frac{F_{,j}}{(F_{,k}F_{,k})^{\frac{1}{2}}}=T\frac{F_{,r}F_{,s}F_{,rs}-F_{,r}F_{,r}F_{,ss}}{(F_{,k}F_{,k})^{\frac{3}{2}}}\cdot\frac{F_{,i}}{(F_{,k}F_{,k})^{\frac{1}{2}}},$$

where $(x_1, x_2, x_3)=(x, y, z)$, $(u_1, u_2, u_3)=(u, v, w)$ and $F_{,i}=\partial F/\partial x_i$. We have refrained from using tensor notation because its particular advantages cannot in general be exploited here.

Condition 2. concerning the tangential stress is satisfied vacuously for an inviscid fluid.

So far we have considered the boundary condition at an interface between two fluids. If the second fluid is absent, the boundary surface for the first fluid is called a "free surface". Usually the pressure above a free surface is assumed to be some given function, say $p_2(x, y, z, t)$, of position and time; in most cases it is taken to be a constant, either an assumed atmospheric pressure or zero. The boundary conditions concerning the stress vector at a free surface are slight modifications of those for an interface, and can be obtained by setting $\mu_2=0$, $\lambda_2=0$. The result is again somewhat unwieldy in its complete form[1]. For an incompressible fluid it is:

$$\left.\begin{aligned}(\bar{p}-p)\,F_x+\mu\,\{2u_x F_x+(u_y+v_x)\,F_y+(u_z+w_x)\,F_z\} &= T(R_1^{-1}+R_2^{-1})\,F_x,\\ (\bar{p}-p)\,F_y+\mu\,\{(v_x+u_y)\,F_x+2v_y F_y+(v_z+w_y)\,F_z\} &= T(R_1^{-1}+R_2^{-1})\,F_y,\\ (\bar{p}-p)\,F_z+\mu\,\{(w_x+u_z)\,F_x+(w_y+v_z)\,F_y+2v_z F_z\} &= T(R_1^{-1}+R_2^{-1})\,F_z.\end{aligned}\right\} \quad (3.10)$$

Here we have written $\bar{p}$ for p_2, and μ for μ_1; $\bar{p}, p, u_x, \ldots$ are to be evaluated at $F(x, y, z, t)=0$.

The case with which we shall be chiefly concerned is that of an inviscid fluid without surface tension and with $\bar{p}(x, y, z, t)=p_0$, a constant. In this case the boundary condition reduces to the single equation

$$p(x, y, z, t)=p_0 \tag{3.11}$$

on $F(x, y, z, t)=0$. If the motion is irrotational and incompressible, one may determine p explicitly from (2.10′) so that (3.10) becomes

$$\Phi_t+\tfrac{1}{2}(u^2+v^2+w^2)+g y=A(t) \tag{3.11′}$$

to be satisfied on $F(x, y, z, t)=0$.

In the case of steady motion of an incompressible fluid, the Bernoulli integral (2.10″) still exists even if the motion is rotational. Consequently, in certain two-dimensional problems of steady motion in which the free surface is a streamline one continues to have a boundary condition like (3.10″):

$$\frac{1}{2}(u^2+v^2)+g y+\frac{p_0}{\varrho}=C, \tag{3.11″}$$

to be satisfied on $F(x, y)=0$.

4. Boundary conditions on rigid surfaces. Let the equation of the rigid surface be given by the equation $G(x, y, z, t)=0$. Then in the case of an inviscid fluid the condition to be satisfied on $G=0$ is the same as the kinematic condition (3.1):

$$u\,G_x+v\,G_y+w\,G_z=-G_t, \tag{4.1}$$

i.e., the component of velocity of the fluid normal to the surface must equal the velocity of the rigid surface in the direction of its normal.

If the fluid is viscous, it must stick to a solid boundary and move with it without slippage. An equation of the form $G(x, y, z, t)=0$ is not suitable for

[1] In tensor notation it may be written:

see errata

$$\{(\bar{p}-p)\,\delta_{ij}+\mu\,(u_{i,j}+u_{j,i})\}\,\frac{F_{,j}}{(F_{,k}F_{,k})^{\frac{1}{2}}}=T\,\frac{F_{,r}F_{,s}F_{,rs}-F_{,r}F_{,r}F_{,ss}}{(F_{,k}F_{,k})^{\frac{3}{2}}}\,\frac{F_{,j}}{(F_{,k}F_{,k})^{\frac{1}{2}}}.$$

Here we have written $\bar{p}$ for p_2 and λ, μ for λ_1, μ_1. All variable quantities in the braces are, of course, to be evaluated at the free surface $F=0$.

formulating this statement in equations (e.g., $x^2+y^2+z^2=a^2$ does not distinguish between a rotating and a stationary sphere). Let the surface be given in parametric coordinates by: $x=X(r,s,t)$, $y=Y(r,s,t)$, $z=Z(r,s,t)$, where a given point on the surface corresponds to a given pair of values (r,s). Then the condition for viscous fluids may be written:

$$u=\frac{\partial X}{\partial t}, \qquad v=\frac{\partial Y}{\partial t}, \qquad w=\frac{\partial Z}{\partial t}. \tag{4.2}$$

If a solid boundary penetrates the free surface (or an interface) of a viscous fluid, there will be some difference in treatment of the boundary condition according as the fluid wets the surface or not. In the case of mercury sloshing in a clean glass basin, the fluid pulls free of the surface as it moves up and down, whereas water in the same basin will continue to adhere to any part of the walls already wetted. Furthermore, if surface tension is taken into account, the angle of contact of the free surface with the solid surface will enter into the boundary condition; in the first case mentioned above the angle may vary according as the liquid is rising or falling along the wall[1]. Although attempts to prove very general existence theorems for fluid motion would presumably take such complications into account, they are usually neglected in most solutions of special problems, there being indeed little choice in the matter.

5. Other types of boundary surfaces. Geophysical problems sometimes suggest situations in which there is an interface between a fluid and an elastic medium. This may occur, for example, in the study of the effect of ocean waves on the ocean floor, as in LONGUET-HIGGINS' (1950) theory of microseisms. Other possibilities are suggested by wave motion on a body of water covered with an ice sheet or at an interface between two fluids separated by an elastic membrane or plate. In one series of investigations the ice sheet has been assumed broken into pieces small with respect to the prevalent wave lengths. In this case only the density of the ice layer enters into the modified boundary condition [see PETERS (1950), KELLER and GOLDSTEIN (1953), KELLER and WEITZ (1953), SHAPIRO and SIMPSON (1953)]. Waves in a thin plate over an infinitely deep fluid have been considered briefly by LANDAU and LIFSHITS (1953, pp. 762—763), but with neglect of gravity. GREENHILL (1887, p. 68; 1916) included gravity.

The kinematic boundary condition (3.1) must always hold. The dynamical conditions will depend upon the nature of the assumptions. The matter will not be further considered here.

C. Preliminary remarks and developments.

6. Classification of problems. Most of the theory of water waves is concerned either with elucidating some general aspects of wave motion or with predicting the behavior of waves in the presence of some special configuration of interest to oceanographers, hydraulic engineers, or ship designers. Unfortunately, even some of the apparently simplest problems have proved too difficult to solve in their most complete formulation. Approximations have been necessary, and in many cases the problems which have been solved are those which could be solved by the approximate methods in use. An examination of the theory also shows that many of the concepts and definitions are almost inextricably bound up with these methods of approximation, following rather than preceding the making of the approximation.

[1] See, e.g., R. S. BURDON: Surface tension and the spreading of liquids, pp. 76—82. Cambridge 1949.

The nature of the approximations used in treating a particular problem provides a natural way of classifying it. First there are the assumptions concerning the properties of the fluid: viscous or inviscid, compressible or incompressible, surface tension or not. Although assuming the fluid to be inviscid, incompressible, and without surface tension simplifies the equations, they are still not easily manageable, even for the simplest kinds of problems. Other approximations of a different nature are required. These are in a sense mathematical approximations. Their physical significance is not in restricting the nature of the fluid but in restricting the character of the waves and the boundary configuration. The kind of mathematical approximation used provides another means of classifying problems, and is the principal one which will be used in this article. There are two principal methods of approximation, explained below in Sect. 10, the infinitesimal-wave approximation and the shallow-water approximation. Thus, the development of these two approximate theories and of the exact theory constitutes the bulk of this article.

7. Progressive waves and wave velocity. Standing waves. It will be convenient to call any motion of a fluid in a gravitational field with a free surface or an interface a *wave motion.*

If the velocity components, pressure, and free surface or interface may be expressed in the form

$$\boldsymbol{v} = \boldsymbol{v}(x - c\,t, y, z), \qquad p = p(x - c\,t, y, z), \qquad y = \eta(x - c\,t, z),$$

respectively, then the wave motion will be said to be a *progressive wave* travelling in the direction Ox. In this case a change to a moving coordinate system with $x' = x - ct$, $y' = y$, $z' = z$ reduces the motion to steady motion with respect to the moving coordinate system. With respect to the fixed coordinate system the profile of the free surface or interface is being transported without change of form in the direction Ox with velocity c. It might seem reasonable therefore to call c the velocity of propagation of the progressive wave.

However, STOKES (1849; or 1880, pp. 202ff.) has pointed out that the velocity of propagation of the profile of the free surface does not by itself give a useful definition of wave velocity. Let the fluid be inviscid, either infinitely deep or with a horizontal bottom, and unlimited otherwise. Now let the whole fluid in the progressive wave described above be transported with velocity C (positive or negative) in the direction Ox. Then the motion will still be consistent with the laws of fluid mechanics, the various parts of the fluid will move the same relatively to each other, but the velocity of propagation of the profile will be arbitrary, depending upon the choice of C. What is required for a useful definition of wave velocity is the velocity of propagation of the profile with respect to a coordinate system fixed in some sense in the fluid.

In the case of an infinitely deep fluid, if the axes may be chosen so that as $y \to -\infty$ the velocity relative to these axes vanishes, then one may reasonably measure the profile velocity with respect to these. If the motion far ahead or far behind the disturbance approaches a uniform velocity (possibly zero), then axes moving with the fluid with this velocity may be used. When the disturbance does not behave thus (as in the case of periodic waves) and when the depth is finite, there is no longer an obvious way to select a set of reference axes.

In order to put the problem somewhat differently, let us assume that the wave motion is given as a steady motion with velocity field $\boldsymbol{v}(x, y)$ and free surface $y = \eta(x)$. We wish to find a moving coordinate system $x' = x - u_0 t$, $y' = y$, so that in some sense the relative motion vanishes on the average. We now have the free surface given by $y'=\eta(x'+u_0 t)$ and the relative velocity by $\boldsymbol{v}'(x'+u_0 t, y') = \boldsymbol{v}(x'+u_0 t, y') - u_0 \boldsymbol{i}$. How is u_0 to be chosen? STOKES made two suggestions. One is to define it by the equation

$$\left.\begin{aligned} &\lim_{a\to-\infty,\, b\to\infty} \frac{1}{b-a}\int_a^b dx' \int_{-h}^{\eta(x'+u_0 t)} u'(x'+u_0 t, y')\, dy' \\ &= \lim_{a\to-\infty,\, b\to\infty} \frac{1}{b-a}\int_a^b dx \int_{-h}^{\eta(x)} [u(x,y)-u_0]\, dy = 0, \end{aligned}\right\} \tag{7.1}$$

where $y=-h$ is the equation for the bottom. In case the motion is periodic, with period λ, the defining equation may be written

$$\int_0^\lambda dx \int_{-h}^{\eta(x)} [u(x,y)-u_0)]\, dy = 0. \tag{7.2}$$

If one notes that the mean depth is given by

$$\lim \frac{1}{b-a}\int_a^b [\eta(x)+h]\, dx \quad \text{or} \quad \frac{1}{\lambda}\int_0^\lambda [\eta(x)+h]\, dx,$$

then one sees that, with h' as mean depth,

$$u_0 h' = Q \tag{7.3}$$

where Q is the average discharge rate per unit width. u_0 is thus defined so that the average discharge rate with respect to the (x', y') coordinate system is zero. u_0 is usually denoted by c'.

STOKES' other suggestion was to define u_0 by

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T u'(x'+u_0 t, y')\, dt = 0 \tag{7.4}$$

or

$$\begin{aligned} u_0 &= \lim_{T\to\infty} \frac{1}{T}\int_0^T u(x'+u_0 t, y')\, dt \\ &= \lim_{T\to\infty} \frac{1}{u_0 T}\int_{x'}^{x'+u_0 T} u(x,y)\, dx = \lim_{a\to\infty} \frac{1}{a}\int_{x'}^{x'+a} u(x,y)\, dx. \end{aligned}$$

If u is periodic in x with period λ, one may write

see errata

$$u_0 = \frac{1}{\lambda}\int_{x'}^{x'+\lambda} u(x,y)\, dx. \tag{7.5}$$

In either case, for the definition to be useful u_0 must be independent of x' and y. If u is bounded, it follows easily that $\partial u_0/\partial x' = 0$ for both cases. If the motion is irrotational, $u_y = v_x$ and it follows again that $\partial u_0/\partial y = 0$ if v is bounded. Wave velocity defined in this manner is usually denoted by c. For the two special cases considered earlier, the two definitions coincide.

The definition of wave velocity in cases where the motion cannot be reduced to a steady motion is no longer straightforward. In many cases of interest, the asymptotic behavior of the motion for large positive or negative x allows one to define a wave velocity in a manner similar to that above. In more complicated wave motions one may simply follow the motion of some special phase of the profile, say a crest. This provides, for example, a definition of phase velocity for a cylindrical wave.

A general definition of *standing wave* is somewhat more awkward to formulate than that for a progressive wave. For the case of a plane wave, the free surface $y=\eta(x,t)$ must be periodic in each of x and t, with wave length λ and period τ, say. In addition, the curves in the (x,t)-plane represented by $\eta(x,t)=0$, where $y=0$ is the undisturbed surface, must consist of two sets of curves oscillating about the lines $x=\frac{1}{2}n\lambda$ and $t=\frac{1}{2}n\tau$, $n=0, \pm 1, \ldots$. For progressive waves the curves $\eta(x,t)=0$ consist of a single set of straight lines, all parallel to $x-ct=0$. The prototype for the standing wave is the surface defined by, say, $y=\sin 2\pi x/\lambda \times \cos 2\pi t/\tau$. However, as shown by both PENNEY and PRICE (1952b) and by SEKERZH-ZENKOVICH (1947), neither set of curves $\eta(x,t)=0$ consists of straight lines, or even fixed curves, for standing waves of finite amplitude.

There remains the problem of establishing that progressive and standing waves exist under suitable boundary conditions. For the exact boundary conditions for a perfect fluid, the existence of progressive waves was first established by LEVI-CIVITA (1925) and NEKRASOV (1921, 1922). The existence of standing waves satisfying the exact boundary conditions is apparently an open question.

8. Energy. Let $T(t)$ be a region occupied by a perfect fluid with a boundary $S(t)$ represented by

$$F(x,y,z,t)=0,$$

the representation being chosen so that (F_x, F_y, F_z) is in the direction of the exterior normal. The surface $S(t)$ moves independently of the motion of the fluid. It is assumed that $T(t)$ contains no singularities of $\boldsymbol{v}$ and that surface tension does not act upon the surface $S(t)$ at any time. The energy of the fluid contained in $T(t)$ is given by

$$E=\iiint\limits_{T(t)}\left[\tfrac{1}{2}\varrho(u^2+v^2+w^2)+\varrho g y\right]d\tau. \tag{8.1}$$

For irrotational motion of an inviscid incompressible fluid, one may use (2.10′) and express E by

$$E=\iiint\limits_{T(t)}\left[-p-\varrho\frac{\partial\Phi}{\partial t}\right]d\tau.$$

[Here Φ has been redefined so that $A(t)$ may be set equal to zero.] One may now compute dE/dt by using the general formula:

$$\frac{d}{dt}\iiint\limits_{T(t)} f(x,y,z,t)\,d\tau=\iiint\limits_{T(t)} f_t(x,y,z,t)\,d\tau+\iint\limits_{S(t)} f(x,y,z,t)\frac{-F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\,d\sigma.$$

One finds [cf. F. JOHN (1949, p. 19ff.), which we follow closely here]:

$$\begin{aligned}\frac{dE}{dt}&=\iiint\limits_{T(t)}\varrho\,\mathrm{grad}\,\Phi\cdot\mathrm{grad}\,\Phi_t\,d\tau+\iint\limits_{S(t)}[\varrho\Phi_t+p]\frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\,d\sigma\\&=\iint\limits_{S(t)}\varrho\Phi_t\frac{\partial\Phi}{\partial n}\,d\sigma+\iint\limits_{S(t)}[\varrho\Phi_t+p]\frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\,d\sigma\end{aligned}$$

by GREEN's Theorem and the equation of continuity. Finally,

$$\frac{dE}{dt}=\iint\limits_{S(t)}\left\{\varrho\Phi_t\left[\frac{\partial\Phi}{\partial n}+\frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right]+p\frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right\}d\sigma. \tag{8.2}$$

We recall that $-F_t/\sqrt{F_x^2+F_y^2+F_z^2}$ is the velocity of $S(t)$ in the direction of the exterior normal. Two cases are of special interest. If $S(t)$ is a "physical" boundary, i.e., one moving with the fluid, then the first summand vanishes and one finds

$$\frac{dE}{dt} = -\iint_{S(t)} p\,\frac{\partial\Phi}{\partial n}\,d\sigma \tag{8.3}$$

[cf. LAMB, Hydrodynamics, p. 9, Eq. (5)]. If $S(t)$ is a fixed "geometrical" boundary, then $F_t=0$ and one gets

$$\frac{dE}{dt} = \iint_{S(t)} \varrho\,\Phi_t\,\frac{\partial\Phi}{\partial n}\,d\sigma. \tag{8.4}$$

If one considers any portion of $S(t)$, then the integral of (8.2) taken over this portion and with a minus sign gives the rate of flow of energy through this portion of $S(t)$. In case a part of $S(t)$ is a physical boundary which is fixed, $\partial\Phi/\partial n=0$ and the flow through this part is zero. The same conclusion holds for any portion of $S(t)$ that is a free surface, for then $p=0$.

If one has a progressive wave moving to the right with $\Phi(x, y, z, t) = \varphi(x-ct, y, z)$ and takes S as a region in the fixed plane $x=x_0$, then the rate of flow of energy through S in the positive direction is given by

$$\iint_S \varrho\,c\,\varphi_x^2(x_0-c\,t, y, z)\,dy\,dz \geqq 0, \tag{8.5}$$

i.e., energy always flows in the direction of the wave.

In cases where one is dealing with waves generated by moving bodies, it is frequently possible to choose the region T so that no energy is lost from it, the latter being true only as an average if the motion is periodic in time. As an example, consider a body moving steadily with velocity c in the x-direction in an infinite ocean with horizontal bottom. In addition to the boundary conditions on the body, free surface, and bottom, we assume that the motion vanishes (in the limit) far ahead and to the sides of the body. The surface $S(t)$ may then be chosen as a plane M: $x-ct-a=0$ far ahead, another plane N: $-(x-ct)+b=0$ behind the body, planes R and L: $z=\pm a$ on either side, and the bottom H, the wetted surface of the body B, and the part of the free surface F included between the body and the planes. The energy within this region is clearly constant, and one easily obtains, with $\Phi(x, y, z, t)=\varphi(x-ct, y, z)$:

$$0 = -\iint_B p\,\frac{\partial\varphi}{\partial n}\,d\sigma - \iint_{M+N+R+L} \varrho\,c\,\frac{\partial\varphi}{\partial x}\,\frac{\partial\varphi}{\partial n}\,d\sigma + c\iint_M \left[\frac{1}{2}\,\varrho\,(\varphi_x^2+\varphi_y^2+\varphi_z^2) + \varrho\,g\,y\right]d\sigma - \\ - c\iint_N \left[\frac{1}{2}\,\varrho\,(\varphi_x^2+\varphi_y^2+\varphi_z^2) + \varrho\,g\,y\right]d\sigma.$$

Since on B one has $\partial\varphi/\partial n = c\cos(n, x)$, one finds for the first integral, remembering that $\boldsymbol{n}$ points into the body,

$$-\iint_B p\,\frac{\partial\varphi}{\partial n}\,d\sigma = -c\iint_B p\cos(n, x)\,d\sigma = R\,c,$$

where R is the force on the body. The parts of the second integral over M, R, L vanish as $a\to\infty$ and similarly for the first summand in the third integral. The terms in $\varrho g y$ give

$$\int_{-a}^{a} dz \int_{-h}^{\eta(a,z)} \varrho g y \, dy - \int_{-a}^{a} dz \int_{-h}^{\eta(b,z)} \varrho g y \, dy = \int_{-a}^{a} \tfrac{1}{2} \varrho g \left[\eta^2(a,z) - \eta^2(b,z)\right] dz$$

which, as $a \to \infty$, converges to

$$-\tfrac{1}{2} \varrho g \int_{-\infty}^{\infty} \eta^2(b,z)\, dz .$$

One obtains finally

$$\left.\begin{aligned} R = \tfrac{1}{2} \varrho \int_{-\infty}^{\infty} dz \int_{-h}^{\eta(b,z)} \left[-\varphi_x^2(b,y,z) + \varphi_y^2(b,y,z) + \varphi_z^2(b,y,z)\right] dy \, + \\ + \tfrac{1}{2} g \varrho \int_{-\infty}^{\infty} \eta^2(b,z)\, dz . \end{aligned}\right\} \tag{8.6}$$

This exact formula for resistance will be put into a different form later after linearization of the boundary conditions. Although the plane $x - ct = b$ may be taken at any distance behind the body without destroying the validity of (8.6), it is usually convenient to take it so far behind that asymptotic expressions for φ can be used.

If in (8.1) a part of the surface $S(t)$, say $S_1(t)$, is an interface with another fluid with surface tension acting, then the energy is given by

$$E = \iiint_{T(t)} \left[\tfrac{1}{2} \varrho (u^2 + v^2 + w^2) + \varrho g y\right] d\tau + T \iint_{S_1(t)} d\sigma . \tag{8.7}$$

Let $S_1(t)$ be bounded by the curve $C(t)$ given parametrically by $x(s,t)$, $y(s,t)$, $z(s,t)$ and let $S(t) = S_1(t) + S_2(t)$. Then the formula analogous to (8.2) is

see errata

$$\left.\begin{aligned} \frac{dE}{dt} = \iint_{S(t)} \varrho \, \Phi_t \left[\Phi_n + \frac{F_t}{\sqrt{F_x^2 + F_y^2 + F_z^2}}\right] d\sigma + \iint_{S_2(t)} p \frac{F_t}{\sqrt{F_x^2 + F_y^2 + F_z^2}} \, d\sigma \, + \\ + \iint_{S_1(t)} \left[p + T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right] \frac{F_t}{\sqrt{F_x^2 + F_y^2 + F_z^2}} \, d\sigma + T \int_{C(t)} \begin{vmatrix} F_x & F_y & F_z \\ x_t & y_t & z_t \\ x_s & y_s & z_s \end{vmatrix} \frac{d_s}{\sqrt{F_x^2 + F_y^2 + F_z^2}} . \end{aligned}\right\} \tag{8.8}$$

If $S_1(t)$ is a free surface, then the boundary condition

$$p_0 - p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where p_0 is an assumed constant pressure implies that there is no flux of energy through S_1.

If the motion is two-dimensional, with S_1 given by $y = \eta(x,t)$, $x_1(t) \leqq x \leqq x_2(t)$, then (8.7) becomes

$$E(t) = \iint_{T(t)} \left[\tfrac{1}{2} \varrho (u^2 + v^2) + \varrho g y\right] d\sigma + T \int_{X_1(t)}^{x_2(t)} ds \tag{8.9}$$

and (8.8) becomes

$$\left.\begin{aligned} \frac{dE}{dt} = \int_{S_2} \varrho \, \Phi_t \left[\Phi_n + \frac{F_t}{\sqrt{F_x^2 + F_y^2}}\right] ds + \int_{S_2} p \frac{F_t}{\sqrt{F_x^2 + F_y^2}} \, ds \, - \\ - \int_{x_1(t)}^{x_2(t)} \left[p + \frac{T \eta_{xx}}{[1 + \eta_x^2]^{\frac{3}{2}}}\right] \eta_t \, dx + T \frac{\eta_x \eta_t}{\sqrt{1 + \eta_x^2}} \Bigg|_{x_1}^{x_2} + T \sqrt{1 + \eta_x^2}\, x'(t) \Bigg|_{x_1}^{x_2} . \end{aligned}\right\} \tag{8.10}$$

If S_1 is a free surface, the integral over S_1 may be dropped by suitably redefining p.

9. Momentum. Expressions for rate of change of momentum may be derived which are analogous to those for rate of change of energy. With

$$\boldsymbol{M} = \iiint_{T(t)} \varrho \boldsymbol{v} \, d\tau, \tag{9.1}$$

and otherwise the same notation as in Sect. 8, one finds

$$\left.\begin{aligned} \frac{d\boldsymbol{M}}{dt} &= \iint_S \varrho \left\{ \Phi_t \boldsymbol{n} + \operatorname{grad} \Phi \frac{-F_t}{\sqrt{F_x^2+F_y^2+F_z^2}} \right\} d\sigma, \\ &= -\iint_S \left\{ (p + \varrho g y) \boldsymbol{n} + \varrho \left[\boldsymbol{v}\cdot\boldsymbol{n} + \frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}} \right] \boldsymbol{v} \right\} d\sigma, \\ &= \iint_S \varrho \left\{ \left(\Phi_t + \frac{1}{2} v^2 \right) \boldsymbol{n} - \left[\boldsymbol{v}\cdot\boldsymbol{n} + \frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}} \right] \boldsymbol{v} \right\} d\sigma. \end{aligned}\right\} \tag{9.2}$$

Here the first line of (9.2) is derived by a direct computation of $d\boldsymbol{M}/dt$ with $\boldsymbol{v} = \operatorname{grad} \Phi$; the second is derived analogously to (8.2); the third follows directly by use of (2.10′). Comparison of lines one and three gives the known relation (LEVI-CIVITA):

$$\iint_S \tfrac{1}{2} v^2 \boldsymbol{n} \, d\sigma = \iint_S (\boldsymbol{v}\cdot\boldsymbol{n}) \boldsymbol{v} \, d\sigma. \tag{9.3}$$

Note that in (9.2) and (9.3) $S(t)$ may move in an arbitrary manner as long as the region $T(t)$ contains no singularities and only fluid. If the boundary is physical, the terms in square brackets vanish in (9.2); if the boundary is fixed, then $F_t = 0$.

Let $S_0(t)$ be a physical boundary, possibly the surface of a solid body, and $S(t)$ a closed surface containing it. Applying (9.2) to the region of fluid bounded jointly by S_0 and S, one finds

$$\left.\begin{aligned} F_0 &= \iint_{S_0} (p + \varrho g y) \boldsymbol{n} \, d\sigma \\ &= -\iint_{S_0} \varrho (\Phi_t \boldsymbol{n} + \boldsymbol{v}\cdot\boldsymbol{n}\,\boldsymbol{v}) \, d\sigma + \iint_S \varrho \left(\tfrac{1}{2} v^2 \boldsymbol{n} - \boldsymbol{v}\cdot\boldsymbol{n}\,\boldsymbol{v} \right) d\sigma. \end{aligned}\right\} \tag{9.4}$$

see errata Here F_0 is the hydrodynamic orce fon S_0 and does not include the hydrostatic force.

If singularities are allowed in the region occupied by fluid, they may be enclosed in spheres of small radius and the formula (9.4) applied to the remaining fluid, with S modified to include the spherical surfaces. If the singularities are isolated sources of strengths m_i at the points $\boldsymbol{a}_i$, then by shrinking the spheres about the singularities in a customary fashion [cf. MILNE-THOMSON (1956, pp. 448 to 450)], one obtains the following modification of (9.4):

$$\boldsymbol{F}_0 = -\iint_{S_0} \varrho (\Phi_t \boldsymbol{n} + \boldsymbol{v}\cdot\boldsymbol{n}\,\boldsymbol{v}) \, d\sigma + \sum 4\pi \varrho m_i \boldsymbol{v}_i + \iint_S \varrho \left(\tfrac{1}{2} v^2 \boldsymbol{n} - \boldsymbol{v}\cdot\boldsymbol{n}\,\boldsymbol{v} \right) d\sigma, \tag{9.5}$$

where $\boldsymbol{v}_i$ is the velocity at the point $\boldsymbol{a}_i$ when the source at that point is removed. Other modifications may be derived for other types of singularities.

If the velocity field is such that $r^{1+\varepsilon} v \to 0$ as $r = \sqrt{x^2+y^2+z^2} \to \infty$ for some $\varepsilon > 0$, then the last integral in (9.4) or (9.5) will vanish as S is expanded to infinity, provided the latter can be done without destroying the validity of the formula. In the case of a body moving in a fluid with a free surface, one cannot expand in all directions and must include the contribution of the last integral over the free surface. However, the formulas are still useful in computing the force on an obstacle resulting from waves.

10. Expansion of solutions in powers of a parameter. In their exact form even the simplest problems with surface waves are difficult to solve. If one neglects viscosity and assumes irrotational motion, the problem is reduced to finding solutions of LAPLACE'S equation, which is at least linear in the unknown. However, the problem is still difficult because of the nonlinear boundary condition at the free surface or interface. This lack of linearity deprives one, for example, of the mathematical tool of superposition of solutions; expansion in eigenfunctions or use of GREEN'S functions is not possible.

In order to be able to treat special problems, the equations are approximated by ones which are more tractable. The two principal methods of approximation may each be treated as a perturbation procedure. As was mentioned in Sect. 7, this procedure is not concerned with the assumptions about the nature of the fluid, for example, whether or not viscosity is neglected, but rather with the nature of the motion and its generation. An advantage in using the perturbation procedure is that the assumptions about the motion are displayed in such a way that it is clear how to obtain approximations of higher order. The method has been applied to water-wave problems by SEKERZH-ZENKOVICH (1947, 1951, 1952), K. FRIEDRICHS (1948), KELLER (1948), F. JOHN (1949), LONGUET-HIGGINS (1953b), PETERS and STOKER (1957), and others. As used here the method is purely formal, the nature of the convergence of the perturbation series, whether it be uniform, pointwise, asymptotic or what not, being left open. However, for each method of approximation it is possible to point to several cases in which convergence has been proved: for the infinitesimal-wave approximation, LEVI-CIVITA'S (1925), STRUIK'S (1926) and NEKRASOV'S (1921, 1928) proofs of the existence of a periodic wave of permanent type; and for the shallow-water approximation, FRIEDRICHS and HYERS' (1954) proof of the existence of a solitary wave and LITTMAN'S (1957) proof of the existence of cnoidal waves.

To a certain extent the two methods of approximation have different aims. The infinitesimal-wave approximation fits into a general scheme for approximating nonlinear equations and boundary conditions by linear ones [see SOURIAU (1952) for a discussion]. To apply it, one must know a particular exact solution to start with. In addition, one must be able to select a dimensionless parameter (or parameters), say ε, which helps to determine the exact physical problem and is such that the solutions to the exact problems associated with each value of ε approach (in some sense) the known exact solution when $\varepsilon \to 0$. It is then assumed that the various functions entering into the problem may be expanded into power series in ε. The series are substituted into the equations and boundary conditions and grouped according to powers of ε. The coefficients of each power then yield a sequence of equations and boundary conditions, the coefficients of ε giving the first-order theory, those of ε^2 the second-order theory, etc. As an exact initial solution it is usually most convenient to take either a state of rest or of uniform motion. Various choices of ε will be made in the applications later.

The shallow-water approximation differs in that a change of variable involving the expansion parameter is made initially. This introduces ε into the exact equations. When the power series expansions are introduced into the equations, the resulting equations of the sequence are linear in quantities of the same order, but the equations are too degenerate to determine all these quantities without recourse to the equations of next higher order. This leads to nonlinear equations for the desired functions, but ones of a type which have been intensively investigated. In this case the procedure is perhaps artificial in that the perturbation scheme is devised to lead to a special set of equations for a first-order theory, derived originally by quite different reasoning. However, in doing this it makes clear the nature of the approximation and gives a systematic procedure for finding higher-order approximations. It is instructive, in this connection, to read the usual derivation as given, for example, in LAMB (1932, pp. 254—256) or STOKER (1957, pp. 22—25) (who also gives the one given here).

α) The infinitesimal-wave approximation. We shall derive the equations of motion and the free-surface or interface boundary conditions for this linearized theory without identifying explicitly the parameter ε used in the expansions. Later on, when specific choices are made, the boundary conditions on certain geometric boundaries associated with the choice of ε will be modified to conform with the linearization.

Consider two incompressible viscous fluids in contact along an interface represented by $y=\eta(x, z, t)$. Quantities referring to the upper fluid have subscript 2, those to the lower fluid subscript 1; the coefficient of surface tension is T. Assume the following expansions in the parameter ε:

$$\left.\begin{aligned} \boldsymbol{v}_i(x, y, z, t, \varepsilon) &= \varepsilon\,\boldsymbol{v}_i^{(1)} + \varepsilon^2\,\boldsymbol{v}_i^{(2)} + \cdots, \\ p_i(x, y, \zeta, t, \varepsilon) &= p_i^{(0)} + \varepsilon\, p_i^{(1)} + \varepsilon^2\, p_i^{(2)} + \cdots, \\ \eta(x, z, t, \varepsilon) &= \varepsilon\,\eta^{(1)} + \varepsilon^2\,\eta^{(2)} + \cdots. \end{aligned}\right\} \tag{10.1}$$

Substitute these expansions in Eqs. (2.2), (2.5), (3.1), (3.3), and (3.7), remembering in addition that formal expansions of the following sort, for example, hold:

$$\begin{aligned} u_1(x, \eta(x, z, t), z, t) &= u_1(x, 0, z, t) + \eta\, u_{1y}(x, 0, z, t) + \cdots \\ &= \varepsilon\, u_1^{(1)}(x, 0, z, t) + \varepsilon^2\left[u_1^{(2)}(x, 0, z, t) + \eta^{(1)} u_{1y}^{(1)}(x, 0, z, t)\right] + \cdots. \end{aligned}$$

Collecting first the terms independent of ε, one finds from (2.5) and (3.3)

$$\operatorname{grad}\left(p_i^{(0)} + \varrho_i\, g\, y\right) = 0, \qquad p_2^{(0)}(x, 0, z, t) = p_1^{(0)}(x, 0, z, t). \tag{10.2}$$

Collecting the coefficients of the first power of ε, one finds

$$\left.\begin{aligned} &\frac{\partial u_i^{(1)}}{\partial x} + \frac{\partial v_i^{(1)}}{\partial y} + \frac{\partial w_i^{(1)}}{\partial z} = 0, \qquad i = 1, 2, \\ &\frac{\partial \boldsymbol{v}_i^{(1)}}{\partial t} = -\frac{1}{\varrho_i}\operatorname{grad} p_i^{(1)} + \nu_i\,\Delta\,\boldsymbol{v}_i^{(1)}, \qquad i = 1, 2, \\ &u_1^{(1)}(x, 0, z, t) = u_2^{(1)}(x, 0, z, t), \\ &v_1^{(1)}(x, 0, z, t) = v_2^{(1)}(x, 0, z, t) = \eta_t^{(1)}(x, z, t), \\ &w_1^{(1)}(x, 0, z, t) = w_2^{(1)}(x, 0, z, t), \\ &\mu_1\left(u_{1y}^{(1)}(x, 0, z, t) + v_{1x}^{(1)}\right) = \mu_2\left(u_{2y}^{(1)} + v_{2x}^{(1)}\right), \\ &p_2^{(1)}(x, 0, z, t) - p_1^{(1)} - (\varrho_2 - \varrho_1)\, g\,\eta^{(1)} - 2\left(\mu_2 v_{2y}^{(1)} - \mu_1 v_{1y}^{(1)}\right) = T\left(\eta_{xx}^{(1)} + \eta_{zz}^{(1)}\right), \\ &\mu_1\left(w_{1y}^{(1)}(x, 0, z, t) + v_{1z}^{(1)}\right) = \mu_2\left(w_{2y}^{(1)} + v_{2z}^{(1)}\right). \end{aligned}\right\} \tag{10.3}$$

If the upper fluid is replaced by a given atmospheric pressure distribution $\bar{p}(x, z, t)$, then the equations for the lower fluid become (after dropping the subscripts)

$$\left.\begin{aligned} &\operatorname{grad}\left(p^{(0)} + \varrho\, g\, y\right) = 0, \qquad p^{(0)}(x, 0, z, t) = \bar{p}^{(0)}(x, z, t), \\ &\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} + \frac{\partial w^{(1)}}{\partial z} = 0, \\ &\frac{\partial \boldsymbol{v}^{(1)}}{\partial t} = -\frac{1}{\varrho}\operatorname{grad} p^{(1)} + \nu\,\Delta\,\boldsymbol{v}^{(1)}, \\ &\eta_t^{(1)}(x, z, t) = v^{(1)}(x, 0, z, t), \\ &u_y^{(1)}(x, 0, z, t) + v_x^{(1)} = w_y^{(1)} + v_z^{(1)} = 0, \\ &p^{(1)}(x, 0, z, t) - \varrho\, g\,\eta^{(1)} - 2\mu\, v_y^{(1)} = -T\left(\eta_{xx}^{(1)} + \eta_{zz}^{(1)}\right) + \bar{p}^{(1)}(x, z, t). \end{aligned}\right\} \tag{10.4}$$

For convenience we have assumed above that the expansion for η starts with $\varepsilon\eta^{(1)}$. If we had assumed instead $\eta=\eta^{(0)}+\varepsilon\eta^{(1)}+\cdots$, we would have found from (3.1) and (3.7) the equations

$$\eta_t^{(0)}=\eta_x^{(0)}=\eta_z^{(0)}=0$$

and, hence, $\eta^{(0)}=$const. The zero values of y in the boundary conditions would then be replaced by this constant. Taking the constant equal to zero means that we have taken the undisturbed interface as (x, z)-plane.

The equations above give the linearized equations of motion and boundary conditions at the interface or free surface. If one now proceeds, as we shall not do for this case, to collect coefficients of ε^2, one may obtain the differential equations and boundary conditions for the second-order corrections to be added to the solutions of the linearized equations, and so forth for higher-order corrections. In general the resulting equations are too unwieldy to be useful.

A special case of the linearized equations which is of particular interest is irrotational flow of a perfect fluid. There is then a velocity potential Φ which we assume has the following expansion:

$$\Phi(x, y, z, t, \varepsilon)=\varepsilon\,\Phi^{(1)}+\varepsilon^2\,\Phi^{(2)}+\cdots. \tag{10.5}$$

Condition (2.11) becomes

$$\Delta\,\Phi^{(i)}=0, \qquad i=1, 2, \ldots. \tag{10.6}$$

Let there be two superposed fluids with velocity potentials Φ_1 and Φ_2 describing the motion in each; otherwise the same notation as above. Then condition (3.1) at the interface gives the linearized condition

$$\eta_t^{(1)}(x, z, t)=\Phi_{1y}^{(1)}(x, 0, z, t)=\Phi_{2y}^{(1)}(x, 0, z, t) \tag{10.7}$$

and condition (3.9), together with (2.10′), gives

$$-\varrho_2\,\Phi_{2t}^{(1)}(x, 0, z, t)+\varrho_1\,\Phi_{1t}^{(1)}(x, 0, z, t)+(\varrho_1-\varrho_2)\,g\,\eta^{(1)}(x, z, t)=T(\eta_{xx}^{(1)}+\eta_{zz}^{(1)}). \tag{10.8}$$

The further special case when both the upper fluid and surface tension are missing will be dealt with so often later on that we repeat the boundary conditions for it. We allow, however, a pressure distribution on the free surface, $\bar{p}(x, z, t)=\varepsilon\bar{p}^{(1)}+\varepsilon^2\bar{p}^{(2)}+\cdots$. The first-order boundary conditions are

$$\left.\begin{aligned}&\eta_t^{(1)}(x, z, t)-\Phi_y^{(1)}(x, 0, z, t)=0,\\&g\,\eta^{(1)}(x, z, t)+\Phi_t(x, 0, z, t)+\varrho^{-1}\bar{p}^{(1)}(x, z, t)=0.\end{aligned}\right\} \tag{10.9}$$

Eliminating $\eta^{(1)}$ between the last two equations, one gets

$$g\,\Phi_y^{(1)}(x, 0, z, t)+\Phi_{tt}^{(1)}(x, 0, z, t)+\varrho^{-1}\bar{p}_t^{(1)}(x, z, t)=0. \tag{10.10}$$

The boundary conditions for the second-order corrections are not too long to write down:

$$\left.\begin{aligned}&\eta_t^{(2)}(x, z, t)-\Phi_y^{(2)}(x, 0, z, t)=\eta^{(1)}\,\Phi_{yy}^{(1)}-\eta_x^{(1)}\,\Phi_x^{(1)}-\eta_z^{(1)}\,\Phi_z^{(1)},\\&g\,\eta^{(2)}(x, z, t)+\Phi_t^{(2)}(x, 0, z, t)+\varrho^{-1}\bar{p}^{(2)}(x, z, t)=-\eta^{(1)}\,\Phi_{ty}^{(1)}-\tfrac{1}{2}(\operatorname{grad}\Phi^{(1)})^2.\end{aligned}\right\} \tag{10.11}$$

Eliminating $\eta^{(1)}$ and $\eta^{(2)}$ from (10.11), one finds a counterpart to (10.10):

$$\left.\begin{aligned}&g\,\Phi_y^{(2)}(x, 0, z, t)+\Phi_{tt}^{(2)}+\varrho^{-1}\bar{p}_t^{(2)}=-\frac{\partial}{\partial t}(\operatorname{grad}\Phi^{(1)})^2+\\&\quad+(\Phi_t^{(1)}+\varrho^{-1}\bar{p}^{(1)})\left(\Phi_{yy}^{(1)}+\frac{1}{g}\,\Phi_{tty}^{(1)}\right)-\varrho^{-1}(\Phi_x^{(1)}\,\bar{p}_x^{(1)}+\Phi_z^{(1)}\,\bar{p}_z^{(1)}).\end{aligned}\right\} \tag{10.12}$$

Under certain circumstances the next-to-last term will vanish. The boundary conditions for higher-order corrections will not be worked out in detail. However, they are of the form

$$\left.\begin{aligned} g\,\Phi_y^{(i)}(x,0,z,t)+\Phi_{tt}^{(i)}+\varrho^{-1}p_t^{(i)}&=A_i\{\Phi^{(1)},\dots,\Phi^{(i-1)},\bar{p}^1,\dots,\bar{p}^{(i-1)}\},\\ g\,\eta^{(i)}(x,z,t)+\Phi_t^{(i)}(x,0,z,t)+\varrho^{-1}\bar{p}^{(i)}&=B_i\{\Phi^{(1)},\dots,\Phi^{(i-1)},\bar{p}^{(1)},\dots,\bar{p}^{(i-1)}\},\end{aligned}\right\}\quad(10.13)$$

where A_i and B_i are functionals of the functions in brackets, in this case complicated polynomials of the functions and their derivatives evaluated at $y=0$.

It is useful to have the form of the linearized boundary conditions when certain additional assumptions are made.

First, let us suppose that the $(\bar{x},\bar{y},\bar{z})$-coordinate system is moving with velocity $c(t)$ in the x-direction with respect to the fixed (x,y,z)-coordinate system. Then, from the equation following (2.15) with $\bar{y}=y$, $\bar{z}=z$

$$\Phi_t(x,y,z,t)=\bar{\Phi}_t-c\,\bar{\Phi}_{\bar{x}},\qquad \Phi_{tt}=\bar{\Phi}_{tt}-2c\,\bar{\Phi}_{t\bar{x}}+c^2\,\bar{\Phi}_{\bar{x}\bar{x}}-\dot{c}\,\bar{\Phi}_{\bar{x}},$$

and the boundary conditions become

$$\left.\begin{aligned} g\,\bar{\eta}^{(1)}(\bar{x},\bar{z},t)+\bar{\Phi}_t^{(1)}(\bar{x},0,\bar{z},t)-c\,\bar{\Phi}_{\bar{x}}^{(1)}(\bar{x},0,\bar{z},t)+\varrho^{-1}\bar{p}^{(1)}(\bar{x},\bar{z},t)&=0,\\ \bar{\Phi}_{tt}^{(1)}(\bar{x},0,\bar{z},t)-2c\,\bar{\Phi}_{t\bar{x}}^{(1)}+c^2\,\bar{\Phi}_{\bar{x}\bar{x}}^{(1)}-\dot{c}\,\bar{\Phi}_{\bar{x}}^{(1)}+g\,\bar{\Phi}_y^{(1)}+\varrho^{-1}\bar{p}_t^{(1)}-c\,\varrho^{-1}\bar{p}_{\bar{x}}^{(1)}&=0.\end{aligned}\right\}\quad(10.14)$$

If c is constant and the motion is steady in the moving coordinate system,

$$\Phi(x,y,z,t)=\varphi(x-c\,t,y,z)=\varphi(\bar{x},\bar{y},\bar{z})$$

and the linearized boundary conditions are

$$\left.\begin{aligned} g\,\bar{\eta}^1(\bar{x},\bar{z})-c\,\varphi_{\bar{x}}^{(1)}(\bar{x},0,\bar{z})+\varrho^{-1}\bar{p}^1(\bar{x},\bar{z})&=0,\\ g\,\varphi_{\bar{y}}^{(1)}(\bar{x},0,\bar{z})+c^2\,\varphi_{\bar{x}\bar{x}}^{(1)}(\bar{x},0,\bar{z})-c\,\varrho^{-1}\bar{p}_{\bar{x}}^{(1)}(\bar{x},\bar{z})&=0.\end{aligned}\right\}\quad(10.15)$$

If the motion is steady with respect to a moving coordinate system, one may impose a uniform flow in the opposite direction and then treat the problem as a steady one in an absolute coordinate system, but carrying out the perturbation about the uniform flow. We illustrate this for the case of two-dimensional irrotational flow. Let $\varphi(x,y)$ and $\psi(x,y)$ be the velocity potential and stream function, respectively, and assume expansions of the form

$$\left.\begin{aligned} \varphi(x,y)&=-c\,x+\varepsilon\,\varphi^{(1)}(x,y)+\varepsilon^2\,\varphi^{(2)}+\cdots,\\ \psi(x,y)&=-c\,y+\varepsilon\,\psi^{(1)}(x,y)+\varepsilon^2\,\psi^{(2)}+\cdots,\\ \eta(x)&=\phantom{-c\,y+{}}\varepsilon\,\eta^{(1)}(x)+\varepsilon^2\,\eta^{(2)}+\cdots.\end{aligned}\right\}\quad(10.16)$$

The differential equations $\Delta\varphi=0$, $\Delta\psi=0$, $\varphi_x=\psi_y$, $\varphi_y=-\psi_x$ become

$$\Delta\varphi^{(i)}=0,\qquad \Delta\psi^{(i)}=0,\qquad \varphi_x^{(i)}=\psi_y^{(i)},\qquad \varphi_y^{(i)}=-\psi_x^{(i)}.\quad(10.17)$$

The kinematic condition (3.1) is replaced by

$$\psi(x,\eta(x))=0.$$

Substituting the expansions (10.16) in this equation and in (3.11′), one finds from the coefficients of ε

$$\left.\begin{aligned} -c\,\eta^{(1)}(x)+\psi^{(1)}(x,0)&=0,\\ g\,\eta^{(1)}(x)-c\,\varphi_x^{(1)}(x,0)+\varrho^{-1}\bar{p}^{(1)}(x)&=0.\end{aligned}\right\}\quad(10.18)$$

Eliminating $\eta^{(1)}$ and using the third of Eqs. (10.17), one gets

$$g\psi^{(1)}(x,0) - c^2\psi_y^{(1)}(x,0) + c\,\varrho^{-1}\bar{p}'^{(1)}(x) = 0. \tag{10.19}$$

Collecting the coefficients of ε^2, one obtains after some manipulation

$$\left.\begin{aligned} g\,\psi^{(2)}(x,0) - c^2\psi_y^{(2)} &= \frac{1}{c}\psi^{(1)}\left[c^2\psi_{yy}^{(1)} - g\,\psi_y^{(1)}\right] - \frac{1}{2}c\left[\psi_x^{(1)2} + \psi_y^{(1)2}\right],\\ c\,\eta^{(2)}(x) &= \psi^{(2)}(x,0) + \frac{1}{c}\psi^{(1)}\psi_y^{(1)}; \end{aligned}\right\} \tag{10.20}$$

here we have assumed for simplicity that $\bar{p} = 0$.

β) The shallow-water approximation. This approximation has been widely used by hydraulic engineers in the study of open-channel flow and, in a further simplification, is used for the theory of tides. In deriving the equations from the exact ones we shall follow the method of Friedrichs (1948) and Keller (1948). However, a somewhat different approach to this approximation due to Ursell (1953) is also instructive. Although it is possible to carry through the derivation while taking account of surface tension, this will not be done here. It will be assumed to start with that there are two perfect, incompressible fluids with an interface $y = \eta(x,z,t)$; the bottom fluid is bounded below by a rigid surface $y = -h(x,z)$. Variables pertaining to the lower fluid have subscript 1, those pertaining to the upper fluid subscript 2. The motion will be assumed irrotational.

Before making an expansion in powers of a parameter, it is essential to make a change of variable in which vertical and horizontal distances are stretched by different amounts. Let m be a scale for horizontal measurement and n one for vertical measurement. Define $\varepsilon = n^2/m^2$. Introduce new variables, $\bar{x}, \bar{y}, \bar{z}, \bar{t}$, by the equations

$$\bar{x} = x\sqrt{\varepsilon},\quad \bar{y} = y,\quad \bar{z} = z\sqrt{\varepsilon},\quad \bar{t} = t\sqrt{\varepsilon},\quad \bar{u} = u,\quad \bar{v} = v\sqrt{\varepsilon},\quad \bar{w} = w,\quad \bar{p} = p. \tag{10.21}$$

Eqs. (2.2), (2.6), (2.8), (3.1), (3.9), and (4.1) (with $T = 0$) become:

$$\left.\begin{aligned} &\varepsilon\left(\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{w}}{\partial\bar{z}}\right) + \frac{\partial\bar{v}}{\partial\bar{y}} = 0,\\ &\varepsilon\left(\frac{\partial\bar{u}}{\partial t} + \bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{u}}{\partial\bar{z}} + \frac{1}{\varrho}\frac{\partial\bar{p}}{\partial\bar{x}}\right) + \bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} = 0,\\ &\varepsilon\left(\frac{\partial\bar{v}}{\partial t} + \bar{u}\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{v}}{\partial\bar{z}} + g + \frac{1}{\varrho}\frac{\partial\bar{p}}{\partial y}\right) + \bar{v}\frac{\partial\bar{v}}{\partial\bar{y}} = 0,\\ &\varepsilon\left(\frac{\partial\bar{w}}{\partial t} + \bar{u}\frac{\partial\bar{w}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{w}}{\partial\bar{z}} + \frac{1}{\varrho}\frac{\partial\bar{p}}{\partial\bar{z}}\right) + \bar{v}\frac{\partial\bar{w}}{\partial\bar{y}} = 0,\\ &\frac{\partial\bar{w}}{\partial\bar{y}} = \frac{\partial\bar{v}}{\partial\bar{z}},\qquad \frac{\partial\bar{u}}{\partial\bar{z}} = \frac{\partial\bar{w}}{\partial\bar{x}},\qquad \frac{\partial\bar{v}}{\partial\bar{x}} = \frac{\partial\bar{u}}{\partial\bar{y}},\\ &\varepsilon\left(\bar{u}\frac{\partial\bar{\eta}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{\eta}}{\partial\bar{z}} + \frac{\partial\bar{\eta}}{\partial\bar{t}}\right) - \bar{v} = 0 \quad\text{for } \bar{y} = \bar{\eta}(\bar{x},\bar{z},\bar{t}),\\ &\varepsilon\left(\bar{u}\frac{\partial\bar{h}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{h}}{\partial\bar{z}}\right) + \bar{v} = 0 \qquad\text{for } \bar{y} = -\bar{h}(\bar{x},\bar{z},\bar{t}),\\ &\bar{p}_2(\bar{x},\bar{\eta},\bar{z},\bar{t}) = \bar{p}_1(\bar{x},\bar{\eta},\bar{z},\bar{t}), \end{aligned}\right\} \tag{10.22}$$

where $\bar{u}, \bar{v}, \bar{w}, \bar{p}, \varrho$ possess suppressed subscripts 1 and 2 for the lower and upper fluids respectively, except in the last equation.

Now assume expansions of the form

$$\left.\begin{aligned}\bar{\boldsymbol{v}}_i &= \boldsymbol{v}_i^{(0)} + \varepsilon\, \boldsymbol{v}_i^{(1)} + \varepsilon^2 \boldsymbol{v}_i^{(2)} + \cdots, \qquad i = 1, 2,\\ \bar{p}_i &= p_i^{(0)} + \varepsilon\, p_i^{(1)} + \varepsilon^2 p_i^{(2)} + \cdots, \qquad i = 1, 2,\\ \bar{\eta} &= \eta^{(0)} + \varepsilon\, \eta^{(1)} + \varepsilon^2 \eta^{(2)} + \cdots,\end{aligned}\right\} \tag{10.23}$$

substitute in the Eqs. (10.22) and collect according to powers of ε. (We shall henceforth suppress the bars on $\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{\eta}$). The terms independent of ε give the equations

$$\left.\begin{aligned}&v_y^{(0)} = 0,\\ &v^{(0)} u_y^{(0)} = 0, \quad v^{(0)} v_y^{(0)} = 0, \quad v^{(0)} w_y^{(0)} = 0,\\ &w_y^{(0)} = v_z^{(0)}, \quad u_z^{(0)} = w_x^{(0)}, \quad v_x^{(0)} = u_y^{(0)},\\ &v^{(0)}(x, \eta^{(0)}, z, t) = 0, \quad v_1^{(0)}(x, -h, z, t) = 0,\\ &p_2^{(0)}(x, \eta^{(0)}, z, t) = p_1^{(0)}(x, \eta^{(0)}, z, t).\end{aligned}\right\} \tag{10.24}$$

The first and fourth equations give

$$v^{(0)}(x, y, z, t) \equiv 0. \tag{10.25}$$

The third then states that

$$u_y^{(0)} = w_y^{(0)} = 0 \quad \text{or} \quad u^{(0)} = u^{(0)}(x, z, t), \quad w^{(0)} = w^{(0)}(x, z, t). \tag{10.26}$$

The terms which are coefficients of ε give, after making use of (10.25) and (10.26),

$$\left.\begin{aligned}&u_x^{(0)} + w_z^{(0)} + v_y^{(1)} = 0,\\ &u_t^{(0)} + u^{(0)} u_x^{(0)} + w^{(0)} u_z^{(0)} + p_x^{(0)}/\varrho = 0,\\ &g + p_y^{(0)}/\varrho = 0,\\ &w_t^{(0)} + u^{(0)} w_x^{(0)} + w^{(0)} w_z^{(0)} + p_z^{(0)}/\varrho = 0,\\ &u^{(0)} \eta_x^{(0)} + w^{(0)} \eta_z^{(0)} + \eta_t^{(0)} - v^{(1)} = 0 \qquad \text{for } y = \eta^{(0)}(x, z, t),\\ &u_1^{(0)} h_x + w_1^{(0)} h_y + v_1^{(1)} = 0 \qquad \text{for } y = -h(x, z).\end{aligned}\right\} \tag{10.27}$$

(The equations deriving from irrotationality and the continuity of pressure will be brought in later.) The first and last two equations of (10.27) together with (10.26) give

$$\left.\begin{aligned}v_1^{(1)} &= -y\,(u_{1x}^{(0)} + w_{1z}^{(0)}) - (u_1^{(0)} h)_x - (w_1^{(0)} h)_z,\\ v_2^{(1)} &= -y\,(u_{2x}^{(0)} + w_{2z}^{(0)}) + (u_2^{(0)} \eta^{(0)})_x + (w_2^{(0)} \eta^{(0)})_z - (u_1^{(0)} \eta^{(0)})_x - (w_1^{(0)} \eta^{(0)})_z +\\ &\qquad - (u_1^{(0)} h)_x - (w_1^{(0)} h)_z.\end{aligned}\right\} \tag{10.28}$$

The third equation of (10.27) gives

$$p^{(0)} = -\varrho g y + f(x, z, t).$$

In order to evaluate f, further information is necessary. Here are two cases of interest. 1. If the upper fluid is absent, the condition $p^{(0)}(x, \eta^{(0)}, z, t) = 0$ gives

$$p^{(0)} = -\varrho g y + \varrho g \eta^{(0)}(x, z, t). \tag{10.29}$$

2. If the upper fluid is unbounded above, then, up to an additive constant,

$$\left.\begin{aligned}p_1^{(0)} &= -\varrho_1 g y + (\varrho_1 - \varrho_2) g \eta^{(0)} + k,\\ p_2^{(0)} &= -\varrho_2 g y + k.\end{aligned}\right\} \tag{10.30}$$

If the upper fluid is bounded above by a free surface $y=d(x,z,t)=d^{(0)}+\varepsilon d^{(1)}+\cdots$, then one may satisfy the boundary conditions $p_2^{(0)}(x,d^{(0)},z,t)=0$, $p_2^{(0)}(x,\eta^{(0)},z,t)=p_1^{(0)}(x,\eta^{(0)},z,t)$ with

$$\left.\begin{aligned} p_1^{(0)} &= -\varrho_1 g(y-\eta^{(0)})+\varrho_2 g(d^{(0)}-\eta^{(0)}),\\ p_2^{(0)} &= -\varrho_2 g(y-d^{(0)}). \end{aligned}\right\} \tag{10.31}$$

It is clear from the form of $p^{(0)}$ why the shallow-water approximation is sometimes called the hydrostatic approximation.

The usual equations for the first approximation to the shallow-water theory are those in which only the lower fluid is present. They may now be obtained by substituting (10.29) in the second and fourth equations in (10.27) and (10.28) in the fifth equation. They are (10.25), (10.29), and

$$\left.\begin{aligned} u_t^{(0)}+u^{(0)}u_x^{(0)}+w^{(0)}u_z^{(0)}+g\,\eta_x^{(0)}&=0,\\ w_t^{(0)}+u^{(0)}w_x^{(0)}+w^{(0)}w_z^{(0)}+g\,\eta_z^{(0)}&=0,\\ \eta_t^{(0)}+[u^{(0)}(\eta^{(0)}+h)]_x+[w^{(0)}(\eta^{(0)}+h)]_z&=0. \end{aligned}\right\} \tag{10.32}$$

If one now collects the coefficients of ε^2 and the remaining coefficients of ε, one finds after some reduction

$$\left.\begin{aligned} &u_x^{(1)}+w_z^{(1)}+v_y^{(2)}=0,\\ &u_t^{(1)}+u^{(1)}u_x^{(0)}+u^{(0)}u_x^{(1)}+w^{(1)}u_z^{(0)}+w^{(0)}u_z^{(1)}+v^{(1)}u_y^{(1)}+p_x^{(1)}/\varrho=0,\\ &v_t^{(1)}+u^{(0)}v_x^{(1)}+w^{(0)}v_z^{(1)}+v^{(1)}v_y^{(1)}+p_y^{(1)}/\varrho=0,\\ &w_t^{(1)}+u^{(1)}w_x^{(0)}+u^{(0)}w_x^{(1)}+w^{(1)}w_z^{(0)}+w^{(0)}w_z^{(1)}+v^{(1)}w_y^{(1)}+p_z^{(1)}/\varrho=0,\\ &w_y^{(1)}=v_z^{(1)},\qquad u_z^{(1)}=w_x^{(1)},\qquad v_x^{(1)}=u_y^{(1)},\\ &u^{(0)}\eta_x^{(1)}+u^{(1)}\eta_x^{(0)}+w^{(0)}\eta_z^{(1)}+w^{(1)}\eta_z^{(0)}+\eta_t^{(1)}-\eta^{(1)}v_y^{(1)}-v^{(2)}=0\\ &\qquad\qquad\qquad\text{for}\quad y=\eta^{(0)}(x,z,t),\\ &u_1^{(1)}h_x+w_1^{(1)}h_z+v_1^{(2)}=0\quad\text{for}\quad y=-h(x,z),\\ &p_2^{(1)}-p_1^{(1)}+\eta^{(1)}(p_{2y}^{(0)}-p_{1y}^{(0)})=0\quad\text{for}\quad y=\eta^{(0)}(x,z,t). \end{aligned}\right\} \tag{10.33}$$

Some relations can be derived immediately from these equations. For the sake of brevity we introduce the following functions:

$$\begin{aligned} &A_i(x,z,t)=u_{ix}^{(0)}+w_{iz}^{(0)},\qquad C_i(x,z,t)=(u_i^{(0)}\eta^{(0)})_x+(w_i^{(0)}\eta^{(0)})_z,\quad i=1,2,\\ &B_1(x,z,t)=-(u_1^{(0)}h)_x-(w_1^{(0)}h)_z,\qquad B_2=C_2-C_1+B_1. \end{aligned}$$

Eqs. (10.28) may then be written

$$v_i^{(1)}=-y\,A_i+B_i,\qquad i=1,2. \tag{10.28'}$$

Then the fifth, first, and third equations of (10.33) give

$$\left.\begin{aligned} u^{(1)}&=-\tfrac{1}{2}y^2A_x+y\,B_x+r(x,z,t),\\ w^{(1)}&=-\tfrac{1}{2}y^2A_z+y\,B_z+s(x,z,t),\\ r_z&=s_x,\\ v^{(2)}&=\tfrac{1}{6}y^3(A_{xx}+A_{zz})-\tfrac{1}{2}y^2(B_{xx}+B_{zz})-y(r_x+s_z)+l(x,z,t),\\ p^{(1)}/\varrho&=\tfrac{1}{2}y^2[A^2+u^{(0)}A_x+w^{(0)}A_z+A_t]+\\ &\quad+y[A\,B+u^{(0)}B_x+w^{(0)}B_z+B_t]+q(x,z,t), \end{aligned}\right\} \tag{10.34}$$

see errata

where we have suppressed the subscripts indicating the fluid. The rest of the equations and the boundary conditions are still available to determine the unknown functions. We carry this out only for the case the upper fluid is missing. Then the last condition in (10.32) becomes $p^{(1)}(x, \eta^{(0)}, z, t) = \varrho g \eta^{(1)}$, which allows one to determine $q(x, z, t)$ after $\eta^{(1)}$ is found. The next-to-the-last condition in (10.32) determines $l(x, z, t)$. The equations for r, s and $\eta^{(1)}$ are

see errata

$$\left.\begin{aligned} &u^{(0)} r_x + w^{(0)} r_z + r_t + u_x^{(0)} r + u_z^{(0)} s = -q_x - B B_x, \\ &u^{(0)} s_x + w^{(0)} s_z + s_t + w_x^{(0)} r + w_z^{(0)} s = -q_z - B B_z, \\ &u^{(0)} \eta_x^{(1)} + w^{(0)} \eta_z^{(1)} + \eta_t^{(1)} + A \eta^{(1)} = [v^{(2)} - u^{(1)} \eta_x^{(0)} - w^{(1)} \eta_z^{(0)}]_{y=\eta^{(0)}}, \\ &r_z = s_x, \end{aligned}\right\} \tag{10.35}$$

where

$$\begin{aligned} q(x, z, t) &= g \eta^{(1)} - \tfrac{1}{2} \eta^{(0)2} [A^2 + u^{(0)} A_x + w^{(0)} A_z + A_t] - \\ &\quad - \eta^{(0)} [A B + u^{(0)} B_x + w^{(0)} B_z + B_t], \\ l(x, z, t) &= -[u^{(1)} h_x + w^{(1)} h_z]_{y=-h} - \tfrac{1}{6} \eta^{(0)3} [A_{xx} + A_{zz}] + \\ &\quad + \tfrac{1}{2} \eta^{(0)2} [B_{xx} + B_{zz}] - \eta^{(0)} (r_x + s_z). \end{aligned}$$

The solutions to these equations give the second-order corrections to the first-order shallow-water theory.

The equations resulting from the coefficients of ε^3 have been given by KELLER (1948) for two dimensions, but will not be reproduced here.

The Eqs. (10.32) for the first-order theory are nonlinear. In the theory of tides and seiches it is customary to simplify further by linearizing them in a manner similar to that used in deriving the equations for the infinitesimal-wave theory. Let $y = 0$ be the surface of the undisturbed water and assume that one may make further expansions in a small parameter α: $u^{(0)} = \alpha u^{(01)} + \cdots$, $w^{(0)} = \alpha w^{(01)} + \cdots$, $\eta^{(0)} = \alpha \eta^{(01)}, \ldots$. After some easy manipulations one finds for the linearized approximation to (10.32) the equations

see errata

$$\left.\begin{aligned} u_t^{(01)} + g \eta_x^{(01)} = 0, \qquad w_t^{(01)} + g \eta_z^{(01)} = 0, \\ \eta_{tt}^{(01)} - g [\eta_x^{(01)} h]_x - g [\eta_z^{(01)} h]_z = 0. \end{aligned}\right\} \tag{10.36}$$

If the bottom is flat, the equation for $\eta^{(01)}$ becomes the simple wave equation.

D. Theory of infinitesimal waves.

This chapter will deal with special solutions of the linearized equations derived in Sect. 10α. This approximate theory has been very fruitful in its application to problems with various boundary configurations; the linear character of both the equations and boundary conditions allows one to use easily found simple solutions to construct other solutions satisfying special boundary conditions. The derivation of the equations in Sect. 10α suggests the limitations of their use in physical problems: If L and V are a typical length and velocity associated with the physical problem, then, when the perturbation parameter ε is small, the surface elevation and velocities (or their deviation from a uniform flow) should be small with respect to L and V respectively. The smallness may not be uniform, but the quantities in question should approach zero point-wise with ε except at singular points.

11. The fundamental equations. With few exceptions, this chapter will be concerned with the solution of a problem in potential theory. Let the (x, z)-plane be at the undisturbed free surface. We shall be seeking a function $\Phi(x, y, z, t)$, the velocity potential of the motion, satisfying the conditions [cf. Eq. (10.10)]

$$\left.\begin{aligned}&\Delta\Phi = \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0\\&\Phi_{tt}(x, 0, z, t) + g\,\Phi_y(x, 0, z, t) = -\varrho^{-1}\bar{p}_t(x, z, t),\\&\Phi_n = V_n \quad \text{on solid boundaries,}\end{aligned}\right\} \tag{11.1}$$

where $\Delta\Phi = 0$ is to be satisfied at all nonsingular points of the fluid in the region $y < 0$ and V_n is the normal velocity of the solid boundary at a given point. $\bar{p}(x, z, t)$ is a given pressure distribution on the free surface; in many problems it will be 0. The form of the free surface is given by:

$$\eta(x, z, t) = -\frac{1}{g}\,\Phi_t(x, 0, z, t) - \frac{1}{\varrho g}\,\bar{p}(x, z, t). \tag{11.2}$$

Two special cases occur frequently. If the motion is steady in a coordinate system moving with constant velocity c in the x-direction, then with x, y, z as moving coordinates, the free-surface boundary condition and equation of the surface are given by [cf. Eq. (10.15)]

$$\left.\begin{aligned}&\varphi_y(x, 0, z) + \frac{c^2}{g}\,\varphi_{xx}(x, 0, z) = \frac{c}{\varrho g}\,\bar{p}_x(x, z),\\&\eta(x, z) = \frac{c}{g}\,\varphi_x(x, 0, z) - \frac{1}{\varrho g}\,\bar{p}(x, z).\end{aligned}\right\} \tag{11.3}$$

If Φ and $\bar{p}$ are harmonic functions of the time, i.e.

$$\Phi(x, y, z, t) = \varphi_1(x, y, z)\cos\sigma t + \varphi_2(x, y, z)\sin\sigma t = \mathrm{Re}\,\varphi(x, y, z)\,e^{-i\sigma t},$$

where

$$\varphi(x, y, z) = \varphi_1(x, y, z) + i\,\varphi_2(x, y, z)$$

and similarly for $\bar{p}$, then the free-surface condition and equation of the surface become

$$\left.\begin{aligned}&\varphi_{1y}(x, 0, z) - \frac{\sigma^2}{g}\,\varphi_1(x, 0, z) = -\frac{\sigma}{\varrho g}\,\bar{p}_2(x, z),\\&\varphi_{2y}(x, 0, z) - \frac{\sigma^2}{g}\,\varphi_2(x, 0, z) = \frac{\sigma}{\varrho g}\,\bar{p}_1(x, z),\\&\eta(x, z, t) = \frac{\sigma}{g}\,[\varphi_1(x,0,z)\sin\sigma t - \varphi_2(x,0,z)\cos\sigma t] -\\&\qquad - \frac{1}{\varrho g}\,[\bar{p}_1(x, z)\cos\sigma t + \bar{p}_2(x, z)\sin\sigma t].\end{aligned}\right\} \tag{11.4}$$

In the few cases where we consider superposed fluids, viscous fluids or surface tension, we shall refer back to Sect. 10 for the equations.

Use of complex variables. For two-dimensional irrotational motion, it is frequently advantageous to use complex variables. Let

$$z = x + i\,y, \qquad f(z, t) = \Phi(x, y, t) + i\,\Psi(x, y, t),$$

where Φ and Ψ are velocity potential and stream function, respectively. (It should be clear from context whether z is being used for $x + iy$ or one of the horizontal coordinates.) Since the equations relating Φ and Ψ,

$$\Phi_x = \Psi_y, \qquad \Phi_y = -\Psi_x,$$

are just the Cauchy-Riemann equations, the function $f(z, t)$ is an analytic function of z for all points z for which Φ_x and Φ_y exist. $f(z, t)$ will be called the "complex potential". The "complex velocity" is given by

$$w(z,t) = f'(z,t) = u - i\,v.$$

The boundary condition at the free surface in (11.1) can be expressed in the following equation in $f(z,t)$:

$$\mathrm{Re}\left\{i\,g\,f'(z,t) + \frac{d^2}{dt^2} f(z,t)\right\} = -\frac{1}{\varrho}\,p_t(x,t) \quad \text{for } y = 0. \tag{11.5}$$

The first equation of (11.3) becomes

$$\mathrm{Re}\,\{i\,g\,f'(z) + c^2 f''(z)\} = \frac{c}{\varrho}\,p'(x) \quad \text{for } y = 0. \tag{11.6}$$

However, Eq. (10.19) shows that this may also be taken in the form

$$\mathrm{Re}\,\{i\,g\,f(z) + c^2 f'(z)\} = \frac{c}{\varrho}\,p(x) \quad \text{for } y = 0.$$

If one may express $f(z,t) = f_1(z)\cos\sigma t + f_2(z)\sin\sigma t$, then the first of Eqs. (11.4) becomes

$$\mathrm{Re}\,\{i\,g\,f_k'(z) - \sigma^2 f_k(z)\} = (-1)^k \frac{\sigma}{\varrho}\,p_{k-(-1)^k}(x) \quad \text{for } y = 0,\ k = 1, 2. \tag{11.7}$$

We note that in order to express $f(z,t)$ in a manner analogous to that used for Φ immediately preceding (11.4) one must introduce a second complex unit j which does not "interact" with i. Thus let $f(z) = f_1(z) + j\,f_2(z)$. Then $f(z,t) = \mathrm{Re}_j\, f(z)\,\mathrm{e}^{-j\sigma t}$.

If $f(z)$ is an analytic function satisfying any one of the conditions (11.5) to (11.7) with $p \equiv 0$, then $f^{(n)}(z)$ will also satisfy it.

12. Other boundary conditions. The boundary conditions given in Sect. 11 will not ordinarily be sufficient to ensure a unique solution to the problems in which the fluid occupies an unbounded region. An additional condition at infinity must be imposed upon the potential function. In certain cases the proper additional condition is fairly clear from the physical problem. For example, for a body moving steadily in an infinite ocean undisturbed except for the body, it seems reasonable to impose the condition that the fluid motion vanish far ahead of and far below the body. For the fluid motion produced by a stationary but steadily oscillating body, it seems reasonable to impose vanishing of the motion far below the body, but outgoing waves at infinity on all sides, if the body does not extend to infinity in some horizontal direction, the so-called "radiation condition".

If the body is not bounded in a horizontal direction, one may easily see that the radiation condition stated above cannot be expected to be satisfied. For example, suppose that waves are being generated by some type of oscillation of a vertical half-plane, say $z = 0$, $x > 0$, in which the oscillatory motion of the half-plane is independent of x. Then one will expect the generated waves to behave like outgoing plane waves from the two sides of the plane as $x \to \infty$; these will not satisfy the radiation condition in the direction Ox. On the other hand, one might expect that the influence of the edge at $x = 0$ would show up as waves satisfying the radiation condition. The formulation of proper boundary conditions in situations of this sort has been discussed by PETERS and STOKER (1954); see also STOKER (1956, 1957, p. 109ff).

In diffraction problems one customarily prescribes the form of an incoming wave and then seeks the scattered wave. The preceding remarks concerning the boundary conditions for waves generated by an oscillating body apply also to the scattered wave.

In more complicated physical situations it is not always clear what boundary conditions should be imposed at infinity, and errors have been made. For example, for a body which is both oscillating with a fixed frequency σ and moving with a steady average velocity c, one might reasonably expect no motion far ahead if c is large, but a radiation condition if c is small. However, the formulation of the boundary condition cannot be completed until the problem is partly solved.

The proper formulation of the boundary conditions at infinity can frequently be obtained by a method recommended by Havelock (1917, 1949a) and used also by Brard (1948a, b), Stoker (1953, 1954), Stoker and Peters (1957), De Prima and Wu (1957), Wu (1957) and others. It consists in formulating an initial-value problem for which the desired steady-state problem is the limit as $t \to \infty$. For the initial-value problem the boundary condition at infinity is that the fluid motion vanishes everywhere. However, even though this procedure may produce the desired solution, it is not always obvious what boundary conditions at infinity in the steady-state problem would have produced it.

13. Some mathematical solutions. Some of the mathematical solutions to be derived in this section will provide solutions, without further modification, to certain physical problems; others, although apparently not acceptable physically, will provide fundamental solutions which can be used in constructing solutions to other more complicated physical problems. In all cases the fluid is assumed unbounded in a horizontal direction and either infinitely deep or with a horizontal bottom $y = -h$; the pressure on the free surface is taken to be zero everywhere. The solutions without singularities are obtained by the method of separation of variables, and are all harmonic in t. It will not be necessary to carry along the subscripts of (11.4).

α) Separation of the y-variable. Assume that one may express φ by

$$\varphi(x, y, z) = Y(y)\,\varphi(x, z).$$

Then $\Delta_3\varphi = \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0$ becomes, after separation,

$$\Delta_2\varphi + A\,\varphi = 0, \qquad Y'' - A\,Y = 0.$$

The two cases $A = m^2 > 0$ and $A = -m^2 < 0$ lead to different solutions.

$A > 0$. In this case $\varphi(x, z)$ satisfies the wave equation

$$\Delta_2\varphi + m^2\,\varphi = 0$$

and Y is given by

$$Y = a\,e^{my} + b\,e^{-my}.$$

If the fluid is infinitely deep and $\varphi_y(x, y, z)$ is to remain bounded as $y \to -\infty$, one must have $b = 0$. Then condition (11.4) requires

$$m = \frac{\sigma^2}{g}$$

and $\varphi(x, y, z)$ is of the form

$$\varphi(x, y, z) = e^{\sigma^2 y/g}\,\varphi(x, z). \tag{13.1}$$

If the fluid is of finite depth h, the boundary condition $\varphi_y(x, -h, z) = 0$ requires Y to take the form

$$Y = a\cosh m(y + h)$$

and condition (11.4) becomes

$$m \tanh m h = \frac{\sigma^2}{g},$$

an equation with two real solutions, say $\pm m_0$. In this case, one has

$$\varphi(x, y, z) = \cosh m_0 (y + h)\, \varphi(x, z). \tag{13.2}$$

We note that, if $h_1 < h_2$ then $\sigma^2/g < m_0^{(2)} < m_0^{(1)}$. Also $m_0/h^{\frac{1}{2}} \to \sigma/g^{\frac{1}{2}}$ as $h \to 0$ and $m_0 \to \sigma^2/g$ as $h \to \infty$.

$A < 0$. In this case $\varphi(x, z)$ satisfies

$$\Delta_2 \varphi - m^2 \varphi = 0$$

and Y is given by

$$Y = a \cos m y + b \sin m y.$$

Condition (11.4) restricts Y further to

$$Y = C\left(m \cos m y + \frac{\sigma^2}{g} \sin m y\right).$$

If the fluid is infinitely deep, requiring φ_y to remain bounded imposes no further restriction. If the fluid is of depth h, then $\varphi_y(x, -h, z) = 0$ requires m to satisfy the equation

$$m \tan m h = -\frac{\sigma^2}{g},$$

an equation with an infinite number of real solutions, $\pm m_1, \pm m_2, \ldots$. In this latter case one may conveniently take Y in the form

$$Y = C \cos m (y + h).$$

The roots m_k satisfy $\frac{1}{2}(2k-1)\,\pi/h < m_k < k\pi/h$. For fixed h, $m_k h \to k\pi$ as $k \to \infty$; for fixed k, $m_k h \to k\pi$ as $h \to 0$, and $m_k h \to \frac{1}{2}(2k-1)\,\pi$ as $h \to \infty$.

For these two cases one finds then for $\varphi(x, y, z)$ the forms:

infinite depth:

$$\varphi(x, y, z) = C\left(m \cos m y + \frac{\sigma^2}{g} \sin m y\right) \varphi(x, z); \tag{13.3}$$

finite depth:

$$\varphi(x, y, z) = C \cos m_i (y + h)\, \varphi(x, z). \tag{13.4}$$

β) Further separation of variables. We now assume $\varphi(x, z) = X(x)\, Z(z)$ and substitute in each of the two equations for φ given above.

$A > 0$. In this case substitution in $\Delta\varphi + m^2\varphi = 0$ gives

$$X'' + (m^2 - k^2)\, X = 0, \qquad Z'' + k^2 Z = 0.$$

(The equations obtained by replacing k^2 by $-k^2$ will give the solution obtained below for $A < 0$, with x and z interchanged.) The solution for Z is

$$Z = f \cos k z + g \sin k z = B \cos(k z + \gamma).$$

The solution for X depends upon the sign of $m^2 - k^2$:

$$k^2 < m^2: \quad X = c \cos x\sqrt{m^2 - k^2} + d \sin x \sqrt{m^2 - k^2};$$

$$k^2 = m^2: \quad X = c\, x + d;$$

$$k^2 > m^2: \quad X = c\, e^{x\sqrt{k^2 - m^2}} + d\, e^{-x\sqrt{k^2 - m^2}}.$$

$A<0$. Substitution in $\Delta_2\varphi - m^2\varphi = 0$ gives

$$X'' - (k^2+m^2)\,X = 0, \qquad Z'' + k^2 Z = 0,$$

which gives Z as above and

$$X = c\,\mathrm{e}^{x\sqrt{k^2+m^2}} + d\,\mathrm{e}^{-x\sqrt{k^2+m^2}}.$$

(Substituting $-k^2$ for k^2 would give the solutions corresponding to $A>0$ with x and z interchanged.) We may accumulate the preceding results to obtain the following fundamental solutions:

for infinite depth:

$$\left.\begin{aligned}
&\mathrm{e}^{\nu y}\left(a\cos x\sqrt{\nu^2-k^2} + b\sin x\sqrt{\nu^2-k^2}\right)\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2<\nu^2,\\
&\mathrm{e}^{\nu y}(a x + b)\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2=\nu^2,\\
&\mathrm{e}^{\nu y}\left(a\,\mathrm{e}^{x\sqrt{k^2-\nu^2}} + b\,\mathrm{e}^{-x\sqrt{k^2-\nu^2}}\right)\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2>\nu^2,\\
&(m\cos m y + \nu\sin m y)\left(a\,\mathrm{e}^{x\sqrt{k^2+m^2}} + b\,\mathrm{e}^{-x\sqrt{k^2+m^2}}\right)\cos(kz+\gamma)\cos(\sigma t+\tau),
\end{aligned}\right\} \quad (13.5)$$

where $\nu = \sigma^2/g$;

for finite depth:

$$\left.\begin{aligned}
&\cosh m_0(y+h)\left(a\cos x\sqrt{m_0^2-k^2} + b\sin x\sqrt{m_0^2-k^2}\right)\times\\
&\qquad\times\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2<m_0^2,\\
&\cosh m_0(y+h)(a x + b)\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2=m_0^2,\\
&\cosh m_0(y+h)\left(a\,\mathrm{e}^{x\sqrt{k^2-m_0^2}} + b\,\mathrm{e}^{-x\sqrt{k^2-m_0^2}}\right)\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2>m_0^2,\\
&\cos m_i(y+h)\left(a\,\mathrm{e}^{x\sqrt{k^2+m_i^2}} + b\,\mathrm{e}^{-x\sqrt{k^2+m_i^2}}\right)\cos(kz+\gamma)\cos(\sigma t+\tau),
\end{aligned}\right\} \quad (13.6)$$

where

$$m_0\tanh m_0 h - \frac{\sigma^2}{g} = 0 \quad\text{and}\quad m_i\tan m_i h + \frac{\sigma^2}{g} = 0.$$

The corresponding solutions for two dimensions may be obtained by setting $k=0$ and deleting the second and third equations in each group.

For either set of solutions only the first in each is bounded for all values of the variables for which $y\leqq 0$ or $-h\leqq y\leqq 0$. For two-dimensional motion it has been shown by WEINSTEIN (1927, 1949) that the only function harmonic in $-h<y<0$ and satisfying (11.4) and $\varphi_y(x,-h)=0$ for which both φ and φ_y are bounded in $-h\leqq y\leqq 0$ is $\varphi = A\cosh m(y+h)\sin(mx+\alpha)$. KELDYSH (1935) and STOKER (1947, pp. 7—9) have proved a similar theorem for the lower half-plane: If φ and $\varphi_x^2+\varphi_y^2$ are bounded for $y\leqq 0$ as $x^2+y^2\to\infty$, the only φ satisfying (11.4) and harmonic everywhere in the half-plane $y\leqq 0$ is $A\,\mathrm{e}^{ky}\sin(kx+\alpha)$. WEINSTEIN'S theorem has been generalized by JOHN (1950, p. 59) to three dimensions: If $\varphi(x,y,z)$ satisfies (11.4), $\varphi_y(x,-h,z)=0$,

$$\lim_{R\to\infty}\varphi(R\cos\alpha, y, R\sin\alpha)\,R^{-\frac{1}{2}}\,\mathrm{e}^{-m_1 R} = 0$$

and is harmonic everywhere in $-h\leqq y\leqq 0$, then $\varphi(x,y,z)$ is of the form (13.2) with $\varphi(x,z)$ an everywhere regular solution of

$$\Delta_2\varphi + m_0^2\,\varphi = 0.$$

The condition at infinity is necessary, as the solution derived below in (13.8),

$$\varphi = I_0(m_1 R)\cos m_1(y+h),$$

shows. The corresponding theorem for infinite depth was proved by KOCHIN (1940).

The equations for $\varphi(x, y)$ may also be separated in polar coordinates (R, α), $x = R\cos\alpha$, $z = R\sin\alpha$. We give only the solutions:

infinite depth:

$$\left.\begin{aligned} &e^{\nu y}\left[A\,J_n(\nu R) + B\,Y_n(\nu R)\right]\cos(n\alpha + d)\cos(\sigma t + \tau),\\ &(m\cos m y + \nu\sin m y)\left[A\,I_n(m R) + B\,K_n(m R)\right]\cos(n\alpha + d)\cos(\sigma t + \tau),\end{aligned}\right\} \quad (13.7)$$

where $\nu = \sigma^2/g$ and n is an integer;

finite depth:

$$\left.\begin{aligned} &\cosh m_0(y + h)\left[A\,J_n(m_0 R) + B\,Y_n(m_0 R)\right]\cos(n\alpha + \delta)\cos(\sigma t + \tau),\\ &\cos m_i(y + h)\left[A\,I_n(m_i R) + B\,K_n(m_i R)\right]\cos(n\alpha + \delta)\cos(\sigma t + \tau), \quad i \geqq 1,\end{aligned}\right\} \quad (13.8)$$

where $m_0\tanh m_0 h - \sigma^2/g = 0$, $m_i\tan m_i h + \sigma^2/g = 0$ and n is an integer. Here J_n, Y_n, I_n, K_n are Bessel functions (we use WATSON's notation). Y_n and K_n are both singular at $R = 0$ but approach zero as $R \to \infty$; J_n and I_n are both finite at $R = 0$; J_n approaches zero as $R \to \infty$, I_n increases exponentially.

γ) Singular solutions. In this section we shall find solutions of the problems set in Sect. 11 which have singularities of simple type at a single point. We shall indicate proofs only for the case of simple sources, i.e. singularities of the
see errata for $[(x-\alpha)^2 + (y-b)^2 + (z-c)^2]^{-\frac{1}{2}}$ or $\log[(x-a)^2 + (z-b)^2]^{\frac{1}{2}}$. We shall consider first the case of a stationary source of pulsating strength, then the case of a moving source. Three-dimensional problems are treated first.

Source of pulsating strength in three dimensions. Let (a, b, c) be in the lower half-space. We wish to find a function

$$\Phi(x, y, z, t) = \varphi_1(x, y, z)\cos\sigma t + \varphi_2(x, y, z)\sin\sigma t$$

defined for $y \leqq 0$ except at (a, b, c) and satisfying

$$\left.\begin{aligned} &1.\ \Delta\varphi_i = 0 \text{ except at } (a, b, c),\ i = 1, 2,\\ &2.\ \varphi_{iy}(x, 0, z) - \nu\,\varphi_i(x, 0, z) = 0,\ i = 1, 2,\ \nu = \frac{\sigma^2}{g},\\ &3.\ \Phi(x, y, z, t) = r^{-1}\cos\sigma t + \Phi_0(x, y, z, t),\\ &\quad\ \text{where } \Phi_0 \text{ is harmonic in the whole region } y < 0,\\ &4.\ \lim_{y \to -\infty}\operatorname{grad}\varphi_i = 0,\ i = 1, 2,\\ &5.\ \lim_{R\to\infty}\sqrt{R}\left(\frac{\partial\varphi_1}{\partial R} + \nu\,\varphi_2\right) = 0,\ \lim_{R\to\infty}\sqrt{R}\left(\frac{\partial\varphi_2}{\partial R} - \nu\,\varphi_1\right) = 0.\end{aligned}\right\} \quad (13.9)$$

Here $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ and $R^2 = (x-a)^2 + (z-c)^2$. Condition 5,
see errata usually called the "radiation condition", requires the waves at infinity to ben progressing outwards and imposes a uniqueness which would not otherwise be present. However, other such conditions could be imposed.

We assume that a solution Φ can be found in the form

$$\Phi(x, y, z, t) = \left[r^{-1} + \varphi_0(x, y, z)\right]\cos\sigma t + \varphi_2(x, y, z)\sin\sigma t. \quad (13.10)$$

φ_2 will be determined at the end so as to satisfy 5. Denote the double Fourier transform in x and z of φ by $\tilde{\varphi}$:

$$\varphi(x, y, z) = \frac{1}{2\pi}\int_0^\infty\int_{-\pi}^{\pi}\tilde{\varphi}(k, \vartheta, y)\,e^{ik(x\cos\vartheta + z\sin\vartheta)}\,d\vartheta\,dk.$$

Then condition 1 applied to φ_0 becomes after transforming

$$\tilde{\varphi}_{0yy} - k^2 \tilde{\varphi}_0 = 0$$

or

$$\tilde{\varphi}_0 = A_0(k, \vartheta)\, e^{yk} \tag{13.11}$$

where we have used 4. to discard the other solution. From the known integral

$$(x^2 + y^2 + z^2)^{-\frac{1}{2}} = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} e^{-k|y|}\, e^{ik(x\cos\vartheta + z\sin\vartheta)}\, d\vartheta\, dk \tag{13.12}$$

one may compute

$$\widetilde{r^{-1}} = e^{-k|y-b|}\, e^{-ik(a\cos\vartheta + c\sin\vartheta)}. \tag{13.13}$$

Substituting $\tilde{\varphi}_0 + \widetilde{r^{-1}}$ in the transform of condition 2 gives

$$A_0(k, \vartheta) = \frac{k+\nu}{k-\nu}\, e^{kb}\, e^{-ik(a\cos\vartheta + c\sin\vartheta)}. \tag{13.14}$$

We now have, formally,

$$\varphi_0(x, y, z) = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} \frac{k+\nu}{k-\nu}\, e^{k(y+b)}\, e^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]}\, d\vartheta\, dk.$$

Since the integrand has a singularity at $k = \nu$, the integral is not meaningful without further definition. We shall interpret the integral as a Cauchy principal value. Then

$$\left.\begin{aligned} \varphi_1(x, y, z) &= \frac{1}{r} + \frac{1}{2\pi} \mathrm{PV} \int_0^\infty \int_{-\pi}^{\pi} \frac{k+\nu}{k-\nu}\, e^{k(y+b)}\, e^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]}\, d\vartheta\, dk, \\ &= \frac{1}{r} + \frac{1}{r_1} + \frac{\nu}{\pi} \mathrm{PV} \int_0^\infty \int_{-\pi}^{\pi} \frac{1}{k-\nu}\, e^{k(y+b)}\, e^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]}\, d\vartheta\, dk, \end{aligned}\right\} \tag{13.15}$$

where $r_1^2 = (x-a)^2 + (y+b)^2 + (z-c)^2$. The second equation may be derived easily from the first one by use of (13.12) suitably modified. φ_1 satisfies 1., 2. and 4.; φ_0 is harmonic in the whole region.

In order to satisfy 5. we shall first find the asymptotic form of φ_1 for large R. With polar coordinates

$$x - a = R\cos\alpha, \qquad z - c = R\sin\alpha,$$

one may write (13.15) as

$$\begin{aligned} \varphi_1(x, y, z) = \varphi_1(R, \alpha, y) &= \frac{1}{r} + \frac{1}{r_1} + \frac{\nu}{\pi} \mathrm{PV} \int_0^\infty \int_{-\pi}^{\pi} \frac{1}{k-\nu}\, e^{k(y+b)}\, e^{ikR\cos(\vartheta-\alpha)}\, d\vartheta\, dk, \\ &= \frac{1}{r} + \frac{1}{r_1} + \frac{4\nu}{\pi} \mathrm{PV} \int_0^\infty \int_0^{\frac{1}{2}\pi} \frac{1}{k-\nu}\, e^{k(y+b)} \cos(kR\cos\vartheta)\, d\vartheta\, dk, \\ &= \frac{1}{r} + \frac{1}{r_1} + \frac{4\nu}{\pi} \mathrm{PV} \int_0^\infty \int_0^1 \frac{1}{\sqrt{1-\lambda^2}}\, \frac{1}{k-\nu}\, e^{k(y+b)} \cos Rk\lambda\, d\lambda\, dk, \\ &= \frac{1}{r} + \frac{1}{r_1} + 2\nu\, \mathrm{PV} \int_0^\infty \frac{1}{k-\nu}\, e^{k(y+b)}\, J_0(kR)\, dk. \end{aligned}$$