Preface to the online edition of 'Surface Waves'

Since its first publication in 1960, the article *Surface Waves* by John V. Wehausen and Edmund V. Laitone has been an inspirational resource for students and research workers in the various fields of science and engineering where water waves are important. This reference has been cited frequently in the literature of these fields, and even after the passage of nearly half a century it continues to serve a unique role. This may be attributed to its encyclopedic scope and to the scholarly efforts of the authors.

Surface Waves was originally published in Volume IX of the *Encyclopedia of Physics*, together with five somewhat shorter articles on other branches of fluid mechanics (by M. Schiffer, H. Cabannes, R. E. Meyer, R. Timman, and D. Gilbarg). This volume has been available mainly through university libraries, with limited distribution to individuals. It has been out of print for many years.

The importance of *Surface Waves* was anticipated in a review written by Professor F. Ursell for *Mathematical Reviews* (Review 10417, Volume 22, No. 10B, 1961, pp 1776-7). The following paragraphs are quoted from Ursell's review:

> Wehausen's contribution is a review article, and a most valuable and thorough one. Can it be that Wehausen has read critically all the 700 works listed in the bibliography? From sample tests the reviewer is inclined to think that he has. It is impossible to give more than a brief discussion of the contents. Chapter A outlines the scope of the work. Chapter B gives the exact equations and boundary conditions for viscous and inviscid fluids. Chapter C discusses with care the schemes of approximation leading to the classical infinitesimal-wave and shallow-water theories, and also contains certain exact consideratons on wave velocity, momentum, and energy. Chapter D, which forms the greater part of the review, is devoted almost entirely to the potential theory of infinitesimal waves. The velocity potential $\phi(x, y, z, t)$ satisfies Laplace's equation with simple linear boundary conditions. The general theory of this system is not yet understood. Thus for the important case of time-periodic motion the general uniqueness problem is still unsolved, and only a few partial results are known. A picture of surface-wave behaviour is, however, beginning to emerge from the solutions of a variety of boundary-value problems which are described in the review. Chapter F describes the known theory relating to the exact nonlinear equation, a part of the subject which will be less familiar to readers than the theory of infinitesimal waves.
>
> Among omissions (probably due to lack of space) is the comparison with experiments which have shown that much of the theoretical work on inviscid fluids is directly applicable to real fluids. It has also been noted that several sections contain no reference to any other author and are presumably due to Wehausen himself. One of the most noteworthy is section 15 on group velocity and the propagation of disturbances and of energy. That energy propagates with the group velocity has appeared to many students as an unexpected coincidence. It will appear less so after Wehausen's discussion.
>
> Wehausen deserves to be congratulated on a scholarly and well-written review. We now turn briefly to Chapter E on shallow-water waves, by Laitone. This author has preferred to concentrate on a few aspects rather than to give a survey of all that is known. (Particularly on some non-linear aspects our knowledge at present is slight.) The treatment is thorough and interesting.
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> To sum up, this article on surface waves is a most worthy contribution to the *Encyclopedia of Physics* which many workers in fluid mechanics would be glad to possess. Unfortunately the price of the complete volume is too high for a wide distribution. The publishers would perform a service by separate publication of this article, perhaps after a lapse of some time.

After a lapse of some forty years, Ursell's review seems prophetic. We have all been impressed by the quality and importance of *Surface Waves*, which has provided concise and authoritative references for much of our own work. To facilitate its use by the generations that are following us, we have developed a digitally scanned edition suitable for distribution online. We first proposed this concept to Professor Wehausen, and received his approval. He took the vital step of obtaining the electronic rights of the article from the original publisher Springer Verlag, which has also approved the the establishment of the online site. The essential principle, which we share with Professor Wehausen, is that the Online Edition of *Surface Waves* should be freely available to all interested persons, throughout the world. Individuals may download part or all of the article, redistribute it in digital format, and print hardcopies for personal or academic use. The electronic rights of the article *Surface Waves* have been assigned by Professor Wehausen and the Estate of Professor Laitone to the Regents of the University of California. Unauthorized commercialization of current and future online versions is strictly prohibited. The Committee on Surface Waves will continue to have editorial rights of the online article.

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Surface Waves.

By

JOHN V. WEHAUSEN and EDMUND V. LAITONE[1]

With 56 Figures.

A. Introduction.

The various problems of fluid motion treated in this article have in common the property that the fluid is subject to a gravitational force. In addition, in almost all cases they also have in common the presence of surfaces separating two fluids of different densities or, if only one fluid is present, of so-called free surfaces. However, not all fluid flows falling into this category are treated here: tidal motion is treated in Vol. XLVIII in the article by A. DEFANT. The observed properties of ocean waves and their generation by wind are treated in the article by H. U. ROLL, also in Vol. XLVIII. Closely related problems concerning flows with free surfaces are treated in the article by D. GILBARG in this volume.

The subject of water waves engaged many of the mathematicians and mathematical physicists of the last century. Moreover, the last several years have brought a renewed interest in the theory of water waves. In addition to this extensive literature on theoretical aspects of the subject, there have also been many experimental investigations, usually carried out by hydraulic engineers. Hydraulic engineers have also produced an extensive literature, both theoretical and experimental, on open channel flow, flow over weirs and through sluice-gates, etc.; included is a considerable literature on numerical and graphical methods of solving the equations involved. Oceanographers have produced their own literature, usually emphasizing different aspects of the subject. The theory of ship waves has produced its own literature.

All this material is pertinent to this article. Clearly some selection has to be made. We have followed roughly the following rules: Fundamental results are derived in full. The treatments of various special problems are selected so as to exemplify particular methods, other methods being mentioned only by literature citation. Experimental results are not usually reproduced, but references are given. Numerical methods of solving equations are not treated at all. The more special problems of hydraulic engineering are also not treated. Geophysical aspects which are omitted have already been mentioned.

Several excellent expositions of the theory of waves or of various parts of it already exist. We mention the following[2]: LAMB [1932, Chaps. VIII (pp. 250 to 362) and IX (pp. 363–475)]; BASSET [1888, Chap. XVII (pp. 144–187)]; WIEN [1900, Chap. V (pp. 166–224)]; KOCHIN, KIBEL', and ROZE [1948, Chap. 8 (pp. 394–526)]; MILNE-THOMSON [1956, Chap. XIV (pp. 374–431)]; AIRY (1845); BOUASSE (1924); AUERBACH (1931); THORADE (1931); SRETENSKII (1936); KHRISTIANOVICH (1938); KEULEGAN (1950); ECKART (1951); and STOKER (1957). The last cited book by STOKER gives an up-to-date account of much of the

[1] Chaps. A, B, C, D, F, G were prepared by J. V. WEHAUSEN, Chap. E by E. V. LAITONE. The former is much indebted to the Office of Naval Research, U.S. Navy, for support during the preparation of his chapters.

[2] References are collected at the end and identified in the text by author and date.

fundamental theory. For observation of waves of many kinds, Cornish (1910, see errata 1934) und Miche (1954) should be consulted. Shuleikin [1953, part 3 (pp. 213 to 292)] contains a general discussion of topics of interest in oceanography. Russell and Macmillan (1952) give a rather nontechnical discussion of ocean waves. A volume published by the Society of Naval Architects of Japan (Zôsen Kyôkai) contains expository papers on various aspects of water-wave theory related to ships [see Maruo (1957), Jinnaka (1957), Nishiyama (1957), Bessho (1957), and Inui (1957)].

For extensive bibliographies one should consult Thorade (1931, pp. 195 to 211); Sretenskii (1936, pp. 294—303); Kampé de Fériet (1932, pp. 225—229); and Stoker (1957, pp. 545—560). Sretenskii (1950, 1951) in a survey of the accomplishments of the USSR during the years 1917—1947 has given a rather complete bibliography of Russian papers during those years. Takao Inui (1954) has included a valuable bibliography of Japanese papers in a survey of Japanese contributions to the theory of ship waves. An interesting early history of the subject may be found in a paper by St. Venant and Flamant (1887). The treatise by the Weber brothers (1825) is still of interest for its content, and especially for its many references to and summaries of the early papers on water waves. The section on waves in the article on hydrodynamics by Love (1914), as modified by Appell, Beghin and Villat, in the *Encyclopédie des sciences mathématiques* gives brief indications of the contents of many of the papers published up to about 1912.

B. Mathematical formulation.

1. Coordinate systems and conventions. In the mathematical description of waves one may, as in fluid mechanics in general, describe the motion by describing either the paths of individual fluid particles ("Lagrangian" description) or the velocity (and acceleration) field in the region occupied by fluid at a given moment ("Eulerian" description). Generally, but not always, the Eulerian description will be used.

Rectangular coordinates may be used conveniently for almost all problems. The y-axis will be taken directed oppositely to the force of gravity, the x-axis and z-axis so as to form a right-handed system (i.e., if the y-axis is toward the top of the page and the x-axis is toward the right, the z-axis will point toward the reader). This is a somewhat unconventional choice for the z-axis, but has the obvious advantage that in two-dimensional problems one can delete z-dependent terms from the equations, have conventional (x, y) coordinates, and set $z = x + iy$ without ambiguity when complex-variable methods are convenient.

It seems hardly worth while to try to formulate rules concerning when a moving coordinate system is preferable to a fixed one. However, use of a moving coordinate system is clearly convenient in those cases where it allows one to formulate a problem in a time-independent manner.

The following well-established convention with regard to use of certain letters will be adhered to. The components of the velocity vector $\boldsymbol{v}$ will be denoted by u, v, w the pressure by p and the density by ϱ. The coefficient of viscosity of the fluid will be denoted by μ, the coefficient of kinematic viscosity, μ/ϱ, by ν. The acceleration resulting from gravity is denoted by g.

In the Eulerian formulation one seeks $\boldsymbol{v}, p$ and ϱ as functions of x, y, z, t i.e., at any instant t one seeks a vector function and two scalar functions defined on the region occupied by fluid at that instant. In the Lagrangian system one focuses attention on the trajectories of individual particles in the fluid: if a, b, c

are the coordinates of a particle at time $t=0$, then one seeks the position $x(a, b, c, t)$, $y(a, b, c, t)$, $z(a, b, c, t)$ of this point at a later time t. One may pass from one system to the other by means of the equations

$$\frac{dx}{dt} = u(x, y, z, t), \quad \frac{dy}{dt} = v(x, y, z, t), \quad \frac{dz}{dt} = w(x, y, z, t) \tag{1.1}$$

with $x=a$, $y=b$, $z=c$ at $t=0$ as initial conditions.

2. Equations of motion. Derivations of the fundamental equations describing fluid motion are available in many places (e.g., Vol. VIII, Part 1 of this Encyclopedia). The equations are reproduced here for convenience of reference.

The equation of continuity in Eulerian coordinates is

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho u)}{\partial x} + \frac{\partial(\varrho v)}{\partial y} + \frac{\partial(\varrho w)}{\partial z} = 0. \tag{2.1}$$

If the fluid is incompressible, but not necessarily homogeneous, $d\varrho/dt=0$ (but not necessarily $\partial\varrho/\partial t=0$) and Eq. (2.1) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.2}$$

In Lagrangian coordinates this may be written

$$\varrho(x, y, z, t)\, D = \varrho(a, b, c, 0) \tag{2.3}$$

where

$$D = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial z} \end{vmatrix}.$$

For an incompressible fluid $\varrho(x, y, z, t) = \varrho(a, b, c, 0)$ and (2.3) becomes

$$D = 1. \tag{2.4}$$

The dynamical equations take different forms according as one does or does not try to take account of viscosity. The Navier-Stokes equations for the motion of an incompressible viscous fluid, when the only external force is that of gravity, are as follows in Eulerian coordinates:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= - \frac{1}{\varrho}\frac{\partial p}{\partial x} + \frac{\mu}{\varrho}\Delta u, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -g - \frac{1}{\varrho}\frac{\partial p}{\partial y} + \frac{\mu}{\varrho}\Delta v, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= - \frac{1}{\varrho}\frac{\partial p}{\partial z} + \frac{\mu}{\varrho}\Delta w. \end{aligned}\right\} \tag{2.5}$$

If viscosity is neglected, the last two terms on the right side of the equations are to be deleted and one obtains the equations for an "ideal" fluid:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= - \frac{1}{\varrho}\frac{\partial p}{\partial x}, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -g - \frac{1}{\varrho}\frac{\partial p}{\partial y}, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= - \frac{1}{\varrho}\frac{\partial p}{\partial z}. \end{aligned}\right\} \tag{2.6}$$

In Lagrangian coordinates the latter equations become:

$$\left.\begin{aligned}
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a}+\left(g+\frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial a}+\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a}&=-\frac{1}{\varrho}\frac{\partial p}{\partial a},\\
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b}+\left(g+\frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial b}+\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b}&=-\frac{1}{\varrho}\frac{\partial p}{\partial b},\\
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c}+\left(g+\frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial c}+\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c}&=-\frac{1}{\varrho}\frac{\partial p}{\partial c}.
\end{aligned}\right\}\tag{2.7}$$

The equations of two-dimensional motion result if one deletes all terms containing z, w, and c.

The motion is called irrotational if it satisfies the additional equations

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=0,\qquad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0,\qquad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0,\tag{2.8}$$

or, in two-dimensional motion,

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0.\tag{2.8'}$$

In the case of irrotational motion there exists a potential function $\Phi(x, y, z, t)$ such that

$$u=\frac{\partial\Phi}{\partial x},\qquad v=\frac{\partial\Phi}{\partial y},\qquad w=\frac{\partial\Phi}{\partial z}.\tag{2.9}$$

It is a classical theorem of hydrodynamics [cf. LAMB (1932, §§ 17, 33)] that, if the motion of an inviscid fluid with $\varrho=\varrho(p)$ is irrotational at any instant, it is so thereafter. In particular, a motion started from rest is irrotational.

If $\varrho=\varrho(p)$ is the equation of state, the following integral of the equations of motion exists for irrotational motion:

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}(u^2+v^2+w^2)+g y+P=A(t)\tag{2.10}$$

where

$$P=\int_{p_0}^{p}\varrho^{-1}\,dp$$

and $A(t)$ is an arbitrary function of t. If the fluid is incompressible, the usual case in this article, ϱ is independent of p and the integral becomes:

see errata

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}+(u^2+v^2+w^2)+g y+\frac{p-p_0}{\varrho}=A(t).\tag{2.10'}$$

In this case one obtains also from (2.2) and (2.9)

$$\Delta\Phi\equiv\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0.\tag{2.11}$$

Even if the motion is not irrotational, there still exists an integral like (2.10) if the motion is steady, the so-called Bernoulli integral:

$$\tfrac{1}{2}(u^2+v^2+w^2)+g y+P=C.\tag{2.10''}$$

Here C is constant along a single streamline:

$$\frac{dx}{dt}=u(x,y,z),\qquad \frac{dy}{dt}=v(x,y,z),\qquad \frac{dz}{dt}=w(x,y,z),$$

but may vary from one streamline to another.

There will be occasion in the following to treat problems in moving coordinate systems. Let $Oxyz$ be a fixed coordinate system and $\bar{O}\bar{x}\bar{y}\bar{z}$ be a system moving with respect to $Oxyz$ but without rotation. Let $\boldsymbol{v}_0$ be the vector $\frac{d}{dt}O\bar{O}$, the velocity of a particle referred to $Oxyz$ be $\boldsymbol{v}$ and to $\bar{O}\bar{x}\bar{y}\bar{z}$ be $\bar{\boldsymbol{v}}$. Then $\boldsymbol{v}=\bar{\boldsymbol{v}}+\boldsymbol{v}_0$. We shall generally want either to describe the absolute motion $\boldsymbol{v}$ with respect to the moving coordinate system $\bar{O}\bar{x}\bar{y}\bar{z}$ or the relative motion $\bar{\boldsymbol{v}}$ with respect to this coordinate system. In either case the continuity equation remains the same in form

$$\frac{\partial\varrho}{\partial t}+\frac{\partial(\varrho u)}{\partial\bar{x}}+\frac{\partial(\varrho v)}{\partial\bar{y}}+\frac{\partial(\varrho w)}{\partial\bar{z}}=0,\qquad \varrho=\varrho(\bar{x},\bar{y},\bar{z},t) \tag{2.12}$$

or

$$\frac{\partial\varrho}{\partial t}+\frac{\partial(\varrho \bar{u})}{\partial\bar{x}}+\frac{\partial(\varrho \bar{v})}{\partial\bar{y}}+\frac{\partial(\varrho \bar{w})}{\partial\bar{z}}=0,\qquad \varrho=\varrho(\bar{x},\bar{y},\bar{z},t). \tag{2.13}$$

The dynamical equations for an ideal fluid for the absolute motion described in the moving coordinate system are:

$$\left.\begin{aligned}
\frac{\partial u}{\partial t}+(u-u_0)\frac{\partial u}{\partial\bar{x}}+(v-v_0)\frac{\partial u}{\partial\bar{y}}+(w-w_0)\frac{\partial u}{\partial\bar{z}}&=\qquad-\frac{1}{\varrho}\frac{\partial p}{\partial\bar{x}},\\
\frac{\partial v}{\partial t}+(u-u_0)\frac{\partial v}{\partial\bar{x}}+(v-v_0)\frac{\partial v}{\partial\bar{y}}+(w-w_0)\frac{\partial v}{\partial\bar{z}}&=-g-\frac{1}{\varrho}\frac{\partial p}{\partial\bar{y}},\\
\frac{\partial w}{\partial t}+(u-u_0)\frac{\partial w}{\partial\bar{x}}+(v-v_0)\frac{\partial w}{\partial\bar{y}}+(w-w_0)\frac{\partial w}{\partial\bar{z}}&=\qquad-\frac{1}{\varrho}\frac{\partial p}{\partial\bar{z}}.
\end{aligned}\right\} \tag{2.14}$$

The dynamical equations for the relative motion are:

$$\left.\begin{aligned}
\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}}+\bar{w}\frac{\partial\bar{u}}{\partial\bar{z}}&=\qquad-\frac{1}{\varrho}\frac{\partial p}{\partial\bar{x}}-\dot{u}_0,\\
\frac{\partial\bar{v}}{\partial t}+\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}+\bar{w}\frac{\partial\bar{v}}{\partial\bar{z}}&=-g-\frac{1}{\varrho}\frac{\partial p}{\partial\bar{y}}-\dot{v}_0,\\
\frac{\partial\bar{w}}{\partial t}+\bar{u}\frac{\partial\bar{w}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{w}}{\partial\bar{y}}+\bar{w}\frac{\partial\bar{w}}{\partial\bar{z}}&=\qquad-\frac{1}{\varrho}\frac{\partial p}{\partial\bar{z}}-\dot{w}_0.
\end{aligned}\right\} \tag{2.15}$$

Let us suppose that the motion is irrotational and let $\Phi(x,y,z,t)$ be the velocity potential for the absolute motion in the fixed coordinate system. Let

$$\Phi(x,y,z,t)=\Phi\left(\bar{x}+\int^t u_0\,dt,\bar{y}+\int^t v_0\,dt,\bar{z}+\int^t w_0\,dt,t\right)=\bar{\Phi}(\bar{x},\bar{y},\bar{z},t).$$

Then $\bar{\Phi}$ is the velocity potential for the absolute motion in the moving coordinate system:

$$\frac{\partial\bar{\Phi}}{\partial\bar{x}}=u,\qquad\frac{\partial\bar{\Phi}}{\partial\bar{y}}=v,\qquad\frac{\partial\bar{\Phi}}{\partial\bar{z}}=w.$$

The integral (2.10) becomes:

$$\frac{\partial\bar{\Phi}}{\partial t}+\frac{1}{2}\left[(u-u_0)^2+(v-v_0)^2+(w-w_0)^2\right]+g\bar{y}+P=\bar{A}(t), \tag{2.16}$$

where $\bar{A}(t)=A(t)+\frac{1}{2}(u_0^2+v_0^2+w_0^2)-g\int^t v_0\,dt$. If one defines $\bar{\bar{\Phi}}$ by

$$\bar{\bar{\Phi}}(\bar{x},\bar{y},\bar{z},t)=\bar{\Phi}(\bar{x},\bar{y},\bar{z},t)-u_0\bar{x}-v_0\bar{y}-w_0\bar{z},$$

then $\bar{\bar{\Phi}}$ is the velocity potential for the relative motion:

$$\frac{\partial\bar{\bar{\Phi}}}{\partial\bar{x}}=\bar{u},\qquad\frac{\partial\bar{\bar{\Phi}}}{\partial\bar{y}}=\bar{v},\qquad\frac{\partial\bar{\bar{\Phi}}}{\partial\bar{z}}=\bar{w},$$

and the integral (2.10) may be written:

$$\frac{\partial \bar{\bar{\Phi}}}{\partial t} + \dot{u}_0 \bar{x} + \dot{v}_0 \bar{y} + \dot{w}_0 \bar{z} + \frac{1}{2}(\bar{u}^2 + \bar{v}^2 + \bar{w}^2) + g\bar{y} + P = \bar{A}(t). \tag{2.17}$$

The more general equations when the system $\bar{O}\bar{x}\bar{y}\bar{z}$ is also rotating will not be necessary for this article.

3. Boundary conditions at an interface. Let us now suppose that we are given two immiscible fluids with a common boundary surface, $S(t)$. The one fluid, with density ϱ_1 and viscosity μ_1, will occupy region $R_1(t)$; the other, with density ϱ_2 and viscosity μ_2, the region $R_2(t)$. Let $F(x, y, z, t) = 0$ describe the surface $S(t)$; we assume $F_x^2 + F_y^2 + F_z^2 > 0$ (where $F_x = \partial F/\partial x$, etc.).

The first condition which the surface $S(t)$ must satisfy is a kinematic one. As the surface moves, the velocity of a point (x, y, z) on the surface in the direction of the normal to the surface is given by $-F_t/\sqrt{F_x^2 + F_y^2 + F_z^2}$. Here one takes the normal in the direction (F_x, F_y, F_z). A particle of fluid at the same point of the surface at that instant will have a velocity component in the direction of the surface normal given by $\frac{u F_x + v F_y + w F_z}{\sqrt{F_x^2 + F_y^2 + F_z^2}} = v_n$. For $S(t)$ to be a bounding surface means, of course, that there can be no transfer of matter across the surface. Consequently the following equation must be satisfied:

$$u F_x + v F_y + w F_z = - F_t, \tag{3.1}$$

where we have used the assumption $F_x^2 + F_y^2 + F_z^2 > 0$ in dropping the denominators. If one defines the "material derivative" by the equation

$$\frac{DF}{Dt} = u F_x + v F_y + w F_z + F_t,$$

then (3.1) is the same as

$$\frac{DF}{Dt} = 0. \tag{3.1'}$$

This condition must be satisfied by any bounding surface, whether an interface or a rigid boundary[1].

There are further dynamical conditions to be satisfied at an interface. Let us first consider the general case of viscous fluids with surface tension at the interface. The following assumptions are made:

1. The effect of surface tension as one passes through the interface is to produce a discontinuity in the normal stress proportional to the mean curvature of the boundary surface.

2. For viscous fluids the tangential stress must be continuous as one passes through the interface.

3. For viscous fluids the tangential component of the velocity must be continuous as one passes through the interface.

In order to formulate these statements in mathematical language, we introduce the following notation. Let $g(x, y, z)$ be some function defined in both R_1 and R_2 and let (x_0, y_0, z_0) be a point of the interface S. Assuming that the following limit exists, we shall write

$$g_i(x_0, y_0, z_0) = \lim g(x, y, z) \quad \text{as} \quad (x, y, z) \to (x_0, y_0, z_0), (x, y, z) \text{ in } R_i,$$

[1] For further discussion of this condition see C. TRUESDELL: Bull. Tech. Univ. Istanbul **3** (1950), No. 1, 71–78 (1951); L. LICHTENSTEIN: Grundlagen der Hydromechanik, pp. 159 to 170, 234ff. Berlin: Springer 1929.

and

$$[g(x_0,y_0,z_0)] = g_2(x_0,y_0,z_0) - g_1(x_0,y_0,z_0).$$

Let the components of the stress tensor be denoted by

$$\begin{matrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}. \end{matrix}$$

Consider an element of area of the surface S at a point (x, y, z) of S. Let the unit normal vector to S at (x, y, z) be (l, m, n). Then the surface element will have associated with it the stress vector with components:

$$\sigma_{xx}l + \sigma_{xy}m + \sigma_{xz}n, \quad \sigma_{yx}l + \sigma_{yy}m + \sigma_{yz}n, \quad \sigma_{zx}l + \sigma_{zy}m + \sigma_{zz}n.$$

Let R_1 and R_2 be the principal radii of curvature of S at (x, y, z). Then 1. and 2. are combined in the one equation

$$\left.\begin{aligned} [\sigma_{xx}l + \sigma_{xy}m + \sigma_{xz}n] &= T(R_1^{-1} + R_2^{-1})\, l, \\ [\sigma_{yx}l + \sigma_{yy}m + \sigma_{yz}n] &= T(R_1^{-1} + R_2^{-1})\, m, \\ [\sigma_{zx}l + \sigma_{zy}m + \sigma_{zz}n] &= T(R_1^{-1} + R_2^{-1})\, n, \end{aligned}\right\} \tag{3.2}$$

where T is a constant of proportionality depending upon the two fluids (and their temperatures, but this will not be considered here). T is called the coefficient of surface tension[1].

The kinematic condition imposed in (3.1) is clearly equivalent to continuity of the normal component of the velocity as one passes through S. Consequently, the condition 3. above may be combined with this to give

$$u_1 = u_2, \quad v_1 = v_2, \quad w_1 = w_2. \tag{3.3}$$

In the linearized theory of viscosity the stress tensor for an incompressible fluid is given by

$$\left.\begin{matrix} p - 2\mu u_x & -\mu(u_y + v_x) & -\mu(u_z + w_x) \\ -\mu(v_x + u_y) & p - 2\mu v_y & -\mu(v_z + w_y) \\ -\mu(w_x + u_z) & \mu(w_y + v_z) & p - 2\mu w_z. \end{matrix}\right\} \tag{3.4}$$

see errata

The geometric quantity $R_1^{-1} + R_2^{-1}$ is given by the formula[2]

$$\left.\begin{aligned} \frac{1}{R_1} + \frac{1}{R_2} &= -\frac{\partial}{\partial x}\frac{F_x}{\sqrt{F_x^2+F_y^2+F_z^2}} - \frac{\partial}{\partial y}\frac{F_y}{\sqrt{F_x^2+F_y^2+F_z^2}} - \frac{\partial}{\partial z}\frac{F_z}{\sqrt{F_x^2+F_y^2+F_z^2}} \\ &= -\frac{F_{xx}(F_y^2+F_z^2) + F_{yy}(F_x^2+F_z^2) + F_{zz}(F_x^2+F_y^2) - 2(F_{xy}F_xF_y + F_{yz}F_yF_z + F_{zx}F_zF_x)}{[F_x^2+F_y^2+F_z^2]^{\frac{3}{2}}}. \end{aligned}\right\} \tag{3.5}$$

The sign is so selected that, if it is positive, the direction of increase of the normal component of the stress vector at the interface is in the direction

$$(l, m, n) = \left(\frac{F_x}{\sqrt{F_x^2+F_y^2+F_z^2}}, \frac{F_y}{\sqrt{F_x^2+F_y^2+F_z^2}}, \frac{F_z}{\sqrt{F_x^2+F_y^2+F_z^2}}\right). \tag{3.6}$$

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[1] For an air-water interface $T = 72.8$ dynes/cm at 20° C, for mercury-air interface $T = 485$ dynes/cm at 20° C, for a mercury-water interface $T = 412$ dynes/cm, for benzene-air $T = 28.9$ dynes/cm at 20° C, for liquid helium-helium vapor $T = 0.24$ dynes/cm at −270° C.

[2] See, e.g., A. DUSCHEK and W. MAYER: Lehrbuch der Differentialgeometrie, Vol. I, pp. 150–152. Leipzig u. Berlin: Teubner 1930.

In the case of a surface given by $y=\eta(x,z)$ Eq. (3.5) becomes

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{\eta_{xx}(1+\eta_z^2)+\eta_{zz}(1+\eta_x^2)-2\eta_{xz}\eta_x\eta_z}{(1+\eta_x^2+\eta_z^2)^{\frac{3}{2}}}, \tag{3.5'}$$

where the direction of increase is upwards. In the case of two-dimensional motion this simplifies further to the well-known formula

$$\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}. \tag{3.5''}$$

If one now substitutes (3.4) to (3.6) in (3.2), one obtains the general boundary condition at the interface. The result is unwieldy in its general form[1].

If the interface is given by $y=\eta(x,z)$, the boundary condition becomes

$$\left.\begin{aligned}
[p]\,\eta_x-\{2[\mu u_x]\,\eta_x-[\mu(u_y+v_x)]+[\mu(u_z+w_x)]\,\eta_z\} &= T(R_1^{-1}+R_2^{-1})\,\eta_x,\\
[p]+\{[\mu(v_x+u_y)]\,\eta_x-2[\mu v_y)+[\mu(v_z+w_y)]\,\eta_z\} &= T(R_1^{-1}+R_2^{-1}),\\
[p]\,\eta_z-\{[\mu(w_x+u_z)]\,\eta_x-[\mu(w_y+v_z)]+2[\mu w_z]\,\eta_z\} &= T(R_1^{-1}+R_2^{-1})\,\eta_z
\end{aligned}\right\} \tag{3.7}$$

with $R_1^{-1}+R_2^{-1}$ given by (3.5'). Here fluid$_1$ is the lower and fluid$_2$ the upper fluid. For two-dimensional motion the equations take the following form:

$$\left.\begin{aligned}
[p]\,\eta'(x)-\{2[\mu u_x]\,\eta'(x)-[\mu(u_y+v_x)]\} &= T\frac{\eta''(x)}{(1+\eta'(x)^2)^{\frac{3}{2}}}\,\eta'(x),\\
[p]+\{[\mu(u_y+v_x)]\,\eta'(x)-2[\mu v_y]\} &= T\frac{\eta''(x)}{(1+\eta'(x)^2)^{\frac{3}{2}}}.
\end{aligned}\right\} \tag{3.8}$$

One may also write this condition in terms of the components of the stress vector normal and tangential to the interface:

see errata

$$\left.\begin{aligned}
[p]-2\,\frac{[\mu u_x]\,\eta'^2-[\mu(u_y+v_x)]\,\eta'+[\mu v_y]}{(1+\eta'^2)^{\frac{1}{2}}} &= T\frac{\eta''(x)}{(1+\eta'^2)^{\frac{3}{2}}},\\
\frac{2[\mu(u_x-v_y)]\,\eta'+[\mu(u_y+v_x)]\,(\eta'^2-1)}{(1+\eta'^2)^{\frac{1}{2}}} &= 0.
\end{aligned}\right\} \tag{3.8'}$$

If surface tension is to be neglected, one obtains the resulting boundary condition by setting $T=0$ in the various equations above. In this case, Eq. (3.2) simply states the continuity of the stress vector as one passes through the interface.

If viscosity is neglected, but not necessarily surface tension, the condition on the stress vector becomes simply

$$[p]=T(R_1^{-1}+R_2^{-1}), \tag{3.9}$$

where, of course, the mean curvature is still given by (3.5). The other boundary condition (3.3) changes more drastically upon neglecting viscosity: Condition 3. stating the continuity of the tangential component of velocity is abandoned. The continuity of the normal component, i.e. (3.1), is still retained, of course.

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[1] In tensor notation the condition is somewhat more perspicuous:

$$\{[p]\,\delta_{ij}-[\mu(u_{i,j}+u_{j,i})]\}\frac{F_{,j}}{(F_{,k}F_{,k})^{\frac{1}{2}}}=T\,\frac{F_{,r}F_{,s}F_{,rs}-F_{,r}F_{,r}F_{,ss}}{(F_{,k}F_{,k})^{\frac{3}{2}}}\cdot\frac{F_{,i}}{(F_{,k}F_{,k})^{\frac{1}{2}}},$$

where $(x_1, x_2, x_3)=(x, y, z)$, $(u_1, u_2, u_3)=(u, v, w)$ and $F_{,i}=\partial F/\partial x_i$. We have refrained from using tensor notation because its particular advantages cannot in general be exploited here.

Condition 2. concerning the tangential stress is satisfied vacuously for an inviscid fluid.

So far we have considered the boundary condition at an interface between two fluids. If the second fluid is absent, the boundary surface for the first fluid is called a "free surface". Usually the pressure above a free surface is assumed to be some given function, say $p_2(x, y, z, t)$, of position and time; in most cases it is taken to be a constant, either an assumed atmospheric pressure or zero. The boundary conditions concerning the stress vector at a free surface are slight modifications of those for an interface, and can be obtained by setting $\mu_2=0$, $\lambda_2=0$. The result is again somewhat unwieldy in its complete form[1]. For an incompressible fluid it is:

$$\left.\begin{aligned}(\bar{p}-p)\,F_x+\mu\,\{2u_xF_x+(u_y+v_x)\,F_y+(u_z+w_x)\,F_z\}&=T(R_1^{-1}+R_2^{-1})\,F_x,\\ (\bar{p}-p)\,F_y+\mu\,\{(v_x+u_y)\,F_x+2v_yF_y+(v_z+w_y)\,F_z\}&=T(R_1^{-1}+R_2^{-1})\,F_y,\\ (\bar{p}-p)\,F_z+\mu\,\{(w_x+u_z)\,F_x+(w_y+v_z)\,F_y+2v_zF_z\}&=T(R_1^{-1}+R_2^{-1})\,F_z.\end{aligned}\right\}\qquad(3.10)$$

Here we have written $\bar{p}$ for p_2, and μ for μ_1; $\bar{p}, p, u_x, \ldots$ are to be evaluated at $F(x, y, z, t)=0$.

The case with which we shall be chiefly concerned is that of an inviscid fluid without surface tension and with $\bar{p}(x, y, z, t)=p_0$, a constant. In this case the boundary condition reduces to the single equation

$$p(x, y, z, t)=p_0 \qquad (3.11)$$

on $F(x, y, z, t)=0$. If the motion is irrotational and incompressible, one may determine p explicitly from (2.10′) so that (3.10) becomes

$$\Phi_t+\tfrac{1}{2}(u^2+v^2+w^2)+gy=A(t) \qquad (3.11')$$

to be satisfied on $F(x, y, z, t)=0$.

In the case of steady motion of an incompressible fluid, the Bernoulli integral (2.10″) still exists even if the motion is rotational. Consequently, in certain two-dimensional problems of steady motion in which the free surface is a streamline one continues to have a boundary condition like (3.10″):

$$\frac{1}{2}(u^2+v^2)+gy+\frac{p_0}{\varrho}=C, \qquad (3.11'')$$

to be satisfied on $F(x, y)=0$.

4. Boundary conditions on rigid surfaces. Let the equation of the rigid surface be given by the equation $G(x, y, z, t)=0$. Then in the case of an inviscid fluid the condition to be satisfied on $G=0$ is the same as the kinematic condition (3.1):

$$u\,G_x+v\,G_y+w\,G_z=-G_t, \qquad (4.1)$$

i.e., the component of velocity of the fluid normal to the surface must equal the velocity of the rigid surface in the direction of its normal.

If the fluid is viscous, it must stick to a solid boundary and move with it without slippage. An equation of the form $G(x, y, z, t)=0$ is not suitable for

[1] In tensor notation it may be written:

see errata

$$\{(\bar{p}-p)\,\delta_{ij}+\mu\,(u_{i,j}+u_{j,i})\}\,\frac{F_{,j}}{(F_{,k}F_{,k})^{\frac{1}{2}}}=T\,\frac{F_{,r}F_{,s}F_{,rs}-F_{,r}F_{,r}F_{,ss}}{(F_{,k}F_{,k})^{\frac{3}{2}}}\,\frac{F_{,j}}{(F_{,k}F_{,k})^{\frac{1}{2}}}.$$

Here we have written $\bar{p}$ for p_2 and λ, μ for λ_1, μ_1. All variable quantities in the braces are, of course, to be evaluated at the free surface $F=0$.

formulating this statement in equations (e.g., $x^2+y^2+z^2=a^2$ does not distinguish between a rotating and a stationary sphere). Let the surface be given in parametric coordinates by: $x=X(r,s,t)$, $y=Y(r,s,t)$, $z=Z(r,s,t)$, where a given point on the surface corresponds to a given pair of values (r, s). Then the condition for viscous fluids may be written:

$$u=\frac{\partial X}{\partial t}, \quad v=\frac{\partial Y}{\partial t}, \quad w=\frac{\partial Z}{\partial t}. \tag{4.2}$$

If a solid boundary penetrates the free surface (or an interface) of a viscous fluid, there will be some difference in treatment of the boundary condition according as the fluid wets the surface or not. In the case of mercury sloshing in a clean glass basin, the fluid pulls free of the surface as it moves up and down, whereas water in the same basin will continue to adhere to any part of the walls already wetted. Furthermore, if surface tension is taken into account, the angle of contact of the free surface with the solid surface will enter into the boundary condition; in the first case mentioned above the angle may vary according as the liquid is rising or falling along the wall[1]. Although attempts to prove very general existence theorems for fluid motion would presumably take such complications into account, they are usually neglected in most solutions of special problems, there being indeed little choice in the matter.

5. Other types of boundary surfaces. Geophysical problems sometimes suggest situations in which there is an interface between a fluid and an elastic medium. This may occur, for example, in the study of the effect of ocean waves on the ocean floor, as in LONGUET-HIGGINS' (1950) theory of microseisms. Other possibilities are suggested by wave motion on a body of water covered with an ice sheet or at an interface between two fluids separated by an elastic membrane or plate. In one series of investigations the ice sheet has been assumed broken into pieces small with respect to the prevalent wave lengths. In this case only the density of the ice layer enters into the modified boundary condition [see PETERS (1950), KELLER and GOLDSTEIN (1953), KELLER and WEITZ (1953), SHAPIRO and SIMPSON (1953)]. Waves in a thin plate over an infinitely deep fluid have been considered briefly by LANDAU and LIFSHITS (1953, pp. 762—763), but with neglect of gravity. GREENHILL (1887, p. 68; 1916) included gravity.

The kinematic boundary condition (3.1) must always hold. The dynamical conditions will depend upon the nature of the assumptions. The matter will not be further considered here.

C. Preliminary remarks and developments.

6. Classification of problems. Most of the theory of water waves is concerned either with elucidating some general aspects of wave motion or with predicting the behavior of waves in the presence of some special configuration of interest to oceanographers, hydraulic engineers, or ship designers. Unfortunately, even some of the apparently simplest problems have proved too difficult to solve in their most complete formulation. Approximations have been necessary, and in many cases the problems which have been solved are those which could be solved by the approximate methods in use. An examination of the theory also shows that many of the concepts and definitions are almost inextricably bound up with these methods of approximation, following rather than preceding the making of the approximation.

[1] See, e.g., R. S. BURDON: Surface tension and the spreading of liquids, pp. 76—82. Cambridge 1949.

The nature of the approximations used in treating a particular problem provides a natural way of classifying it. First there are the assumptions concerning the properties of the fluid: viscous or inviscid, compressible or incompressible, surface tension or not. Although assuming the fluid to be inviscid, incompressible, and without surface tension simplifies the equations, they are still not easily manageable, even for the simplest kinds of problems. Other approximations of a different nature are required. These are in a sense mathematical approximations. Their physical significance is not in restricting the nature of the fluid but in restricting the character of the waves and the boundary configuration. The kind of mathematical approximation used provides another means of classifying problems, and is the principal one which will be used in this article. There are two principal methods of approximation, explained below in Sect. 10, the infinitesimal-wave approximation and the shallow-water approximation. Thus, the development of these two approximate theories and of the exact theory constitutes the bulk of this article.

7. Progressive waves and wave velocity. Standing waves. It will be convenient to call any motion of a fluid in a gravitational field with a free surface or an interface a *wave motion.*

If the velocity components, pressure, and free surface or interface may be expressed in the form

$$\boldsymbol{v} = \boldsymbol{v}(x - c\,t, y, z), \qquad p = p(x - c\,t, y, z), \qquad y = \eta(x - c\,t, z),$$

respectively, then the wave motion will be said to be a *progressive wave* travelling in the direction Ox. In this case a change to a moving coordinate system with $x' = x - ct$, $y' = y$, $z' = z$ reduces the motion to steady motion with respect to the moving coordinate system. With respect to the fixed coordinate system the profile of the free surface or interface is being transported without change of form in the direction Ox with velocity c. It might seem reasonable therefore to call c the velocity of propagation of the progressive wave.

However, STOKES (1849; or 1880, pp. 202ff.) has pointed out that the velocity of propagation of the profile of the free surface does not by itself give a useful definition of wave velocity. Let the fluid be inviscid, either infinitely deep or with a horizontal bottom, and unlimited otherwise. Now let the whole fluid in the progressive wave described above be transported with velocity C (positive or negative) in the direction Ox. Then the motion will still be consistent with the laws of fluid mechanics, the various parts of the fluid will move the same relatively to each other, but the velocity of propagation of the profile will be arbitrary, depending upon the choice of C. What is required for a useful definition of wave velocity is the velocity of propagation of the profile with respect to a coordinate system fixed in some sense in the fluid.

In the case of an infinitely deep fluid, if the axes may be chosen so that as $y \to -\infty$ the velocity relative to these axes vanishes, then one may reasonably measure the profile velocity with respect to these. If the motion far ahead or far behind the disturbance approaches a uniform velocity (possibly zero), then axes moving with the fluid with this velocity may be used. When the disturbance does not behave thus (as in the case of periodic waves) and when the depth is finite, there is no longer an obvious way to select a set of reference axes.

In order to put the problem somewhat differently, let us assume that the wave motion is given as a steady motion with velocity field $\boldsymbol{v}(x, y)$ and free surface $y = \eta(x)$. We wish to find a moving coordinate system $x' = x - u_0 t$, $y' = y$, so that in some sense the relative motion vanishes on the average. We now have the free surface given by $y'=\eta(x'+u_0 t)$ and the relative velocity by $\boldsymbol{v}'(x'+u_0 t, y') = \boldsymbol{v}(x'+u_0 t, y') - u_0\,\boldsymbol{i}$. How is u_0 to be chosen? STOKES made two suggestions. One is to define it by the equation

$$\left.\begin{aligned} &\lim_{a\to-\infty,\, b\to\infty} \frac{1}{b-a}\int_a^b dx' \int_{-h}^{\eta(x'+u_0 t)} u'(x'+u_0 t, y')\, dy' \\ &= \lim_{a\to-\infty,\, b\to\infty} \frac{1}{b-a}\int_a^b dx \int_{-h}^{\eta(x)} [u(x,y)-u_0]\, dy = 0, \end{aligned}\right\} \tag{7.1}$$

where $y=-h$ is the equation for the bottom. In case the motion is periodic, with period λ, the defining equation may be written

$$\int_0^\lambda dx \int_{-h}^{\eta(x)} [u(x,y)-u_0)]\, dy = 0. \tag{7.2}$$

If one notes that the mean depth is given by

$$\lim \frac{1}{b-a}\int_a^b [\eta(x)+h]\, dx \quad \text{or} \quad \frac{1}{\lambda}\int_0^\lambda [\eta(x)+h]\, dx,$$

then one sees that, with h' as mean depth,

$$u_0 h' = Q \tag{7.3}$$

where Q is the average discharge rate per unit width. u_0 is thus defined so that the average discharge rate with respect to the (x', y') coordinate system is zero. u_0 is usually denoted by c'.

STOKES' other suggestion was to define u_0 by

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T u'(x'+u_0 t, y')\, dt = 0 \tag{7.4}$$

or

$$\begin{aligned} u_0 &= \lim_{T\to\infty} \frac{1}{T}\int_0^T u(x'+u_0 t, y')\, dt \\ &= \lim_{T\to\infty} \frac{1}{u_0 T}\int_{x'}^{x'+u_0 T} u(x,y)\, dx = \lim_{a\to\infty} \frac{1}{a}\int_{x'}^{x'+a} u(x,y)\, dx. \end{aligned}$$

If u is periodic in x with period λ, one may write

see errata

$$u_0 = \frac{1}{\lambda}\int_{x'}^{x'+\lambda} u(x,y)\, dx. \tag{7.5}$$

In either case, for the definition to be useful u_0 must be independent of x' and y. If u is bounded, it follows easily that $\partial u_0/\partial x' = 0$ for both cases. If the motion is irrotational, $u_y = v_x$ and it follows again that $\partial u_0/\partial y = 0$ if v is bounded. Wave velocity defined in this manner is usually denoted by c. For the two special cases considered earlier, the two definitions coincide.

The definition of wave velocity in cases where the motion cannot be reduced to a steady motion is no longer straightforward. In many cases of interest, the asymptotic behavior of the motion for large positive or negative x allows one to define a wave velocity in a manner similar to that above. In more complicated wave motions one may simply follow the motion of some special phase of the profile, say a crest. This provides, for example, a definition of phase velocity for a cylindrical wave.

A general definition of *standing wave* is somewhat more awkward to formulate than that for a progressive wave. For the case of a plane wave, the free surface $y=\eta(x,t)$ must be periodic in each of x and t, with wave length λ and period τ, say. In addition, the curves in the (x,t)-plane represented by $\eta(x,t)=0$, where $y=0$ is the undisturbed surface, must consist of two sets of curves oscillating about the lines $x=\frac{1}{2}n\lambda$ and $t=\frac{1}{2}n\tau$, $n=0, \pm 1, \ldots$. For progressive waves the curves $\eta(x,t)=0$ consist of a single set of straight lines, all parallel to $x-ct=0$. The prototype for the standing wave is the surface defined by, say, $y=\sin 2\pi x/\lambda \times \cos 2\pi t/\tau$. However, as shown by both PENNEY and PRICE (1952b) and by SEKERZH-ZENKOVICH (1947), neither set of curves $\eta(x,t)=0$ consists of straight lines, or even fixed curves, for standing waves of finite amplitude.

There remains the problem of establishing that progressive and standing waves exist under suitable boundary conditions. For the exact boundary conditions for a perfect fluid, the existence of progressive waves was first established by LEVI-CIVITA (1925) and NEKRASOV (1921, 1922). The existence of standing waves satisfying the exact boundary conditions is apparently an open question.

8. Energy. Let $T(t)$ be a region occupied by a perfect fluid with a boundary $S(t)$ represented by

$$F(x, y, z, t) = 0,$$

the representation being chosen so that (F_x, F_y, F_z) is in the direction of the exterior normal. The surface $S(t)$ moves independently of the motion of the fluid. It is assumed that $T(t)$ contains no singularities of $\boldsymbol{v}$ and that surface tension does not act upon the surface $S(t)$ at any time. The energy of the fluid contained in $T(t)$ is given by

$$E = \iiint_{T(t)} \left[\tfrac{1}{2}\varrho(u^2+v^2+w^2) + \varrho g y\right] d\tau. \tag{8.1}$$

For irrotational motion of an inviscid incompressible fluid, one may use (2.10′) and express E by

$$E = \iiint_{T(t)} \left[-p - \varrho \frac{\partial \Phi}{\partial t}\right] d\tau.$$

[Here Φ has been redefined so that $A(t)$ may be set equal to zero.] One may now compute dE/dt by using the general formula:

$$\frac{d}{dt}\iiint_{T(t)} f(x,y,z,t)\,d\tau = \iiint_{T(t)} f_t(x,y,z,t)\,d\tau + \iint_{S(t)} f(x,y,z,t)\frac{-F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\,d\sigma.$$

One finds [cf. F. JOHN (1949, p. 19ff.), which we follow closely here]:

$$\begin{aligned}\frac{dE}{dt} &= \iiint_{T(t)} \varrho \operatorname{grad}\Phi \cdot \operatorname{grad}\Phi_t\,d\tau + \iint_{S(t)} [\varrho\Phi_t + p]\frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\,d\sigma \\ &= \iint_{S(t)} \varrho\Phi_t \frac{\partial\Phi}{\partial n}\,d\sigma + \iint_{S(t)} [\varrho\Phi_t + p]\frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\,d\sigma\end{aligned}$$

by GREEN'S Theorem and the equation of continuity. Finally,

$$\frac{dE}{dt} = \iint_{S(t)} \left\{\varrho\Phi_t\left[\frac{\partial\Phi}{\partial n} + \frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right] + p\frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right\} d\sigma. \tag{8.2}$$

We recall that $-F_t/\sqrt{F_x^2+F_y^2+F_z^2}$ is the velocity of $S(t)$ in the direction of the exterior normal. Two cases are of special interest. If $S(t)$ is a "physical" boundary, i.e., one moving with the fluid, then the first summand vanishes and one finds

$$\frac{dE}{dt} = -\iint\limits_{S(t)} p\,\frac{\partial\Phi}{\partial n}\,d\sigma \tag{8.3}$$

[cf. LAMB, Hydrodynamics, p. 9, Eq. (5)]. If $S(t)$ is a fixed "geometrical" boundary, then $F_t=0$ and one gets

$$\frac{dE}{dt} = \iint\limits_{S(t)} \varrho\,\Phi_t\,\frac{\partial\Phi}{\partial n}\,d\sigma. \tag{8.4}$$

If one considers any portion of $S(t)$, then the integral of (8.2) taken over this portion and with a minus sign gives the rate of flow of energy through this portion of $S(t)$. In case a part of $S(t)$ is a physical boundary which is fixed, $\partial\Phi/\partial n=0$ and the flow through this part is zero. The same conclusion holds for any portion of $S(t)$ that is a free surface, for then $p=0$.

If one has a progressive wave moving to the right with $\Phi(x, y, z, t) = \varphi(x-ct, y, z)$ and takes S as a region in the fixed plane $x=x_0$, then the rate of flow of energy through S in the positive direction is given by

$$\iint\limits_S \varrho\,c\,\varphi_x^2(x_0-c\,t, y, z)\,dy\,dz \geqq 0, \tag{8.5}$$

i.e., energy always flows in the direction of the wave.

In cases where one is dealing with waves generated by moving bodies, it is frequently possible to choose the region T so that no energy is lost from it, the latter being true only as an average if the motion is periodic in time. As an example, consider a body moving steadily with velocity c in the x-direction in an infinite ocean with horizontal bottom. In addition to the boundary conditions on the body, free surface, and bottom, we assume that the motion vanishes (in the limit) far ahead and to the sides of the body. The surface $S(t)$ may then be chosen as a plane M: $x-ct-a=0$ far ahead, another plane N: $-(x-ct)+b=0$ behind the body, planes R and L: $z=\pm a$ on either side, and the bottom H, the wetted surface of the body B, and the part of the free surface F included between the body and the planes. The energy within this region is clearly constant, and one easily obtains, with $\Phi(x, y, z, t)=\varphi(x-ct, y, z)$:

$$0 = -\iint\limits_B p\,\frac{\partial\varphi}{\partial n}\,d\sigma - \iint\limits_{M+N+R+L} \varrho\,c\,\frac{\partial\varphi}{\partial x}\,\frac{\partial\varphi}{\partial n}\,d\sigma + c\iint\limits_M \left[\frac{1}{2}\,\varrho\,(\varphi_x^2+\varphi_y^2+\varphi_z^2)+\varrho\,g\,y\right]d\sigma -$$
$$- c\iint\limits_N \left[\frac{1}{2}\,\varrho\,(\varphi_x^2+\varphi_y^2+\varphi_z^2)+\varrho\,g\,y\right]d\sigma.$$

Since on B one has $\partial\varphi/\partial n=c\cos(n, x)$, one finds for the first integral, remembering that $\boldsymbol{n}$ points into the body,

$$-\iint\limits_B p\,\frac{\partial\varphi}{\partial n}\,d\sigma = -c\iint\limits_B p\cos(n, x)\,d\sigma = R\,c,$$

where R is the force on the body. The parts of the second integral over M, R, L vanish as $a\to\infty$ and similarly for the first summand in the third integral. The terms in $\varrho g y$ give

$$\int_{-a}^{a} dz \int_{-h}^{\eta(a,z)} \varrho g y\, dy - \int_{-a}^{a} dz \int_{-h}^{\eta(b,z)} \varrho g y\, dy = \int_{-a}^{a} \tfrac{1}{2} \varrho g \left[\eta^2(a,z) - \eta^2(b,z)\right] dz$$

which, as $a \to \infty$, converges to

$$-\tfrac{1}{2} \varrho g \int_{-\infty}^{\infty} \eta^2(b,z)\, dz .$$

One obtains finally

$$\left.\begin{aligned} R = \tfrac{1}{2} \varrho \int_{-\infty}^{\infty} dz \int_{-h}^{\eta(b,z)} \left[-\varphi_x^2(b,y,z) + \varphi_y^2(b,y,z) + \varphi_z^2(b,y,z)\right] dy + \\ + \tfrac{1}{2} g \varrho \int_{-\infty}^{\infty} \eta^2(b,z)\, dz . \end{aligned}\right\} \quad (8.6)$$

This exact formula for resistance will be put into a different form later after linearization of the boundary conditions. Although the plane $x - ct = b$ may be taken at any distance behind the body without destroying the validity of (8.6), it is usually convenient to take it so far behind that asymptotic expressions for φ can be used.

If in (8.1) a part of the surface $S(t)$, say $S_1(t)$, is an interface with another fluid with surface tension acting, then the energy is given by

$$E = \iiint_{T(t)} \left[\tfrac{1}{2} \varrho (u^2 + v^2 + w^2) + \varrho g y\right] d\tau + T \iint_{S_1(t)} d\sigma . \quad (8.7)$$

Let $S_1(t)$ be bounded by the curve $C(t)$ given parametrically by $x(s,t)$, $y(s,t)$, $z(s,t)$ and let $S(t) = S_1(t) + S_2(t)$. Then the formula analogous to (8.2) is

see errata

$$\left.\begin{aligned} \frac{dE}{dt} = \iint_{S(t)} \varrho \Phi_t \left[\Phi_n + \frac{F_t}{\sqrt{F_x^2 + F_y^2 + F_z^2}}\right] d\sigma + \iint_{S_2(t)} p \frac{F_t}{\sqrt{F_x^2 + F_y^2 + F_z^2}}\, d\sigma + \\ + \iint_{S_1(t)} \left[p + T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right] \frac{F_t}{\sqrt{F_x^2 + F_y^2 + F_z^2}}\, d\sigma + T \int_{C(t)} \begin{vmatrix} F_x & F_y & F_z \\ x_t & y_t & z_t \\ x_s & y_s & z_s \end{vmatrix} \frac{d_s}{\sqrt{F_x^2 + F_y^2 + F_z^2}} . \end{aligned}\right\} \quad (8.8)$$

If $S_1(t)$ is a free surface, then the boundary condition

$$p_0 - p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where p_0 is an assumed constant pressure implies that there is no flux of energy through S_1.

If the motion is two-dimensional, with S_1 given by $y = \eta(x,t)$, $x_1(t) \leqq x \leqq x_2(t)$, then (8.7) becomes

$$E(t) = \iint_{T(t)} \left[\tfrac{1}{2} \varrho (u^2 + v^2) + \varrho g y\right] d\sigma + T \int_{X_1(t)}^{x_2(t)} ds \quad (8.9)$$

and (8.8) becomes

$$\left.\begin{aligned} \frac{dE}{dt} = \int_{S_2} \varrho \Phi_t \left[\Phi_n + \frac{F_t}{\sqrt{F_x^2 + F_y^2}}\right] ds + \int_{S_2} p \frac{F_t}{\sqrt{F_x^2 + F_y^2}}\, ds - \\ - \int_{x_1(t)}^{x_2(t)} \left[p + \frac{T \eta_{xx}}{[1 + \eta_x^2]^{\frac{3}{2}}}\right] \eta_t\, dx + T \frac{\eta_x \eta_t}{\sqrt{1 + \eta_x^2}} \Bigg|_{x_1}^{x_2} + T \sqrt{1 + \eta_x^2}\, x'(t) \Bigg|_{x_1}^{x_2} . \end{aligned}\right\} \quad (8.10)$$

If S_1 is a free surface, the integral over S_1 may be dropped by suitably redefining p.

9. Momentum. Expressions for rate of change of momentum may be derived which are analogous to those for rate of change of energy. With

$$\boldsymbol{M} = \iiint_{T(t)} \varrho\, \boldsymbol{v}\, d\tau, \tag{9.1}$$

and otherwise the same notation as in Sect. 8, one finds

$$\left.\begin{aligned}\frac{d\boldsymbol{M}}{dt} &= \iint_S \varrho \left\{\Phi_t \boldsymbol{n} + \operatorname{grad} \Phi \frac{-F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right\} d\sigma,\\ &= -\iint_S \left\{(p+\varrho g y)\,\boldsymbol{n} + \varrho\left[\boldsymbol{v}\cdot\boldsymbol{n} + \frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right]\boldsymbol{v}\right\} d\sigma,\\ &= \iint_S \varrho \left\{\left(\Phi_t + \frac{1}{2}v^2\right)\boldsymbol{n} - \left[\boldsymbol{v}\cdot\boldsymbol{n} + \frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right]\boldsymbol{v}\right\} d\sigma.\end{aligned}\right\} \tag{9.2}$$

Here the first line of (9.2) is derived by a direct computation of $d\boldsymbol{M}/dt$ with $\boldsymbol{v} = \operatorname{grad}\Phi$; the second is derived analogously to (8.2); the third follows directly by use of (2.10′). Comparison of lines one and three gives the known relation (LEVI-CIVITA):

$$\iint_S \tfrac{1}{2} v^2 \boldsymbol{n}\, d\sigma = \iint_S (\boldsymbol{v}\cdot\boldsymbol{n})\, \boldsymbol{v}\, d\sigma. \tag{9.3}$$

Note that in (9.2) and (9.3) $S(t)$ may move in an arbitrary manner as long as the region $T(t)$ contains no singularities and only fluid. If the boundary is physical, the terms in square brackets vanish in (9.2); if the boundary is fixed, then $F_t = 0$.

Let $S_0(t)$ be a physical boundary, possibly the surface of a solid body, and $S(t)$ a closed surface containing it. Applying (9.2) to the region of fluid bounded jointly by S_0 and S, one finds

$$\left.\begin{aligned}F_0 &= \iint_{S_0} (p + \varrho g y)\, \boldsymbol{n}\, d\sigma\\ &= -\iint_{S_0} \varrho(\Phi_t \boldsymbol{n} + \boldsymbol{v}\cdot\boldsymbol{n}\,\boldsymbol{v})\, d\sigma + \iint_S \varrho\left(\tfrac{1}{2}v^2\boldsymbol{n} - \boldsymbol{v}\cdot\boldsymbol{n}\,\boldsymbol{v}\right) d\sigma.\end{aligned}\right\} \tag{9.4}$$

see errata
Here F_0 is the hydrodynamic orce fon S_0 and does not include the hydrostatic force.

If singularities are allowed in the region occupied by fluid, they may be enclosed in spheres of small radius and the formula (9.4) applied to the remaining fluid, with S modified to include the spherical surfaces. If the singularities are isolated sources of strengths m_i at the points $\boldsymbol{a}_i$, then by shrinking the spheres about the singularities in a customary fashion [cf. MILNE-THOMSON (1956, pp. 448 to 450)], one obtains the following modification of (9.4):

$$\boldsymbol{F}_0 = -\iint_{S_0} \varrho(\Phi_t \boldsymbol{n} + \boldsymbol{v}\cdot\boldsymbol{n}\,\boldsymbol{v})\, d\sigma + \sum 4\pi\varrho\, m_i \boldsymbol{v}_i + \iint_S \varrho\left(\tfrac{1}{2}v^2\boldsymbol{n} - \boldsymbol{v}\cdot\boldsymbol{n}\,\boldsymbol{v}\right) d\sigma, \tag{9.5}$$

where $\boldsymbol{v}_i$ is the velocity at the point $\boldsymbol{a}_i$ when the source at that point is removed. Other modifications may be derived for other types of singularities.

If the velocity field is such that $r^{1+\varepsilon} v \to 0$ as $r = \sqrt{x^2+y^2+z^2} \to \infty$ for some $\varepsilon > 0$, then the last integral in (9.4) or (9.5) will vanish as S is expanded to infinity, provided the latter can be done without destroying the validity of the formula. In the case of a body moving in a fluid with a free surface, one cannot expand in all directions and must include the contribution of the last integral over the free surface. However, the formulas are still useful in computing the force on an obstacle resulting from waves.

10. Expansion of solutions in powers of a parameter. In their exact form even the simplest problems with surface waves are difficult to solve. If one neglects viscosity and assumes irrotational motion, the problem is reduced to finding solutions of Laplace's equation, which is at least linear in the unknown. However, the problem is still difficult because of the nonlinear boundary condition at the free surface or interface. This lack of linearity deprives one, for example, of the mathematical tool of superposition of solutions; expansion in eigenfunctions or use of Green's functions is not possible.

In order to be able to treat special problems, the equations are approximated by ones which are more tractable. The two principal methods of approximation may each be treated as a perturbation procedure. As was mentioned in Sect. 7, this procedure is not concerned with the assumptions about the nature of the fluid, for example, whether or not viscosity is neglected, but rather with the nature of the motion and its generation. An advantage in using the perturbation procedure is that the assumptions about the motion are displayed in such a way that it is clear how to obtain approximations of higher order. The method has been applied to water-wave problems by Sekerzh-Zenkovich (1947, 1951, 1952), K. Friedrichs (1948), Keller (1948), F. John (1949), Longuet-Higgins (1953b), Peters and Stoker (1957), and others. As used here the method is purely formal, the nature of the convergence of the perturbation series, whether it be uniform, pointwise, asymptotic or what not, being left open. However, for each method of approximation it is possible to point to several cases in which convergence has been proved: for the infinitesimal-wave approximation, Levi-Civita's (1925), Struik's (1926) and Nekrasov's (1921, 1928) proofs of the existence of a periodic wave of permanent type; and for the shallow-water approximation, Friedrichs and Hyers' (1954) proof of the existence of a solitary wave and Littman's (1957) proof of the existence of cnoidal waves.

To a certain extent the two methods of approximation have different aims. The infinitesimal-wave approximation fits into a general scheme for approximating nonlinear equations and boundary conditions by linear ones [see Souriau (1952) for a discussion]. To apply it, one must know a particular exact solution to start with. In addition, one must be able to select a dimensionless parameter (or parameters), say ε, which helps to determine the exact physical problem and is such that the solutions to the exact problems associated with each value of ε approach (in some sense) the known exact solution when $\varepsilon \to 0$. It is then assumed that the various functions entering into the problem may be expanded into power series in ε. The series are substituted into the equations and boundary conditions and grouped according to powers of ε. The coefficients of each power then yield a sequence of equations and boundary conditions, the coefficients of ε giving the first-order theory, those of ε^2 the second-order theory, etc. As an exact initial solution it is usually most convenient to take either a state of rest or of uniform motion. Various choices of ε will be made in the applications later.

The shallow-water approximation differs in that a change of variable involving the expansion parameter is made initially. This introduces ε into the exact equations. When the power series expansions are introduced into the equations, the resulting equations of the sequence are linear in quantities of the same order, but the equations are too degenerate to determine all these quantities without recourse to the equations of next higher order. This leads to nonlinear equations for the desired functions, but ones of a type which have been intensively investigated. In this case the procedure is perhaps artificial in that the perturbation scheme is devised to lead to a special set of equations for a first-order theory, derived originally by quite different reasoning. However, in doing this it makes clear the nature of the approximation and gives a systematic procedure for finding higher-order approximations. It is instructive, in this connection, to read the usual derivation as given, for example, in LAMB (1932, pp. 254—256) or STOKER (1957, pp. 22—25) (who also gives the one given here).

α) The infinitesimal-wave approximation. We shall derive the equations of motion and the free-surface or interface boundary conditions for this linearized theory without identifying explicitly the parameter ε used in the expansions. Later on, when specific choices are made, the boundary conditions on certain geometric boundaries associated with the choice of ε will be modified to conform with the linearization.

Consider two incompressible viscous fluids in contact along an interface represented by $y=\eta(x,z,t)$. Quantities referring to the upper fluid have subscript 2, those to the lower fluid subscript 1; the coefficient of surface tension is T. Assume the following expansions in the parameter ε:

$$\left.\begin{aligned} \boldsymbol{v}_i(x,y,z,t,\varepsilon) &= \varepsilon\,\boldsymbol{v}_i^{(1)} + \varepsilon^2\,\boldsymbol{v}_i^{(2)} + \cdots, \\ p_i(x,y,\zeta,t,\varepsilon) &= p_i^{(0)} + \varepsilon\,p_i^{(1)} + \varepsilon^2\,p_i^{(2)} + \cdots, \\ \eta(x,z,t,\varepsilon) &= \varepsilon\,\eta^{(1)} + \varepsilon^2\,\eta^{(2)} + \cdots. \end{aligned}\right\} \tag{10.1}$$

Substitute these expansions in Eqs. (2.2), (2.5), (3.1), (3.3), and (3.7), remembering in addition that formal expansions of the following sort, for example, hold:

$$\begin{aligned} u_1(x,\eta(x,z,t),z,t) &= u_1(x,0,z,t) + \eta\,u_{1y}(x,0,z,t) + \cdots \\ &= \varepsilon\,u_1^{(1)}(x,0,z,t) + \varepsilon^2\left[u_1^{(2)}(x,0,z,t) + \eta^{(1)}\,u_{1y}^{(1)}(x,0,z,t)\right] + \cdots. \end{aligned}$$

Collecting first the terms independent of ε, one finds from (2.5) and (3.3)

$$\operatorname{grad}(p_i^{(0)} + \varrho_i\,g\,y) = 0, \qquad p_2^{(0)}(x,0,z,t) = p_1^{(0)}(x,0,z,t). \tag{10.2}$$

Collecting the coefficients of the first power of ε, one finds

$$\left.\begin{aligned} &\frac{\partial u_i^{(1)}}{\partial x} + \frac{\partial v_i^{(1)}}{\partial y} + \frac{\partial w_i^{(1)}}{\partial z} = 0, \qquad i = 1, 2, \\ &\frac{\partial \boldsymbol{v}_i^{(1)}}{\partial t} = -\frac{1}{\varrho_i}\operatorname{grad} p_i^{(1)} + \nu_i\,\Delta\,\boldsymbol{v}_i^{(1)}, \qquad i = 1, 2, \\ &u_1^{(1)}(x,0,z,t) = u_2^{(1)}(x,0,z,t), \\ &v_1^{(1)}(x,0,z,t) = v_2^{(1)}(x,0,z,t) = \eta_t^{(1)}(x,z,t), \\ &w_1^{(1)}(x,0,z,t) = w_2^{(1)}(x,0,z,t), \\ &\mu_1(u_{1y}^{(1)}(x,0,z,t) + v_{1x}^{(1)}) = \mu_2(u_{2y}^{(1)} + v_{2x}^{(1)}), \\ &p_2^{(1)}(x,0,z,t) - p_1^{(1)} - (\varrho_2 - \varrho_1)\,g\,\eta^{(1)} - 2(\mu_2\,v_{2y}^{(1)} - \mu_1\,v_{1y}^{(1)}) = T(\eta_{xx}^{(1)} + \eta_{zz}^{(1)}), \\ &\mu_1(w_{1y}^{(1)}(x,0,z,t) + v_{1z}^{(1)}) = \mu_2(w_{2y}^{(1)} + v_{2z}^{(1)}). \end{aligned}\right\} \tag{10.3}$$

If the upper fluid is replaced by a given atmospheric pressure distribution $\bar{p}(x,z,t)$, then the equations for the lower fluid become (after dropping the subscripts)

$$\left.\begin{aligned} &\operatorname{grad}(p^{(0)} + \varrho\,g\,y) = 0, \qquad p^{(0)}(x,0,z,t) = \bar{p}^{(0)}(x,z,t), \\ &\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} + \frac{\partial w^{(1)}}{\partial z} = 0, \\ &\frac{\partial \boldsymbol{v}^{(1)}}{\partial t} = -\frac{1}{\varrho}\operatorname{grad} p^{(1)} + \nu\,\Delta\,\boldsymbol{v}^{(1)}, \\ &\eta_t^{(1)}(x,z,t) = v^{(1)}(x,0,z,t), \\ &u_y^{(1)}(x,0,z,t) + v_x^{(1)} = w_y^{(1)} + v_z^{(1)} = 0, \\ &p^{(1)}(x,0,z,t) - \varrho\,g\,\eta^{(1)} - 2\mu\,v_y^{(1)} = -T(\eta_{xx}^{(1)} + \eta_{zz}^{(1)}) + \bar{p}^{(1)}(x,z,t). \end{aligned}\right\} \tag{10.4}$$

For convenience we have assumed above that the expansion for η starts with $\varepsilon\eta^{(1)}$. If we had assumed instead $\eta=\eta^{(0)}+\varepsilon\eta^{(1)}+\cdots$, we would have found from (3.1) and (3.7) the equations

$$\eta_t^{(0)}=\eta_x^{(0)}=\eta_z^{(0)}=0$$

and, hence, $\eta^{(0)}=$const. The zero values of y in the boundary conditions would then be replaced by this constant. Taking the constant equal to zero means that we have taken the undisturbed interface as (x, z)-plane.

The equations above give the linearized equations of motion and boundary conditions at the interface or free surface. If one now proceeds, as we shall not do for this case, to collect coefficients of ε^2, one may obtain the differential equations and boundary conditions for the second-order corrections to be added to the solutions of the linearized equations, and so forth for higher-order corrections. In general the resulting equations are too unwieldy to be useful.

A special case of the linearized equations which is of particular interest is irrotational flow of a perfect fluid. There is then a velocity potential Φ which we assume has the following expansion:

$$\Phi(x, y, z, t, \varepsilon)=\varepsilon\,\Phi^{(1)}+\varepsilon^2\,\Phi^{(2)}+\cdots. \tag{10.5}$$

Condition (2.11) becomes

$$\Delta\Phi^{(i)}=0,\qquad i=1, 2, \ldots. \tag{10.6}$$

Let there be two superposed fluids with velocity potentials Φ_1 and Φ_2 describing the motion in each; otherwise the same notation as above. Then condition (3.1) at the interface gives the linearized condition

$$\eta_t^{(1)}(x, z, t)=\Phi_{1y}^{(1)}(x, 0, z, t)=\Phi_{2y}^{(1)}(x, 0, z, t) \tag{10.7}$$

and condition (3.9), together with (2.10′), gives

$$-\varrho_2\,\Phi_{2t}^{(1)}(x, 0, z, t)+\varrho_1\,\Phi_{1t}^{(1)}(x, 0, z, t)+(\varrho_1-\varrho_2)\,g\,\eta^{(1)}(x, z, t)=T(\eta_{xx}^{(1)}+\eta_{zz}^{(1)}). \tag{10.8}$$

The further special case when both the upper fluid and surface tension are missing will be dealt with so often later on that we repeat the boundary conditions for it. We allow, however, a pressure distribution on the free surface, $\bar{p}(x, z, t)=\varepsilon\bar{p}^{(1)}+\varepsilon^2\bar{p}^{(2)}+\cdots$. The first-order boundary conditions are

$$\left.\begin{aligned}&\eta_t^{(1)}(x, z, t)-\Phi_y^{(1)}(x, 0, z, t)=0,\\ &g\,\eta^{(1)}(x, z, t)+\Phi_t(x, 0, z, t)+\varrho^{-1}\bar{p}^{(1)}(x, z, t)=0.\end{aligned}\right\} \tag{10.9}$$

Eliminating $\eta^{(1)}$ between the last two equations, one gets

$$g\,\Phi_y^{(1)}(x, 0, z, t)+\Phi_{tt}^{(1)}(x, 0, z, t)+\varrho^{-1}\bar{p}_t^{(1)}(x, z, t)=0. \tag{10.10}$$

The boundary conditions for the second-order corrections are not too long to write down:

$$\left.\begin{aligned}&\eta_t^{(2)}(x, z, t)-\Phi_y^{(2)}(x, 0, z, t)=\eta^{(1)}\,\Phi_{yy}^{(1)}-\eta_x^{(1)}\,\Phi_x^{(1)}-\eta_z^{(1)}\,\Phi_z^{(1)},\\ &g\,\eta^{(2)}(x, z, t)+\Phi_t^{(2)}(x, 0, z, t)+\varrho^{-1}\bar{p}^{(2)}(x, z, t)=-\eta^{(1)}\,\Phi_{ty}^{(1)}-\tfrac{1}{2}(\operatorname{grad}\Phi^{(1)})^2.\end{aligned}\right\} \tag{10.11}$$

Eliminating $\eta^{(1)}$ and $\eta^{(2)}$ from (10.11), one finds a counterpart to (10.10):

$$\left.\begin{aligned}&g\,\Phi_y^{(2)}(x, 0, z, t)+\Phi_{tt}^{(2)}+\varrho^{-1}\bar{p}_t^{(2)}=-\frac{\partial}{\partial t}(\operatorname{grad}\Phi^{(1)})^2+\\ &\quad+(\Phi_t^{(1)}+\varrho^{-1}\bar{p}^{(1)})\left(\Phi_{yy}^{(1)}+\frac{1}{g}\,\Phi_{tty}^{(1)}\right)-\varrho^{-1}(\Phi_x^{(1)}\,\bar{p}_x^{(1)}+\Phi_z^{(1)}\,\bar{p}_z^{(1)}).\end{aligned}\right\} \tag{10.12}$$

Under certain circumstances the next-to-last term will vanish. The boundary conditions for higher-order corrections will not be worked out in detail. However, they are of the form

$$\left.\begin{aligned} g\,\Phi_y^{(i)}(x,0,z,t)+\Phi_{tt}^{(i)}+\varrho^{-1}p_t^{(i)} &= A_i\{\Phi^{(1)},\ldots,\Phi^{(i-1)},\bar{p}^1,\ldots,\bar{p}^{(i-1)}\},\\ g\,\eta^{(i)}(x,z,t)+\Phi_t^{(i)}(x,0,z,t)+\varrho^{-1}\bar{p}^{(i)} &= B_i\{\Phi^{(1)},\ldots,\Phi^{(i-1)},\bar{p}^{(1)},\ldots,\bar{p}^{(i-1)}\},\end{aligned}\right\}\quad(10.13)$$

where A_i and B_i are functionals of the functions in brackets, in this case complicated polynomials of the functions and their derivatives evaluated at $y=0$.

It is useful to have the form of the linearized boundary conditions when certain additional assumptions are made.

First, let us suppose that the $(\bar{x},\bar{y},\bar{z})$-coordinate system is moving with velocity $c(t)$ in the x-direction with respect to the fixed (x,y,z)-coordinate system. Then, from the equation following (2.15) with $\bar{y}=y$, $\bar{z}=z$

$$\Phi_t(x,y,z,t)=\bar{\Phi}_t-c\,\bar{\Phi}_{\bar{x}},\qquad \Phi_{tt}=\bar{\Phi}_{tt}-2c\,\bar{\Phi}_{t\bar{x}}+c^2\,\bar{\Phi}_{\bar{x}\bar{x}}-\dot{c}\,\bar{\Phi}_{\bar{x}},$$

and the boundary conditions become

$$\left.\begin{aligned} g\,\bar{\eta}^{(1)}(\bar{x},\bar{z},t)+\bar{\Phi}_t^{(1)}(\bar{x},0,\bar{z},t)-c\,\bar{\Phi}_{\bar{x}}^{(1)}(\bar{x},0,\bar{z},t)+\varrho^{-1}\bar{p}^{(1)}(\bar{x},\bar{z},t) &=0,\\ \bar{\Phi}_{tt}^{(1)}(\bar{x},0,\bar{z},t)-2c\,\bar{\Phi}_{t\bar{x}}^{(1)}+c^2\,\bar{\Phi}_{\bar{x}\bar{x}}^{(1)}-\dot{c}\,\bar{\Phi}_{\bar{x}}^{(1)}+g\,\bar{\Phi}_y^{(1)}+\varrho^{-1}\bar{p}_t^{(1)}-c\,\varrho^{-1}\bar{p}_{\bar{x}}^{(1)} &=0.\end{aligned}\right\}\quad(10.14)$$

If c is constant and the motion is steady in the moving coordinate system,

$$\Phi(x,y,z,t)=\varphi(x-ct,y,z)=\varphi(\bar{x},\bar{y},\bar{z})$$

and the linearized boundary conditions are

$$\left.\begin{aligned} g\,\bar{\eta}^1(\bar{x},\bar{z})-c\,\varphi_{\bar{x}}^{(1)}(\bar{x},0,\bar{z})+\varrho^{-1}\bar{p}^1(\bar{x},\bar{z}) &=0,\\ g\,\varphi_{\bar{y}}^{(1)}(\bar{x},0,\bar{z})+c^2\,\varphi_{\bar{x}\bar{x}}^{(1)}(\bar{x},0,\bar{z})-c\,\varrho^{-1}\bar{p}_{\bar{x}}^{(1)}(\bar{x},\bar{z}) &=0.\end{aligned}\right\}\quad(10.15)$$

If the motion is steady with respect to a moving coordinate system, one may impose a uniform flow in the opposite direction and then treat the problem as a steady one in an absolute coordinate system, but carrying out the perturbation about the uniform flow. We illustrate this for the case of two-dimensional irrotational flow. Let $\varphi(x,y)$ and $\psi(x,y)$ be the velocity potential and stream function, respectively, and assume expansions of the form

$$\left.\begin{aligned} \varphi(x,y) &= -c\,x+\varepsilon\,\varphi^{(1)}(x,y)+\varepsilon^2\,\varphi^{(2)}+\cdots,\\ \psi(x,y) &= -c\,y+\varepsilon\,\psi^{(1)}(x,y)+\varepsilon^2\,\psi^{(2)}+\cdots,\\ \eta(x) &= \qquad\quad \varepsilon\,\eta^{(1)}(x)\quad+\varepsilon^2\,\eta^{(2)}+\cdots.\end{aligned}\right\}\quad(10.16)$$

The differential equations $\Delta\varphi=0$, $\Delta\psi=0$, $\varphi_x=\psi_y$, $\varphi_y=-\psi_x$ become

$$\Delta\varphi^{(i)}=0,\qquad \Delta\psi^{(i)}=0,\qquad \varphi_x^{(i)}=\psi_y^{(i)},\qquad \varphi_y^{(i)}=-\psi_x^{(i)}.\quad(10.17)$$

The kinematic condition (3.1) is replaced by

$$\psi(x,\eta(x))=0.$$

Substituting the expansions (10.16) in this equation and in (3.11′), one finds from the coefficients of ε

$$\left.\begin{aligned} -c\,\eta^{(1)}(x)+\psi^{(1)}(x,0) &=0,\\ g\,\eta^{(1)}(x)-c\,\varphi_x^{(1)}(x,0)+\varrho^{-1}\bar{p}^{(1)}(x) &=0.\end{aligned}\right\}\quad(10.18)$$

Eliminating $\eta^{(1)}$ and using the third of Eqs. (10.17), one gets

$$g\psi^{(1)}(x,0) - c^2\psi_y^{(1)}(x,0) + c\varrho^{-1}\bar{p}'^{(1)}(x) = 0. \tag{10.19}$$

Collecting the coefficients of ε^2, one obtains after some manipulation

$$\left.\begin{aligned} g\psi^{(2)}(x,0) - c^2\psi_y^{(2)} &= \frac{1}{c}\psi^{(1)}\left[c^2\psi_{yy}^{(1)} - g\psi_y^{(1)}\right] - \frac{1}{2}c\left[\psi_x^{(1)2} + \psi_y^{(1)2}\right],\\ c\eta^{(2)}(x) &= \psi^{(2)}(x,0) + \frac{1}{c}\psi^{(1)}\psi_y^{(1)}; \end{aligned}\right\} \tag{10.20}$$

here we have assumed for simplicity that $\bar{p}=0$.

β) The shallow-water approximation. This approximation has been widely used by hydraulic engineers in the study of open-channel flow and, in a further simplification, is used for the theory of tides. In deriving the equations from the exact ones we shall follow the method of FRIEDRICHS (1948) and KELLER (1948). However, a somewhat different approach to this approximation due to URSELL (1953) is also instructive. Although it is possible to carry through the derivation while taking account of surface tension, this will not be done here. It will be assumed to start with that there are two perfect, incompressible fluids with an interface $y=\eta(x,z,t)$; the bottom fluid is bounded below by a rigid surface $y=-h(x,z)$. Variables pertaining to the lower fluid have subscript 1, those pertaining to the upper fluid subscript 2. The motion will be assumed irrotational.

Before making an expansion in powers of a parameter, it is essential to make a change of variable in which vertical and horizontal distances are stretched by different amounts. Let m be a scale for horizontal measurement and n one for vertical measurement. Define $\varepsilon = n^2/m^2$. Introduce new variables, $\bar{x}, \bar{y}, \bar{z}, \bar{t}$, by the equations

$$\bar{x} = x\sqrt{\varepsilon},\quad \bar{y}=y,\quad \bar{z}=z\sqrt{\varepsilon},\quad \bar{t}=t\sqrt{\varepsilon},\quad \bar{u}=u,\quad \bar{v}=v\sqrt{\varepsilon},\quad \bar{w}=w,\quad \bar{p}=p. \tag{10.21}$$

Eqs. (2.2), (2.6), (2.8), (3.1), (3.9), and (4.1) (with $T=0$) become:

$$\left.\begin{aligned}
&\varepsilon\left(\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{w}}{\partial\bar{z}}\right) + \frac{\partial\bar{v}}{\partial\bar{y}} = 0,\\
&\varepsilon\left(\frac{\partial\bar{u}}{\partial t} + \bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{u}}{\partial\bar{z}} + \frac{1}{\varrho}\frac{\partial\bar{p}}{\partial\bar{x}}\right) + \bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} = 0,\\
&\varepsilon\left(\frac{\partial\bar{v}}{\partial t} + \bar{u}\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{v}}{\partial\bar{z}} + g + \frac{1}{\varrho}\frac{\partial\bar{p}}{\partial y}\right) + \bar{v}\frac{\partial\bar{v}}{\partial\bar{y}} = 0,\\
&\varepsilon\left(\frac{\partial\bar{w}}{\partial t} + \bar{u}\frac{\partial\bar{w}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{w}}{\partial\bar{z}} + \frac{1}{\varrho}\frac{\partial\bar{p}}{\partial\bar{z}}\right) + \bar{v}\frac{\partial\bar{w}}{\partial\bar{y}} = 0,\\
&\frac{\partial\bar{w}}{\partial\bar{y}} = \frac{\partial\bar{v}}{\partial\bar{z}},\qquad \frac{\partial\bar{u}}{\partial\bar{z}} = \frac{\partial\bar{w}}{\partial\bar{x}},\qquad \frac{\partial\bar{v}}{\partial\bar{x}} = \frac{\partial\bar{u}}{\partial\bar{y}},\\
&\varepsilon\left(\bar{u}\frac{\partial\bar{\eta}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{\eta}}{\partial\bar{z}} + \frac{\partial\bar{\eta}}{\partial\bar{t}}\right) - \bar{v} = 0 \quad\text{for } \bar{y} = \bar{\eta}(\bar{x},\bar{z},\bar{t}),\\
&\varepsilon\left(\bar{u}\frac{\partial\bar{h}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{h}}{\partial\bar{z}}\right) + \bar{v} = 0 \qquad\text{for } \bar{y} = -\bar{h}(\bar{x},\bar{z},\bar{t}),\\
&\bar{p}_2(\bar{x},\bar{\eta},\bar{z},\bar{t}) = \bar{p}_1(\bar{x},\bar{\eta},\bar{z},\bar{t}),
\end{aligned}\right\} \tag{10.22}$$

where $\bar{u}, \bar{v}, \bar{w}, \bar{p}, \varrho$ possess suppressed subscripts 1 and 2 for the lower and upper fluids respectively, except in the last equation.

Now assume expansions of the form

$$\left.\begin{aligned} \bar{\boldsymbol{v}}_i &= \boldsymbol{v}_i^{(0)} + \varepsilon \boldsymbol{v}_i^{(1)} + \varepsilon^2 \boldsymbol{v}_i^{(2)} + \cdots, \qquad i = 1, 2, \\ \bar{p}_i &= p_i^{(0)} + \varepsilon p_i^{(1)} + \varepsilon^2 p_i^{(2)} + \cdots, \qquad i = 1, 2, \\ \bar{\eta} &= \eta^{(0)} + \varepsilon \eta^{(1)} + \varepsilon^2 \eta^{(2)} + \cdots, \end{aligned}\right\} \tag{10.23}$$

substitute in the Eqs. (10.22) and collect according to powers of ε. (We shall henceforth suppress the bars on $\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{\eta}$). The terms independent of ε give the equations

$$\left.\begin{array}{l} \qquad\qquad v_y^{(0)} = 0, \\ v^{(0)} u_y^{(0)} = 0, \quad v^{(0)} v_y^{(0)} = 0, \quad v^{(0)} w_y^{(0)} = 0, \\ w_y^{(0)} = v_z^{(0)}, \quad u_z^{(0)} = w_x^{(0)}, \quad v_x^{(0)} = u_y^{(0)}, \\ v^{(0)}(x, \eta^{(0)}, z, t) = 0, \quad v_1^{(0)}(x, -h, z, t) = 0, \\ p_2^{(0)}(x, \eta^{(0)}, z, t) = p_1^{(0)}(x, \eta^{(0)}, z, t). \end{array}\right\} \tag{10.24}$$

The first and fourth equations give

$$v^{(0)}(x, y, z, t) \equiv 0. \tag{10.25}$$

The third then states that

$$u_y^{(0)} = w_y^{(0)} = 0 \quad \text{or} \quad u^{(0)} = u^{(0)}(x, z, t), \quad w^{(0)} = w^{(0)}(x, z, t). \tag{10.26}$$

The terms which are coefficients of ε give, after making use of (10.25) and (10.26),

$$\left.\begin{array}{ll} u_x^{(0)} + w_z^{(0)} + v_y^{(1)} = 0, & \\ u_t^{(0)} + u^{(0)} u_x^{(0)} + w^{(0)} u_z^{(0)} + p_x^{(0)}/\varrho = 0, & \\ g + p_y^{(0)}/\varrho = 0, & \\ w_t^{(0)} + u^{(0)} w_x^{(0)} + w^{(0)} w_z^{(0)} + p_z^{(0)}/\varrho = 0, & \\ u^{(0)} \eta_x^{(0)} + w^{(0)} \eta_z^{(0)} + \eta_t^{(0)} - v^{(1)} = 0 & \text{for } y = \eta^{(0)}(x, z, t), \\ u_1^{(0)} h_x + w_1^{(0)} h_y + v_1^{(1)} = 0 & \text{for } y = -h(x, z). \end{array}\right\} \tag{10.27}$$

(The equations deriving from irrotationality and the continuity of pressure will be brought in later.) The first and last two equations of (10.27) together with (10.26) give

$$\left.\begin{aligned} v_1^{(1)} &= -y(u_{1x}^{(0)} + w_{1z}^{(0)}) - (u_1^{(0)} h)_x - (w_1^{(0)} h)_z, \\ v_2^{(1)} &= -y(u_{2x}^{(0)} + w_{2z}^{(0)}) + (u_2^{(0)} \eta^{(0)})_x + (w_2^{(0)} \eta^{(0)})_z - (u_1^{(0)} \eta^{(0)})_x - (w_1^{(0)} \eta^{(0)})_z + \\ &\qquad - (u_1^{(0)} h)_x - (w_1^{(0)} h)_z. \end{aligned}\right\} \tag{10.28}$$

The third equation of (10.27) gives

$$p^{(0)} = -\varrho g y + f(x, z, t).$$

In order to evaluate f, further information is necessary. Here are two cases of interest. 1. If the upper fluid is absent, the condition $p^{(0)}(x, \eta^{(0)}, z, t) = 0$ gives

$$p^{(0)} = -\varrho g y + \varrho g \eta^{(0)}(x, z, t). \tag{10.29}$$

2. If the upper fluid is unbounded above, then, up to an additive constant,

$$\left.\begin{aligned} p_1^{(0)} &= -\varrho_1 g y + (\varrho_1 - \varrho_2) g \eta^{(0)} + k, \\ p_2^{(0)} &= -\varrho_2 g y + k. \end{aligned}\right\} \tag{10.30}$$

If the upper fluid is bounded above by a free surface $y = d(x, z, t) = d^{(0)} + \varepsilon d^{(1)} + \cdots$, then one may satisfy the boundary conditions $p_2^{(0)}(x, d^{(0)}, z, t) = 0$, $p_2^{(0)}(x, \eta^{(0)}, z, t) = p_1^{(0)}(x, \eta^{(0)}, z, t)$ with

$$\left.\begin{aligned} p_1^{(0)} &= -\varrho_1 g(y - \eta^{(0)}) + \varrho_2 g(d^{(0)} - \eta^{(0)}), \\ p_2^{(0)} &= -\varrho_2 g(y - d^{(0)}). \end{aligned}\right\} \tag{10.31}$$

It is clear from the form of $p^{(0)}$ why the shallow-water approximation is sometimes called the hydrostatic approximation.

The usual equations for the first approximation to the shallow-water theory are those in which only the lower fluid is present. They may now be obtained by substituting (10.29) in the second and fourth equations in (10.27) and (10.28) in the fifth equation. They are (10.25), (10.29), and

$$\left.\begin{aligned} u_t^{(0)} + u^{(0)} u_x^{(0)} + w^{(0)} u_z^{(0)} + g\,\eta_x^{(0)} &= 0, \\ w_t^{(0)} + u^{(0)} w_x^{(0)} + w^{(0)} w_z^{(0)} + g\,\eta_z^{(0)} &= 0, \\ \eta_t^{(0)} + [u^{(0)}(\eta^{(0)} + h)]_x + [w^{(0)}(\eta^{(0)} + h)]_z &= 0. \end{aligned}\right\} \tag{10.32}$$

If one now collects the coefficients of ε^2 and the remaining coefficients of ε, one finds after some reduction

$$\left.\begin{aligned} & u_x^{(1)} + w_z^{(1)} + v_y^{(2)} = 0, \\ & u_t^{(1)} + u^{(1)} u_x^{(0)} + u^{(0)} u_x^{(1)} + w^{(1)} u_z^{(0)} + w^{(0)} u_z^{(1)} + v^{(1)} u_y^{(1)} + p_x^{(1)}/\varrho = 0, \\ & v_t^{(1)} + u^{(0)} v_x^{(1)} + w^{(0)} v_z^{(1)} + v^{(1)} v_y^{(1)} + p_y^{(1)}/\varrho = 0, \\ & w_t^{(1)} + u^{(1)} w_x^{(0)} + u^{(0)} w_x^{(1)} + w^{(1)} w_z^{(0)} + w^{(0)} w_z^{(1)} + v^{(1)} w_y^{(1)} + p_z^{(1)}/\varrho = 0, \\ & w_y^{(1)} = v_z^{(1)}, \quad u_z^{(1)} = w_x^{(1)}, \quad v_x^{(1)} = u_y^{(1)}, \\ & u^{(0)} \eta_x^{(1)} + u^{(1)} \eta_x^{(0)} + w^{(0)} \eta_z^{(1)} + w^{(1)} \eta_z^{(0)} + \eta_t^{(1)} - \eta^{(1)} v_y^{(1)} - v^{(2)} = 0 \\ & \qquad\qquad\qquad\qquad \text{for} \quad y = \eta^{(0)}(x, z, t), \\ & u_1^{(1)} h_x + w_1^{(1)} h_z + v_1^{(2)} = 0 \quad \text{for} \quad y = -h(x, z), \\ & p_2^{(1)} - p_1^{(1)} + \eta^{(1)}(p_{2y}^{(0)} - p_{1y}^{(0)}) = 0 \quad \text{for} \quad y = \eta^{(0)}(x, z, t). \end{aligned}\right\} \tag{10.33}$$

Some relations can be derived immediately from these equations. For the sake of brevity we introduce the following functions:

$$\begin{aligned} & A_i(x, z, t) = u_{ix}^{(0)} + w_{iz}^{(0)}, \qquad C_i(x, z, t) = (u_i^{(0)} \eta^{(0)})_x + (w_i^{(0)} \eta^{(0)})_z, \quad i = 1, 2, \\ & B_1(x, z, t) = -(u_1^{(0)} h)_x - (w_1^{(0)} h)_z, \qquad B_2 = C_2 - C_1 + B_1. \end{aligned}$$

Eqs. (10.28) may then be written

$$v_i^{(1)} = -y\,A_i + B_i, \qquad i = 1, 2. \tag{10.28'}$$

Then the fifth, first, and third equations of (10.33) give

$$\left.\begin{aligned} & u^{(1)} = -\tfrac{1}{2} y^2 A_x + y\,B_x + r(x, z, t), \\ & w^{(1)} = -\tfrac{1}{2} y^2 A_z + y\,B_z + s(x, z, t), \\ & r_z = s_x, \\ & v^{(2)} = \tfrac{1}{6} y^3 (A_{xx} + A_{zz}) - \tfrac{1}{2} y^2 (B_{xx} + B_{zz}) - y(r_x + s_z) + l(x, z, t), \\ & p^{(1)}/\varrho = \tfrac{1}{2} y^2 [A^2 + u^{(0)} A_x + w^{(0)} A_z + A_t] + \\ & \qquad + y\,[A\,B + u^{(0)} B_x + w^{(0)} B_z + B_t] + q(x, z, t), \end{aligned}\right\} \tag{10.34}$$

see errata

where we have suppressed the subscripts indicating the fluid. The rest of the equations and the boundary conditions are still available to determine the unknown functions. We carry this out only for the case the upper fluid is missing. see errata Then the last condition in (10.32) becomes $p^{(1)}(x, \eta^{(0)}, z, t) = \varrho g \eta^{(1)}$, which allows one to determine $q(x, z, t)$ after $\eta^{(1)}$ is found. The next-to-the-last condition in (10.32) determines $l(x, z, t)$. The equations for r, s and $\eta^{(1)}$ are

$$\left.\begin{aligned} &u^{(0)} r_x + w^{(0)} r_z + r_t + u_x^{(0)} r + u_z^{(0)} s = -q_x - B B_x,\\ &u^{(0)} s_x + w^{(0)} s_z + s_t + w_x^{(0)} r + w_z^{(0)} s = -q_z - B B_z,\\ &u^{(0)} \eta_x^{(1)} + w^{(0)} \eta_z^{(1)} + \eta_t^{(1)} + A\,\eta^{(1)} = [v^{(2)} - u^{(1)} \eta_x^{(0)} - w^{(1)} \eta_z^{(0)}]_{y=\eta^{(0)}},\\ &r_z = s_x, \end{aligned}\right\} \qquad (10.35)$$

where

$$\begin{aligned} q(x, z, t) &= g\,\eta^{(1)} - \tfrac{1}{2}\eta^{(0)2}\left[A^2 + u^{(0)} A_x + w^{(0)} A_z + A_t\right] - \\ &\quad - \eta^{(0)}\left[A\,B + u^{(0)} B_x + w^{(0)} B_z + B_t\right],\\ l(x, z, t) &= -\left[u^{(1)} h_x + w^{(1)} h_z\right]_{y=-h} - \tfrac{1}{6}\eta^{(0)3}\left[A_{xx} + A_{zz}\right] + \\ &\quad + \tfrac{1}{2}\eta^{(0)2}\left[B_{xx} + B_{zz}\right] - \eta^{(0)}(r_x + s_z). \end{aligned}$$

The solutions to these equations give the second-order corrections to the first-order shallow-water theory.

The equations resulting from the coefficients of ε^3 have been given by KELLER (1948) for two dimensions, but will not be reproduced here.

The Eqs. (10.32) for the first-order theory are nonlinear. In the theory of tides and seiches it is customary to simplify further by linearizing them in a manner similar to that used in deriving the equations for the infinitesimal-wave theory. Let $y=0$ be the surface of the undisturbed water and assume that one may make further expansions in a small parameter α: $u^{(0)} = \alpha u^{(01)} + \cdots$, $w^{(0)} =$ see errata $\alpha w^{(01)} + \cdots$, $\eta^{(0)} = \alpha \eta^{(01)}, \ldots$. After some easy manipulations one finds for the linearized approximation to (10.32) the equations

$$\left.\begin{aligned} u_t^{(01)} + g\,\eta_x^{(01)} = 0, \qquad w_t^{(01)} + g\,\eta_z^{(01)} = 0,\\ \eta_{tt}^{(01)} - g\,[\eta_x^{(01)} h]_x - g\,[\eta_z^{(01)} h]_z = 0. \end{aligned}\right\} \qquad (10.36)$$

If the bottom is flat, the equation for $\eta^{(01)}$ becomes the simple wave equation.

D. Theory of infinitesimal waves.

This chapter will deal with special solutions of the linearized equations derived in Sect. 10α. This approximate theory has been very fruitful in its application to problems with various boundary configurations; the linear character of both the equations and boundary conditions allows one to use easily found simple solutions to construct other solutions satisfying special boundary conditions. The derivation of the equations in Sect. 10α suggests the limitations of their use in physical problems: If L and V are a typical length and velocity associated with the physical problem, then, when the perturbation parameter ε is small, the surface elevation and velocities (or their deviation from a uniform flow) should be small with respect to L and V respectively. The smallness may not be uniform, but the quantities in question should approach zero point-wise with ε except at singular points.

11. The fundamental equations. With few exceptions, this chapter will be concerned with the solution of a problem in potential theory. Let the (x, z)-plane be at the undisturbed free surface. We shall be seeking a function $\Phi(x, y, z, t)$, the velocity potential of the motion, satisfying the conditions [cf. Eq. (10.10)]

$$\left.\begin{aligned} &\Delta\Phi = \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 \\ &\Phi_{tt}(x, 0, z, t) + g\,\Phi_y(x, 0, z, t) = -\varrho^{-1}\bar{p}_t(x, z, t), \\ &\Phi_n = V_n \quad \text{on solid boundaries,} \end{aligned}\right\} \tag{11.1}$$

where $\Delta\Phi = 0$ is to be satisfied at all nonsingular points of the fluid in the region $y < 0$ and V_n is the normal velocity of the solid boundary at a given point. $\bar{p}(x, z, t)$ is a given pressure distribution on the free surface; in many problems it will be 0. The form of the free surface is given by:

$$\eta(x, z, t) = -\frac{1}{g}\Phi_t(x, 0, z, t) - \frac{1}{\varrho g}\bar{p}(x, z, t). \tag{11.2}$$

Two special cases occur frequently. If the motion is steady in a coordinate system moving with constant velocity c in the x-direction, then with x, y, z as moving coordinates, the free-surface boundary condition and equation of the surface are given by [cf. Eq. (10.15)]

$$\left.\begin{aligned} &\varphi_y(x, 0, z) + \frac{c^2}{g}\varphi_{xx}(x, 0, z) = \frac{c}{\varrho g}\bar{p}_x(x, z), \\ &\eta(x, z) = \frac{c}{g}\varphi_x(x, 0, z) - \frac{1}{\varrho g}\bar{p}(x, z). \end{aligned}\right\} \tag{11.3}$$

If Φ and $\bar{p}$ are harmonic functions of the time, i.e.

$$\Phi(x, y, z, t) = \varphi_1(x, y, z)\cos\sigma t + \varphi_2(x, y, z)\sin\sigma t = \operatorname{Re}\varphi(x, y, z)\,e^{-i\sigma t},$$

where

$$\varphi(x, y, z) = \varphi_1(x, y, z) + i\,\varphi_2(x, y, z)$$

and similarly for $\bar{p}$, then the free-surface condition and equation of the surface become

$$\left.\begin{aligned} \varphi_{1y}(x, 0, z) - \frac{\sigma^2}{g}\varphi_1(x, 0, z) &= -\frac{\sigma}{\varrho g}\bar{p}_2(x, z), \\ \varphi_{2y}(x, 0, z) - \frac{\sigma^2}{g}\varphi_2(x, 0, z) &= \frac{\sigma}{\varrho g}\bar{p}_1(x, z), \\ \eta(x, z, t) &= \frac{\sigma}{g}[\varphi_1(x,0,z)\sin\sigma t - \varphi_2(x,0,z)\cos\sigma t] - \\ &\quad - \frac{1}{\varrho g}[\bar{p}_1(x, z)\cos\sigma t + \bar{p}_2(x, z)\sin\sigma t]. \end{aligned}\right\} \tag{11.4}$$

In the few cases where we consider superposed fluids, viscous fluids or surface tension, we shall refer back to Sect. 10 for the equations.

Use of complex variables. For two-dimensional irrotational motion, it is frequently advantageous to use complex variables. Let

$$z = x + i\,y, \qquad f(z, t) = \Phi(x, y, t) + i\,\Psi(x, y, t),$$

where Φ and Ψ are velocity potential and stream function, respectively. (It should be clear from context whether z is being used for $x + iy$ or one of the horizontal coordinates.) Since the equations relating Φ and Ψ,

$$\Phi_x = \Psi_y, \qquad \Phi_y = -\Psi_x,$$

are just the Cauchy-Riemann equations, the function $f(z, t)$ is an analytic function of z for all points z for which Φ_x and Φ_y exist. $f(z, t)$ will be called the "complex potential". The "complex velocity" is given by

$$w(z,t) = f'(z,t) = u - i\,v.$$

The boundary condition at the free surface in (11.1) can be expressed in the following equation in $f(z,t)$:

$$\mathrm{Re}\left\{i\,g\,f'(z,t) + \frac{d^2}{dt^2} f(z,t)\right\} = -\frac{1}{\varrho}\,p_t(x,t) \quad \text{for } y = 0. \tag{11.5}$$

The first equation of (11.3) becomes

$$\mathrm{Re}\,\{i\,g\,f'(z) + c^2 f''(z)\} = \frac{c}{\varrho}\,p'(x) \quad \text{for } y = 0. \tag{11.6}$$

However, Eq. (10.19) shows that this may also be taken in the form

$$\mathrm{Re}\,\{i\,g\,f(z) + c^2 f'(z)\} = \frac{c}{\varrho}\,p(x) \quad \text{for } y = 0.$$

If one may express $f(z,t) = f_1(z)\cos\sigma t + f_2(z)\sin\sigma t$, then the first of Eqs. (11.4) becomes

$$\mathrm{Re}\,\{i\,g\,f_k'(z) - \sigma^2 f_k(z)\} = (-1)^k \frac{\sigma}{\varrho}\,p_{k-(-1)^k}(x) \quad \text{for } y = 0,\ k = 1, 2. \tag{11.7}$$

We note that in order to express $f(z,t)$ in a manner analogous to that used for Φ immediately preceding (11.4) one must introduce a second complex unit j which does not "interact" with i. Thus let $f(z) = f_1(z) + j\,f_2(z)$. Then $f(z,t) = \mathrm{Re}_j\, f(z)\,\mathrm{e}^{-j\sigma t}$.

If $f(z)$ is an analytic function satisfying any one of the conditions (11.5) to (11.7) with $p \equiv 0$, then $f^{(n)}(z)$ will also satisfy it.

12. Other boundary conditions. The boundary conditions given in Sect. 11 will not ordinarily be sufficient to ensure a unique solution to the problems in which the fluid occupies an unbounded region. An additional condition at infinity must be imposed upon the potential function. In certain cases the proper additional condition is fairly clear from the physical problem. For example, for a body moving steadily in an infinite ocean undisturbed except for the body, it seems reasonable to impose the condition that the fluid motion vanish far ahead of and far below the body. For the fluid motion produced by a stationary but steadily oscillating body, it seems reasonable to impose vanishing of the motion far below the body, but outgoing waves at infinity on all sides, if the body does not extend to infinity in some horizontal direction, the so-called "radiation condition".

If the body is not bounded in a horizontal direction, one may easily see that the radiation condition stated above cannot be expected to be satisfied. For example, suppose that waves are being generated by some type of oscillation of a vertical half-plane, say $z = 0$, $x > 0$, in which the oscillatory motion of the half-plane is independent of x. Then one will expect the generated waves to behave like outgoing plane waves from the two sides of the plane as $x \to \infty$; these will not satisfy the radiation condition in the direction Ox. On the other hand, one might expect that the influence of the edge at $x = 0$ would show up as waves satisfying the radiation condition. The formulation of proper boundary conditions in situations of this sort has been discussed by PETERS and STOKER (1954); see also STOKER (1956, 1957, p. 109ff).

In diffraction problems one customarily prescribes the form of an incoming wave and then seeks the scattered wave. The preceding remarks concerning the boundary conditions for waves generated by an oscillating body apply also to the scattered wave.

In more complicated physical situations it is not always clear what boundary conditions should be imposed at infinity, and errors have been made. For example, for a body which is both oscillating with a fixed frequency σ and moving with a steady average velocity c, one might reasonably expect no motion far ahead if c is large, but a radiation condition if c is small. However, the formulation of the boundary condition cannot be completed until the problem is partly solved.

The proper formulation of the boundary conditions at infinity can frequently be obtained by a method recommended by HAVELOCK (1917, 1949a) and used also by BRARD (1948a, b), STOKER (1953, 1954), STOKER and PETERS (1957), DE PRIMA and WU (1957), WU (1957) and others. It consists in formulating an initial-value problem for which the desired steady-state problem is the limit as $t\to\infty$. For the initial-value problem the boundary condition at infinity is that the fluid motion vanishes everywhere. However, even though this procedure may produce the desired solution, it is not always obvious what boundary conditions at infinity in the steady-state problem would have produced it.

13. Some mathematical solutions. Some of the mathematical solutions to be derived in this section will provide solutions, without further modification, to certain physical problems; others, although apparently not acceptable physically, will provide fundamental solutions which can be used in constructing solutions to other more complicated physical problems. In all cases the fluid is assumed unbounded in a horizontal direction and either infinitely deep or with a horizontal bottom $y=-h$; the pressure on the free surface is taken to be zero everywhere. The solutions without singularities are obtained by the method of separation of variables, and are all harmonic in t. It will not be necessary to carry along the subscripts of (11.4).

α) Separation of the y-variable. Assume that one may express φ by

$$\varphi(x, y, z) = Y(y)\,\varphi(x, z).$$

Then $\Delta_3\varphi=\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=0$ becomes, after separation,

$$\Delta_2\varphi + A\,\varphi = 0, \qquad Y'' - A\,Y = 0.$$

The two cases $A=m^2>0$ and $A=-m^2<0$ lead to different solutions.

$A>0$. In this case $\varphi(x,z)$ satisfies the wave equation

$$\Delta_2\varphi + m^2\,\varphi = 0$$

and Y is given by

$$Y = a\,e^{my} + b\,e^{-my}.$$

If the fluid is infinitely deep and $\varphi_y(x,y,z)$ is to remain bounded as $y\to-\infty$, one must have $b=0$. Then condition (11.4) requires

$$m = \frac{\sigma^2}{g}$$

and $\varphi(x,y,z)$ is of the form

$$\varphi(x, y, z) = e^{\sigma^2 y/g}\,\varphi(x, z). \tag{13.1}$$

If the fluid is of finite depth h, the boundary condition $\varphi_y(x,-h,z)=0$ requires Y to take the form

$$Y = a\cosh m\,(y+h)$$

and condition (11.4) becomes

$$m \tanh m h = \frac{\sigma^2}{g},$$

an equation with two real solutions, say $\pm m_0$. In this case, one has

$$\varphi(x, y, z) = \cosh m_0 (y + h)\, \varphi(x, z). \tag{13.2}$$

We note that, if $h_1 < h_2$ then $\sigma^2/g < m_0^{(2)} < m_0^{(1)}$. Also $m_0/h^{\frac{1}{2}} \to \sigma/g^{\frac{1}{2}}$ as $h \to 0$ and $m_0 \to \sigma^2/g$ as $h \to \infty$.

$A < 0$. In this case $\varphi(x, z)$ satisfies

$$\Delta_2 \varphi - m^2 \varphi = 0$$

and Y is given by

$$Y = a \cos m y + b \sin m y.$$

Condition (11.4) restricts Y further to

$$Y = C\left(m \cos m y + \frac{\sigma^2}{g} \sin m y\right).$$

If the fluid is infinitely deep, requiring φ_y to remain bounded imposes no further restriction. If the fluid is of depth h, then $\varphi_y(x, -h, z) = 0$ requires m to satisfy the equation

$$m \tan m h = -\frac{\sigma^2}{g},$$

an equation with an infinite number of real solutions, $\pm m_1, \pm m_2, \ldots$. In this latter case one may conveniently take Y in the form

$$Y = C \cos m (y + h).$$

The roots m_k satisfy $\frac{1}{2}(2k-1)\,\pi/h < m_k < k\pi/h$. For fixed h, $m_k h \to k\pi$ as $k \to \infty$; for fixed k, $m_k h \to k\pi$ as $h \to 0$, and $m_k h \to \frac{1}{2}(2k-1)\,\pi$ as $h \to \infty$.

For these two cases one finds then for $\varphi(x, y, z)$ the forms:

infinite depth:

$$\varphi(x, y, z) = C\left(m \cos m y + \frac{\sigma^2}{g} \sin m y\right) \varphi(x, z); \tag{13.3}$$

finite depth:

$$\varphi(x, y, z) = C \cos m_i (y + h)\, \varphi(x, z). \tag{13.4}$$

β) Further separation of variables. We now assume $\varphi(x, z) = X(x)\, Z(z)$ and substitute in each of the two equations for φ given above.

$A > 0$. In this case substitution in $\Delta \varphi + m^2 \varphi = 0$ gives

$$X'' + (m^2 - k^2)\, X = 0, \qquad Z'' + k^2 Z = 0.$$

(The equations obtained by replacing k^2 by $-k^2$ will give the solution obtained below for $A < 0$, with x and z interchanged.) The solution for Z is

$$Z = f \cos k z + g \sin k z = B \cos (k z + \gamma).$$

The solution for X depends upon the sign of $m^2 - k^2$:

$$k^2 < m^2: \quad X = c \cos x \sqrt{m^2 - k^2} + d \sin x \sqrt{m^2 - k^2};$$
$$k^2 = m^2: \quad X = c x + d;$$
$$k^2 > m^2: \quad X = c\, e^{x\sqrt{k^2 - m^2}} + d\, e^{-x\sqrt{k^2 - m^2}}.$$

$A<0$. Substitution in $\Delta_2\varphi - m^2\varphi = 0$ gives

$$X'' - (k^2+m^2)\,X = 0, \qquad Z'' + k^2 Z = 0,$$

which gives Z as above and

$$X = c\,e^{x\sqrt{k^2+m^2}} + d\,e^{-x\sqrt{k^2+m^2}}.$$

(Substituting $-k^2$ for k^2 would give the solutions corresponding to $A>0$ with x and z interchanged.) We may accumulate the preceding results to obtain the following fundamental solutions:

for infinite depth:

$$\left.\begin{array}{l}
e^{\nu y}\left(a\cos x\sqrt{\nu^2-k^2} + b\sin x\sqrt{\nu^2-k^2}\right)\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2<\nu^2,\\
e^{\nu y}(a x + b)\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2=\nu^2,\\
e^{\nu y}\left(a\,e^{x\sqrt{k^2-\nu^2}} + b\,e^{-x\sqrt{k^2-\nu^2}}\right)\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2>\nu^2,\\
(m\cos m y + \nu\sin m y)\left(a\,e^{x\sqrt{k^2+m^2}} + b\,e^{-x\sqrt{k^2+m^2}}\right)\cos(kz+\gamma)\cos(\sigma t+\tau),
\end{array}\right\}\quad(13.5)$$

where $\nu = \sigma^2/g$;

for finite depth:

$$\left.\begin{array}{l}
\cosh m_0(y+h)\left(a\cos x\sqrt{m_0^2-k^2} + b\sin x\sqrt{m_0^2-k^2}\right)\times\\
\qquad\qquad\times\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2<m_0^2,\\
\cosh m_0(y+h)(a x+b)\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2=m_0^2,\\
\cosh m_0(y+h)\left(a\,e^{x\sqrt{k^2-m_0^2}} + b\,e^{-x\sqrt{k^2-m_0^2}}\right)\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2>m_0^2,\\
\cos m_i(y+h)\left(a\,e^{x\sqrt{k^2+m_i^2}} + b\,e^{-x\sqrt{k^2+m_i^2}}\right)\cos(kz+\gamma)\cos(\sigma t+\tau),
\end{array}\right\}\quad(13.6)$$

where

$$m_0\tanh m_0 h - \frac{\sigma^2}{g} = 0 \quad\text{and}\quad m_i\tan m_i h + \frac{\sigma^2}{g} = 0.$$

The corresponding solutions for two dimensions may be obtained by setting $k=0$ and deleting the second and third equations in each group.

For either set of solutions only the first in each is bounded for all values of the variables for which $y\leqq 0$ or $-h\leqq y\leqq 0$. For two-dimensional motion it has been shown by WEINSTEIN (1927, 1949) that the only function harmonic in $-h<y<0$ and satisfying (11.4) and $\varphi_y(x,-h)=0$ for which both φ and φ_y are bounded in $-h\leqq y\leqq 0$ is $\varphi = A\cosh m(y+h)\sin(mx+\alpha)$. KELDYSH (1935) and STOKER (1947, pp. 7—9) have proved a similar theorem for the lower half-plane: If φ and $\varphi_x^2+\varphi_y^2$ are bounded for $y\leqq 0$ as $x^2+y^2\to\infty$, the only φ satisfying (11.4) and harmonic everywhere in the half-plane $y\leqq 0$ is $A\,e^{ky}\sin(kx+\alpha)$. WEINSTEIN'S theorem has been generalized by JOHN (1950, p. 59) to three dimensions: If $\varphi(x,y,z)$ satisfies (11.4), $\varphi_y(x,-h,z)=0$,

$$\lim_{R\to\infty}\varphi(R\cos\alpha, y, R\sin\alpha)\,R^{-\frac{1}{2}}\,e^{-m_1 R} = 0$$

and is harmonic everywhere in $-h\leqq y\leqq 0$, then $\varphi(x,y,z)$ is of the form (13.2) with $\varphi(x,z)$ an everywhere regular solution of

$$\Delta_2\varphi + m_0^2\varphi = 0.$$

The condition at infinity is necessary, as the solution derived below in (13.8),

$$\varphi = I_0(m_1 R)\cos m_1(y+h),$$

shows. The corresponding theorem for infinite depth was proved by KOCHIN (1940).

The equations for $\varphi(x, y)$ may also be separated in polar coordinates (R, α), $x = R\cos\alpha$, $z = R\sin\alpha$. We give only the solutions:

infinite depth:

$$\left.\begin{aligned} &e^{\nu y}[A\,J_n(\nu R) + B\,Y_n(\nu R)]\cos(n\alpha + d)\cos(\sigma t + \tau),\\ &(m\cos m y + \nu\sin m y)\,[A\,I_n(m R) + B\,K_n(m R)]\cos(n\alpha + d)\cos(\sigma t + \tau),\end{aligned}\right\} \tag{13.7}$$

where $\nu = \sigma^2/g$ and n is an integer;

finite depth:

$$\left.\begin{aligned} &\cosh m_0(y+h)\,[A\,J_n(m_0 R) + B\,Y_n(m_0 R)]\cos(n\alpha + \delta)\cos(\sigma t + \tau),\\ &\cos m_i(y+h)\,[A\,I_n(m_i R) + B\,K_n(m_i R)]\cos(n\alpha + \delta)\cos(\sigma t + \tau),\quad i \geqq 1,\end{aligned}\right\} \tag{13.8}$$

where $m_0\tanh m_0 h - \sigma^2/g = 0$, $m_i\tan m_i h + \sigma^2/g = 0$ and n is an integer. Here J_n, Y_n, I_n, K_n are Bessel functions (we use WATSON's notation). Y_n and K_n are both singular at $R=0$ but approach zero as $R\to\infty$; J_n and I_n are both finite at $R=0$; J_n approaches zero as $R\to\infty$, I_n increases exponentially.

γ) Singular solutions. In this section we shall find solutions of the problems set in Sect. 11 which have singularities of simple type at a single point. We shall indicate proofs only for the case of simple sources, i.e. singularities of the see errata for $[(x-\alpha)^2 + (y-b)^2 + (z-c)^2]^{-\frac{1}{2}}$ or $\log[(x-a)^2 + (z-b)^2]^{\frac{1}{2}}$. We shall consider first the case of a stationary source of pulsating strength, then the case of a moving source. Three-dimensional problems are treated first.

Source of pulsating strength in three dimensions. Let (a, b, c) be in the lower half-space. We wish to find a function

$$\Phi(x, y, z, t) = \varphi_1(x, y, z)\cos\sigma t + \varphi_2(x, y, z)\sin\sigma t$$

defined for $y \leqq 0$ except at (a, b, c) and satisfying

$$\left.\begin{aligned} &1.\ \Delta\varphi_i = 0 \text{ except at } (a, b, c),\ i = 1, 2,\\ &2.\ \varphi_{iy}(x, 0, z) - \nu\varphi_i(x, 0, z) = 0,\ i = 1, 2,\ \nu = \frac{\sigma^2}{g},\\ &3.\ \Phi(x, y, z, t) = r^{-1}\cos\sigma t + \Phi_0(x, y, z, t),\\ &\quad\text{where } \Phi_0 \text{ is harmonic in the whole region } y<0,\\ &4.\ \lim_{y\to-\infty}\operatorname{grad}\varphi_i = 0,\ i = 1, 2,\\ &5.\ \lim_{R\to\infty}\sqrt{R}\left(\frac{\partial\varphi_1}{\partial R} + \nu\varphi_2\right) = 0,\ \lim_{R\to\infty}\sqrt{R}\left(\frac{\partial\varphi_2}{\partial R} - \nu\varphi_1\right) = 0.\end{aligned}\right\} \tag{13.9}$$

Here $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ and $R^2 = (x-a)^2 + (z-c)^2$. Condition 5, see errata usually called the "radiation condition", requires the waves at infinity to ben progressing outwards and imposes a uniqueness which would not otherwise be present. However, other such conditions could be imposed.

We assume that a solution Φ can be found in the form

$$\Phi(x, y, z, t) = [r^{-1} + \varphi_0(x, y, z)]\cos\sigma t + \varphi_2(x, y, z)\sin\sigma t. \tag{13.10}$$

φ_2 will be determined at the end so as to satisfy 5. Denote the double Fourier transform in x and z of φ by $\tilde{\varphi}$:

$$\varphi(x, y, z) = \frac{1}{2\pi}\int_0^\infty\int_{-\pi}^{\pi}\tilde{\varphi}(k, \vartheta, y)\,e^{ik(x\cos\vartheta + z\sin\vartheta)}\,d\vartheta\,dk.$$

Then condition 1 applied to φ_0 becomes after transforming

$$\tilde{\varphi}_{0yy} - k^2 \tilde{\varphi}_0 = 0$$

or

$$\tilde{\varphi}_0 = A_0(k, \vartheta)\, e^{yk} \tag{13.11}$$

where we have used 4. to discard the other solution. From the known integral

$$(x^2 + y^2 + z^2)^{-\frac{1}{2}} = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} e^{-k|y|}\, e^{ik(x\cos\vartheta + z\sin\vartheta)}\, d\vartheta\, dk \tag{13.12}$$

one may compute

$$\widetilde{r^{-1}} = e^{-k|y-b|}\, e^{-ik(a\cos\vartheta + c\sin\vartheta)}. \tag{13.13}$$

Substituting $\tilde{\varphi}_0 + \widetilde{r^{-1}}$ in the transform of condition 2 gives

$$A_0(k, \vartheta) = \frac{k+\nu}{k-\nu}\, e^{kb}\, e^{-ik(a\cos\vartheta + c\sin\vartheta)}. \tag{13.14}$$

We now have, formally,

$$\varphi_0(x, y, z) = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} \frac{k+\nu}{k-\nu}\, e^{k(y+b)}\, e^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]}\, d\vartheta\, dk.$$

Since the integrand has a singularity at $k = \nu$, the integral is not meaningful without further definition. We shall interpret the integral as a Cauchy principal value. Then

$$\left.\begin{aligned} \varphi_1(x, y, z) &= \frac{1}{r} + \frac{1}{2\pi}\, \mathrm{PV} \int_0^\infty \int_{-\pi}^{\pi} \frac{k+\nu}{k-\nu}\, e^{k(y+b)}\, e^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]}\, d\vartheta\, dk, \\ &= \frac{1}{r} + \frac{1}{r_1} + \frac{\nu}{\pi}\, \mathrm{PV} \int_0^\infty \int_{-\pi}^{\pi} \frac{1}{k-\nu}\, e^{k(y+b)}\, e^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]}\, d\vartheta\, dk, \end{aligned}\right\} \tag{13.15}$$

where $r_1^2 = (x-a)^2 + (y+b)^2 + (z-c)^2$. The second equation may be derived easily from the first one by use of (13.12) suitably modified. φ_1 satisfies 1., 2. and 4.; φ_0 is harmonic in the whole region.

In order to satisfy 5. we shall first find the asymptotic form of φ_1 for large R. With polar coordinates

$$x - a = R\cos\alpha, \quad z - c = R\sin\alpha,$$

one may write (13.15) as

$$\begin{aligned} \varphi_1(x, y, z) = \varphi_1(R, \alpha, y) &= \frac{1}{r} + \frac{1}{r_1} + \frac{\nu}{\pi}\, \mathrm{PV} \int_0^\infty \int_{-\pi}^{\pi} \frac{1}{k-\nu}\, e^{k(y+b)}\, e^{ikR\cos(\vartheta-\alpha)}\, d\vartheta\, dk, \\ &= \frac{1}{r} + \frac{1}{r_1} + \frac{4\nu}{\pi}\, \mathrm{PV} \int_0^\infty \int_0^{\frac{1}{2}\pi} \frac{1}{k-\nu}\, e^{k(y+b)} \cos(kR\cos\vartheta)\, d\vartheta\, dk, \\ &= \frac{1}{r} + \frac{1}{r_1} + \frac{4\nu}{\pi}\, \mathrm{PV} \int_0^\infty \int_0^1 \frac{1}{\sqrt{1-\lambda^2}}\, \frac{1}{k-\nu}\, e^{k(y+b)} \cos Rk\lambda\, d\lambda\, dk, \\ &= \frac{1}{r} + \frac{1}{r_1} + 2\nu\, \mathrm{PV} \int_0^\infty \frac{1}{k-\nu}\, e^{k(y+b)}\, J_0(kR)\, dk. \end{aligned}$$

In the next to the last equation we shall change the order of integration, write

$$\cos R\lambda k = \cos R\lambda\nu \cos R\lambda(k-\nu) - \sin R\lambda\nu \sin R\lambda(k-\nu)$$

and use the following theorem from the theory of Fourier integrals[1]: If $f(x)$ is a differentiable function in $[a, \infty]$, if $f''(x_0)$, $x_0 > a$, exists, and if $f(x)/x$ and $f'(x)/x$ are both absolutely integrable in $[a, \infty]$, then, as $R \to \infty$,

$$\int_a^\infty f(x)\frac{\sin R(x-x_0)}{x-x_0}\,dx = \pi f(x_0) + O\left(\frac{1}{R}\right), \quad \mathrm{PV}\int_a^\infty f(x)\frac{\cos R(x-x_0)}{x-x_0}\,dx = O\left(\frac{1}{R}\right). \quad (13.16)$$

Remembering that both r^{-1} and r_1^{-1} are $O(R^{-1})$, one finds

$$\varphi_1(x, y, z) = -4\nu\, e^{\nu(y+b)}\int_0^1 (1-\lambda^2)^{-\frac{1}{2}}\sin R\lambda\nu\, d\lambda + O(R^{-1}).$$

The asymptotic expansion of this integral is well known[2] and we may write

$$\varphi_1(x, y, z) = -2\pi\nu\, e^{\nu(y+b)}\sqrt{\frac{2}{\pi R\nu}}\sin\left(R\nu - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right).$$

If we can find a harmonic function φ_2 satisfying 1., 2. and 4. and having the asymptotic behavior

$$\varphi_2(x, y, z) = 2\pi\nu\, e^{\nu(y+b)}\sqrt{\frac{2}{\pi R\nu}}\cos\left(R\nu - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right),$$

then

$$\varphi_1\cos\sigma t + \varphi_2\sin\sigma t = -2\pi\nu\, e^{\nu(y+b)}\sqrt{\frac{2}{\pi R\nu}}\sin\left(R\nu - \sigma t - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right)$$

will be a solution. The following function fulfils the requirements[3]:

$$\varphi_2(x, y, z) = 2\pi\nu\, e^{\nu(y+b)} J_0(R\nu).$$

We note in passing that φ_1 has the same asymptotic behavior as

$$-2\pi\nu\, e^{\nu(y+b)} Y_0(R\nu).$$

The final result is

$$\left.\begin{aligned}\Phi(x, y, z, t) = &\left[\frac{1}{r} + \mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\, e^{k(y+b)} J_0(kR)\, dk\right]\cos\sigma t + \\ &+ 2\pi\nu\, e^{\nu(y+b)} J_0(\nu R)\sin\sigma t, \qquad \nu = \sigma^2/g.\end{aligned}\right\} \quad (13.17)$$

Haskind (1954), using a derivation having some similarity to that used below for the two-dimensional case, has found the following form for Φ:

$$\left.\begin{aligned}\Phi(x, y, z, t) = &\left[\frac{1}{r} + \frac{1}{r_1} + 2\nu\, e^{\nu y}\int_\infty^y \frac{e^{-\nu y}}{r_1}\, dy - 2\pi\nu\, e^{\nu(y+b)} Y_0(\nu R)\right]\cos\sigma t + \\ &+ 2\pi\nu\, e^{\nu(y+b)} J_0(\nu R)\sin\sigma t.\end{aligned}\right\} \quad (13.17')$$

It is sometimes convenient to use the complex form for the potential, $\varphi\, e^{-i\sigma t}$, with

$$\varphi(x, y, z) = \frac{1}{r} + \mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\, e^{k(y+b)} J_0(kR)\, dk + i\, 2\pi\nu\, e^{\nu(y+b)} J_0(\nu R), \quad (13.17'')$$

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[1] See, e.g.: S. Bochner, Vorlesungen über Fouriersche Integrale, Leipzig, 1932, ch. I and § 8.
[2] See, e.g., A. Erdélyi: Asymptotic expansions, p. 43. Dover, New York 1956.
[3] See the first Eq. (13.7) and G. N. Watson: Bessel functions, p. 199. Cambridge 1949.

for then $\operatorname{Re}\varphi\, e^{-i\sigma t}$ gives (13.17) and $\operatorname{Im}\varphi\, e^{-i\sigma t}$ the source potential for an outgoing wave with singularity of the form $r^{-1}\sin\sigma t$. Eq. (13.17′) may be written analogously. By deforming the path of integration in a familiar way one may also express $\varphi(x, y, z)$ in the form [cf. HAVELOCK (1942, 1955)]:

$$\left.\begin{aligned}\varphi(x,y,z) = \frac{1}{r} + \frac{1}{r_1} - \frac{4\nu}{\pi}\int_0^\infty [\nu\cos k(y+b) - k\sin k(y+b)]\frac{K_0(kR)}{k^2+\nu^2}\,dk - \\ - 2\pi\nu\, e^{\nu(y+b)}\, Y_0(\nu R) + i\,2\pi\nu\, e^{\nu(y+b)} J_0(\nu R).\end{aligned}\right\}\quad(13.17''')$$

In the analogous problem for finite depth h one replaces 4. by 4′. $\varphi_y(x, -h, z) = 0$ and proceeds somewhat similarly. However, in order to satisfy 4′. it is convenient to look for a solution in the form

$$\Phi(x,y,z,t) = [r^{-1} + r_2^{-1} + \varphi_0(x,y,z)]\cos\sigma t + \varphi_2(x,y,z)\sin\sigma t,$$

where

$$r_2^2 = (x-a)^2 + (y+2h+b)^2 + (z-c)^2.$$

Eq. (13.11) then becomes

$$\tilde{\varphi}_0 = A_0(k,\vartheta)\cosh k(y+h)$$

and (13.14), now more complicated because of r^{-1} and r_2^{-1}, becomes

$$A_0(k,\vartheta) = \frac{2(k+\nu)\, e^{-kh}\cosh k(b+h)}{k\sinh kh - \nu\cosh kh}\, e^{-ik(a\cos\vartheta + c\sin\vartheta)}.$$

The final formula for the velocity potential is

$$\left.\begin{aligned}&\Phi(x,y,z,t)\\ &= \left[\frac{1}{r} + \frac{1}{r_2} + \mathrm{PV}\int_0^\infty \frac{2(k+\nu)\, e^{-kh}\cosh k(b+h)\cosh k(y+h)}{k\sinh kh - \nu\cosh kh} J_0(kR)\,dk\right]\cos\sigma t + \\ &+ \frac{2\pi(m_0+\nu)\, e^{-m_0 h}\sinh m_0 h\cosh m_0(b+h)\cosh m_0(y+h)}{\nu h + \sinh^2 m_0 h} J_0(m_0 R)\sin\sigma t,\end{aligned}\right\}\quad(13.18)$$

where $m_0\tanh m_0 h - \nu = 0$, $\nu = \sigma^2/g$. The form of the last term of (13.18) may be altered by using the identities

$$\frac{e^{-m_0 h}\sinh m_0 h}{\nu h + \sinh^2 m_0 h} = \frac{2e^{-m_0 h}\cosh m_0 h}{2m_0 h + \sinh 2m_0 h} = \frac{m_0 - \nu}{m_0^2 h - \nu^2 h + \nu}.$$

JOHN (1950, p. 95) has derived the following series for Φ, the analogue of (13.17‴)

$$\left.\begin{aligned}\Phi(x,y,z,t) = 2\pi\frac{\nu^2 - m_0^2}{hm_0^2 - h\nu^2 + \nu}\cosh m_0(y+h)\cosh m_0(b+h)\,\times \\ \times [Y_0(m_0 R)\cos\sigma t - J_0(m_0 R)\sin\sigma t] + \\ + 4\sum_{k=1}^\infty \frac{m_k^2+\nu^2}{h m_k^2 + h\nu^2 - \nu}\cos m_k(y+h)\cos m_k(b+h)\, K_0(m_k R)\cos\sigma t,\end{aligned}\right\}\quad(13.19)$$

where m_k, $k > 0$, are the positive real roots of $m\tan mh + \nu = 0$. Either expression may also be given in complex form as in (13.17″).

Potential functions satisfying the condition (13.9), but with $r^{-1}\cos\sigma t$ in 3. replaced by a higher-order singularity have been given by THORNE (1953) and HAVELOCK (1955). In fact, THORNE gives a rather complete census of the possible singular solutions for both two and three dimensions and for finite and infinite depth. Included are series expansions as well as integrals. For infinite depth the general expression which includes (13.17) is

$$\left.\begin{aligned}\Phi(x,y,z,t) = \Bigg[\frac{P_n^m(\cos\Theta)}{r^{n+1}} + \frac{(-1)^m}{(n-m)!}\,\mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\,k^n\,e^{k(y+b)}\,J_m(kR)\,dk\Bigg] \times \\ \times \cos m\alpha\cos\sigma t + \frac{(-1)^m}{(n-m)!}\,2\pi\nu^{n+1}\,e^{\nu(y+b)}\,J_m(\nu R)\cos m\alpha\sin\sigma t,\end{aligned}\right\} \quad (13.20)$$

where $\cos\Theta = (y-b)/r$, $x = R\cos\alpha$, $z = R\sin\alpha$. Here P_n^m are the associated Legendre polynomials defined by

$$P_n^m(\mu) = (1-\mu^2)^{m/2}\,\frac{d^m}{d\mu^m}\,P_n(\mu), \qquad m \leqq n.$$

The asymptotic behavior of (13.20) is given by

$$\Phi(x,y,z,t) = \frac{(-1)^{m+1}}{(n-m)!}\,2\pi\nu^{n+1}\,e^{\nu(y+b)}\sqrt{\frac{2}{\pi\nu R}}\,\sin\left(\nu R - \sigma t - \frac{2m+1}{4}\,\pi\right) + O\left(\frac{1}{R}\right).$$

It has been pointed out by both HAVELOCK (1955) and MACCAMY (1954) that solutions can be constructed which vanish much faster than this at infinity. Let the function of (13.20) be denoted by Φ_n. Then $\Phi_{n+1} - \nu(n-m+1)^{-1}\,\Phi_n$ is the following function:

see errata

$$\left.\begin{aligned}\Bigg[\frac{P_{n+1}^m(\cos\Theta)}{r^{n+2}} - \frac{\nu}{n-m+1}\,\frac{P_n^m(\cos\Theta)}{r^{n+1}} + (-1)^m\,\frac{P_{n+1}^m(-\cos\Theta_1)}{r_1^{n+1}} + \\ + (-1)^m\,\frac{\nu}{n-m+1}\,\frac{P_n^m(-\cos\Theta_1)}{r_1^{n+1}}\Bigg]\cos m\alpha\cos\sigma t,\end{aligned}\right\} \quad (13.21)$$

where $\cos\Theta_1 = (y+b)/r_1$, $r_1^2 = (x-a)^2 + (y+b)^2 + (z-c)^2$. For $y=0$ and large R these solutions are $O(R^{-n-1})$ if m and n are both odd, $O(R^{-n-2})$ if one is even and one odd, and $O(R^{-n-3})$ if both are even. Although they have the form of standing waves, they satisfy the radiation condition because they decrease so rapidly with large R.

In addition to the papers cited above, one can find treatments of the submerged source of pulsating strength in KOCHIN (1940), HAVELOCK (1942), JOHN (1950, p. 92ff.), where a detailed discussion is given for the case of finite depth, HASKIND (1944), and LIU (1952). The definition of the improper integral in (13.15) and following is not always the same in these different treatments. In some cases the variable k is treated as complex and the path of integration deflected around the singularity $k=\nu$ by following a small semi-circle in the lower half of the k-plane. The radiation condition is then automatically satisfied if one writes Φ in the complex form $\varphi\,e^{-i\sigma t}$, $\varphi = \varphi_1 + i\varphi_2$. Other treatments achieve the same end by introducing a "fictitious viscosity" $i\mu$ which has the effect of replacing the singularity at $k=\nu$ by one at $k=\nu+i\mu$ and thus placing the path of integration below the singularity. In the end one must find the limit of the solution as $\mu\to 0$. The fictitious viscosity has no relation to real viscosity and may be considered a mathematical device to enable one to interpret an improper integral in a suitable way (for the purpose it seems to be infallible).

Source and vortex of pulsating strength in two dimensions. The two-dimensional problem can be formulated analogously to (13.9), and solutions found in a similar manner. The fundamental singularities will now be of the form $\log r\cos\sigma t$, $r^{-n}\cos n\Theta\cos\sigma t$ and $r^{-n}\sin n\Theta\cos\sigma t$, $n=1,2,\ldots$. The results are given in the paper of THORNE (1953) cited earlier. We shall follow a different method here in order to illustrate the use of complex variables to solve such problems.

We shall consider simultaneously a source of strength Q and a vortex of intensity Γ at the point $c = a + ib$, $b < 0$. In the notation used at the end of Sect. 11, we shall be looking for a function $f(z, t)$ analytic in z and of the form

$$\left.\begin{aligned} f(z,t) &= \left[\frac{\Gamma + iQ}{2\pi i}\log(z-c) + f_0(z)\right]\cos\sigma t + f_2(z)\sin\sigma t, \qquad \operatorname{Im} c < 0, \\ &= f_1(z)\cos\sigma t + f_2(z)\sin\sigma t, \end{aligned}\right\} \tag{13.22}$$

where f_0 and f_2 have no singularities in the lower half-plane. In addition, f_1 and f_2 must each satisfy the free-surface condiction (11.7) which we write

$$\operatorname{Im}\{f_k'(x - i0) + i\nu f_k(x - i0)\} = 0, \qquad \nu = \sigma^2/g, \qquad k = 1, 2. \tag{13.23}$$

Condition 4 of (13.9) will be taken in the somewhat stronger form,

$$|f_k'| \leqq M \quad \text{for} \quad |z| \geqq m \quad \text{and} \quad \lim_{y\to-\infty} |f_k'| = 0, \tag{13.24}$$

where m and M are given constants. The radiation condition becomes:

$$\lim_{x\to\pm\infty} \operatorname{Re}\{f_1' \pm \nu f_2\} = 0, \qquad \lim_{x\to\pm\infty} \operatorname{Re}\{f_2' \mp \nu f_1\} = 0. \tag{13.25}$$

Following a method apparently originally due to LEVI-CIVITA (see TONOLO, 1913), but used frequently by KELDYSH (1935), KOCHIN (e.g., 1939), STOKER (1947), LEWY (1946) and others, we introduce the functions

$$A_k(z) = f_k'(z) + i\nu f_k(z). \tag{13.26}$$

Then (13.23) becomes

$$\operatorname{Im} A_k(x - i0) = 0, \qquad k = 1, 2, \tag{13.27}$$

and (13.22) becomes: the two functions

$$A_0(z) = A_1(z) - \frac{\Gamma + iQ}{2\pi i}\frac{1}{z-c} - \nu\frac{\Gamma + iQ}{2\pi}\log(z-c)$$

and $A_2(z)$ are both regular everywhere in the lower half-plane. A function $A(z)$ with $\operatorname{Im} A(x - i0) = 0$ may be continued into the upper half-plane by defining $A(x+iy) = \overline{A(x-iy)}$, $y > 0$, the bar indicating complex conjugate. Since A_2 is regular in the lower half-plane, the extended function will be regular in the whole plane. In addition, one may derive easily from (13.24) that $|A_2(z)| < C|z| + D$ for sufficiently large $|z|$: then, from the regularity of A_2, such an inequality holds in the whole half-plane and hence in the whole plane after reflection. It then follows from a known generalization of LIOUVILLE'S Theorem[1] that $A_2(z) = az + b$, where a and b are constants. It follows from (13.27) that a and b are real. The differential equation

$$f_2'(z) + i\nu f_2(z) = az + b$$

has the solution

$$f_2(z) = C\,e^{-i\nu z} - \frac{ia}{\nu}z - \frac{ib}{\nu} + \frac{a}{\nu^2}.$$

The condition $\lim_{y\to-\infty} |f_2'| = 0$ requires $a = 0$. Thus, finally

$$f_2(z) = C_2\,e^{-i\nu z} + i B_2, \qquad B_2 \text{ real}.$$

One may set $B_2 = 0$ without loss of generality. Incidentally, this provides a proof of the theorem of STOKER and KELDYSH mentioned earlier [shortly after Eq. (13.6)].

[1] See C. CARATHÉODORY: Theory of functions of a complex variable, Vol. I, § 168. Chelsea, New York 1954.

The function $A_1(z)$, after extension into the upper half-plane, will consist of four singular terms plus a function regular in the whole complex plane, say $A_3(z)$:

$$A_1(z) = \frac{\Gamma + iQ}{2\pi i}\frac{1}{z-c} + \nu\frac{\Gamma+iQ}{2\pi}\log(z-c) - \frac{\Gamma - iQ}{2\pi i}\frac{1}{z-\bar{c}} + \\ + \nu\frac{\Gamma - iQ}{2\pi}\log(z-\bar{c}) + A_3(z).$$

Since A_1 satisfies (13.27), and the four singular terms taken together also have vanishing imaginary part for $y=0$, the same must hold for A_3. Hence A_3 must have the same form as A_2. Substituting the resulting expression for A_1 in (13.26), one has a differential equation for $f_1(z)$. The solution is

$$f_1(z) = \frac{\Gamma+iQ}{2\pi i}\log(z-c) + \frac{\Gamma-iQ}{2\pi i}\log(z-\bar{c}) - \\ - \frac{\Gamma - iQ}{\pi i} e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\bar{c}}\,du + C_1 e^{-i\nu z} + i B_1,$$

where B_1 is real and the path of integration is in the lower half-plane. As in the case f_2, we may set $B_1=0$. C_1 and C_2 must now be chosen to satisfy (13.25). Making use of

$$\int_{-\infty}^{\infty}\frac{e^{i\nu u}}{u-\bar{c}}\,du = 2\pi i\, e^{i\nu\bar{c}},$$

one can show that

$$f_1' + \nu f_2 = -i\nu C_1 e^{-i\nu z} + \nu C_2 e^{-i\nu z} + O(z^{-1}) \quad \text{as} \quad x\to+\infty,$$
$$f_1' - \nu f_2 = -2i(\Gamma - iQ)\, e^{-i\nu(z-\bar{c})} - i\nu C_1 e^{-i\nu z} - \nu C_2 e^{-i\nu z} + O(z^{-1}) \quad \text{as} \quad x\to-\infty.$$

This gives

$$C_1 = -(\Gamma - iQ)\, e^{i\nu\bar{c}}, \qquad C_2 = -i(\Gamma - iQ)\, e^{i\nu\bar{c}}.$$

One may easily verify that this choice of C_1 and C_2 does produce outgoing waves.

If one makes the change of variable $\nu(u-z) = -k(z-\bar{c})$ in the integral term in f_1 and deforms the resulting path to Ox, one finds

$$-e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\bar{c}}\,du = \mathrm{PV}\int_{0}^{\infty}\frac{e^{-ik(z-\bar{c})}}{k-\nu}\,dk + \pi i\, e^{-i\nu(z-\bar{c})}.$$

Substituting this in the expression for f_1, one finally obtains

$$\left.\begin{aligned} f(z,t) = \Bigg[&\frac{\Gamma+iQ}{2\pi i}\log(z-c) + \frac{\Gamma-iQ}{2\pi i}\log(z-\bar{c}) + \\ &+ \frac{\Gamma-iQ}{\pi i}\,\mathrm{PV}\int_0^{\infty}\frac{e^{-ik(z-\bar{c})}}{k-\nu}\,dk\Bigg]\cos\sigma t - i(\Gamma - iQ)\, e^{-i\nu(z-\bar{c})}\sin\sigma t.\end{aligned}\right\} \quad (13.28)$$

Singularities of higher order may be found by differentiating (13.28) with respect to z. The expression for $f'(z,t)$ may be put into a somewhat different form by using

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$$\frac{\Gamma - iQ}{2\pi i}\frac{1}{z-\bar{c}} = \frac{\Gamma - iQ}{2\pi}\int_0^{\infty} e^{-ik(z-c)}\,dk.$$

Then

$$f'(z,t) = \left[\frac{\Gamma + iQ}{2\pi i}\,\frac{1}{z-c} - \frac{\Gamma - iQ}{2\pi}\,\mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\,e^{-ik(z-\bar c)}\,dk\right]\cos\sigma t - \\ -\nu(\Gamma - iQ)\,e^{-i\nu(z-\bar c)}\sin\sigma t. \tag{13.29}$$

One may continue differentiating, using either form for $f'(z,t)$. Thus, from (13.29)

$$f^{(n)}(z,t) = \left[\frac{\Gamma + iQ}{2\pi i}\,\frac{(-1)^{n-1}(n-1)!}{(z-c)^n} - \right. \\ \left. -\frac{\Gamma - iQ}{2\pi}(-i)^{n-1}\,\mathrm{PV}\int_0^\infty k^{n-1}\frac{k+\nu}{k-\nu}\,e^{-ik(z-\bar c)}\,dk\right]\cos\sigma t - \\ -\nu^n(-i)^{n-1}(\Gamma - iQ)\,e^{-i\nu(z-\bar c)}\sin\sigma t. \tag{13.30}$$

By setting $\Gamma = 0$, $z - c = r\,e^{i(\frac{1}{2}\pi-\Theta)} = ir\,e^{-i\Theta}$ (rather than the conventional $r\,e^{i\Theta}$ in order to distinguish easily symmetrical from unsymmetrical solutions) and taking the appropriate real or imaginary part, one finds the following formulas for $\Phi(x,y,t)$:

$$\left.\begin{aligned}
\Phi(x,y,t) &= \left[\frac{Q}{2\pi}\log\frac{r}{r_1} - \frac{Q}{\pi}\,\mathrm{PV}\int_0^\infty \frac{e^{k(y+b)}\cos k(x-a)}{k-\nu}\,dk\right]\cos\sigma t - \\
&\quad - Q\,e^{\nu(y+b)}\cos\nu(x-a)\sin\sigma t,\\
\Phi(x,y,t) &= \left[\frac{Q}{2\pi}\,\frac{\cos n\Theta}{r^n} - \right.\\
&\quad \left. - \frac{(-1)^{n-1}}{(n-1)!}\,\frac{Q}{2\pi}\,\mathrm{PV}\int_0^\infty k^{n-1}\frac{k+\nu}{k-\nu}\,e^{k(y+b)}\cos k(x-a)\,dk\right]\cos\sigma t - \\
&\quad - \frac{(-1)^{n-1}}{(n-1)!}\,Q\,\nu^n e^{\nu(y+b)}\cos\nu(x-a)\sin\sigma t,\\
\Phi(x,y,t) &= \left[\frac{Q}{2\pi}\,\frac{\sin n\Theta}{r^n} + \right.\\
&\quad \left. + \frac{(-1)^{n-1}}{(n-1)!}\,\frac{Q}{2\pi}\,\mathrm{PV}\int_0^\infty k^{n-1}\frac{k+\nu}{k-\nu}\,e^{k(y+b)}\sin k(x-a)\,dk\right]\cos\sigma t + \\
&\quad + \frac{(-1)^{n-1}}{(n-1)!}\,Q\,\nu^n e^{\nu(y+b)}\sin\nu(x-a)\sin\sigma t.
\end{aligned}\right\} \tag{13.31}$$

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In the formula for the logarithmic singularity r_1 may be eliminated and the coefficient of $\cos\sigma t$ written as [see John (1950, p. 100)]:

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$$\frac{Q}{2\pi}\log r + \frac{Q}{2\pi}\,\mathrm{PV}\int_0^\infty\left[\frac{k+\nu}{(k-\nu)k}\,e^{k(y+b)}\cos k(x-a) + \frac{1}{k}\,e^{-k}\right]dk.$$

For water of finite depth the method used above does not work as conveniently because of the difficulty of formulating the boundary condition on the bottom, $\operatorname{Im} f'(x-ih) = 0$, in terms of the function $A(z)$. However, it can be done, yielding a differential-difference equation for $f(z)$ which can be solved by use of

Laplace transforms[1]. The method used for the three-dimensional problem can also be carried through [see HASKIND (1942b), JOHN (1950), and THORNE (1953)].

It is convenient to separate the vortex from the source. The resulting functions are as follows:

vortex:

$$\left.\begin{aligned} f(z,t) = \Big[&\frac{\Gamma}{2\pi i}\log(z-c) - \frac{\Gamma}{2\pi i}\log(z-c_2) - \\ &-\frac{\Gamma}{\pi}\,\mathrm{PV}\!\int_0^\infty \frac{k+\nu}{k}\,\frac{e^{-kh}\sinh k(h+b)\sin k(z-a+ih)}{k\sinh kh - \nu\cosh kh}\,dk\Big]\cos\sigma t - \\ &-\Gamma\frac{\nu+m_0}{m_0}\cdot\frac{e^{-m_0 h}\sinh m_0 h \sinh m_0(h+b)\sin m_0(z-a+ih)}{\nu h + \sinh^2 m_0 h}\sin\sigma t; \end{aligned}\right\} \tag{13.32}$$

source:

$$\left.\begin{aligned} f(z,t) = \Big[&\frac{Q}{2\pi}\log(z-c) + \frac{Q}{2\pi}\log(z-c_2) - \frac{Q}{\pi}\log ih - \\ &-\frac{Q}{\pi}\,\mathrm{PV}\!\int_0^\infty\Big\{\frac{k+\nu}{k}\,\frac{e^{-kh}\cosh k(h+b)\cos k(z-a+ih)}{k\sinh kh-\nu\cosh kh} + \frac{e^{-kh}}{k}\Big\}dk\Big]\cos\sigma t - \\ &-Q\frac{\nu+m_0}{m_0}\,\frac{e^{-m_0 h}\sinh m_0 h\cosh m_0(h+b)\cos m_0(z-a+ih)}{\nu h+\sinh^2 m_0 h}\sin\sigma t. \end{aligned}\right\} \tag{13.33}$$

Here $c_2 = a - ib - 2ih$. The remark following (13.18) concerning the form of the last term of that formula applies also here. The real part of either of these gives the corresponding potential function.

For the source, the integral representation and the series representation analogous to (13.19) are:

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$$\left.\begin{aligned} \Phi(x,y,t) = \Big[&\frac{Q}{2\pi}\log\frac{r}{h} + \frac{Q}{2\pi}\log\frac{r_2}{h} - \\ &-\frac{Q}{\pi}\,\mathrm{PV}\!\int_0^\infty\Big\{\frac{k+\nu}{k}\,\frac{e^{-kh}\cosh k(h+b)\cosh k(y+h)\cos k(x-a)}{k\sinh kh-\nu\cosh kh} - \frac{e^{-kh}}{k}\Big\}dk\Big]\cos\sigma t - \\ &-Q\frac{\nu+m_0}{m_0}\,\frac{e^{-m_0 h}\sinh m_0 h\cosh m_0(h+b)\cosh m_0(y+h)\cos m_0(x-a)}{\nu h+\sinh^2 m_0 h}\sin\sigma t, \\ = Q&\frac{1}{m_0}\,\frac{m_0^2-\nu^2}{h m_0^2 - h\nu^2+\nu}\cosh m_0(y+h)\cosh m_0(b+h)\sin[m_0|x-a|-\sigma t] - \\ &-Q\sum_{k=1}^\infty\frac{1}{m_k}\,\frac{m_k^2+\nu^2}{h m_k^2+h\nu^2-\nu}\cos m_k(y+h)\cos m_k(b+h)\,e^{-m_k|x-a|}\sin\sigma t. \end{aligned}\right\} \tag{13.34}$$

THORNE (1953) gives the potential functions for the higher-order singularities and the function for the logarithmic singularity in a form involving r and r_1 and hence more analogous to the one in (13.31). VOITSENYA (1958) has derived the complex potential for a source-vortex situated in an infinitely deep fluid of density ϱ_1 lying beneath another of density $\varrho_2 < \varrho_1$ and of thickness d.

Source of constant strength in uniform motion: three dimensions. We shall assume the source moving in the direction Ox with constant velocity u_0. Let (x, y, z) be coordinates in a system moving with velocity u_0 in direc-

[1] Cf. S. BOCHNER: Vorlesungen über Fouriersche Integrale, pp. 167–168. Leipzig 1932.

tion Ox and let the source be at (a, b, c), $b<0$. Then, from Sect. 11, we wish to find a function $\varphi(x, y, z)$ satisfying

$$\left.\begin{aligned}
&1.\quad \Delta\varphi = 0 \quad \text{except at } (a, b, c),\\
&2.\quad \varphi_{xx}(x, 0, z) + \nu\,\varphi_y(x, 0, z) = 0, \qquad \nu = g/u_0^2,\\
&3.\quad \varphi(x, y, z) = r^{-1} + \varphi_0(x, y, z),\\
&\qquad\qquad \text{where } \varphi_0 \text{ is harmonic in the region } y<0,\\
&4.\quad \lim_{y\to-\infty} \operatorname{grad}\varphi = 0,\\
&5.\quad \lim_{x\to+\infty} \operatorname{grad}\varphi = 0.
\end{aligned}\right\}\tag{13.35}$$

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For fluid of finite depth h, 4. is replaced by 4'. $\varphi_y(x, -h, z) = a$. Without condition 5, demanding vanishing of the motion far ahead of the source, the solution would not be unique. The profile of the free surface is obtained from $\eta(x, z) = u_0 g^{-1}\varphi_x(x, 0, z)$. Strictly speaking, the solution of (13.35) will represent a sink, i.e. a source of strength -1. However, we shall continue to call such solutions sources.

A solution to this problem may be obtained by methods very similar to those used for the source of pulsating strength. The details will not be repeated, but can be found in HAVELOCK (1932), SRETENSKII (1937), KOCHIN (1937), LUNDE (1951), PETERS and STOKER (1957), TIMMAN and VOSSERS (1955) and elsewhere. The result is

$$\left.\begin{aligned}
\varphi(x,y,z) = {}&\frac{1}{r} - \frac{1}{r_1} - \frac{4\nu}{\pi}\int_0^{\frac12\pi} d\vartheta\,\mathrm{PV}\!\int_0^\infty \frac{e^{k(y+b)}\cos[k(x-a)\cos\vartheta]\cos[k(z-c)\sin\vartheta]}{k\cos^2\vartheta-\nu}\,dk -\\
&-4\nu\int_0^{\frac12\pi} e^{\nu(y+b)\sec^2\vartheta}\sin[\nu(x-a)\sec\vartheta]\cos[\nu(z-c)\sin\vartheta\sec^2\vartheta]\sec^2\vartheta\,d\vartheta,
\end{aligned}\right\}\tag{13.36}$$

where

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2, \quad r_1^2 = (x-a)^2 + (y+b)^2 + (z-c)^2, \quad \nu = g/u_0^2.$$

The potential functions for higher-order singularities are unwieldy and will not be given. The one corresponding to $r^{-n-1}P_n(\cos\Theta)$ can be easily obtained by n-fold differentiation with respect to y, if one remembers that

$$\frac{P_n(\cos\Theta)}{r^{n+1}} = \frac{(-1)^n}{n!}\,\frac{\partial^n}{\partial y^n}\left(\frac{1}{r}\right).$$

The dipole with axis in the direction Ox is obtained by differentiating (13.36) with respect to x and will be used later.

The velocity potential for a source moving in fluid of finite depth has been calculated by SRETENSKII (1937) and by HASKIND (1945b). The form given below is essentially that given by LUNDE (1951):

$$\left.\begin{aligned}
\varphi(x, y, z) = {}&\frac{1}{r} + \frac{1}{r_2} -\\
&-\frac{4}{\pi}\int_0^{\frac12\pi} d\vartheta\,\mathrm{PV}\!\int_0^\infty \frac{e^{-kh}\cosh k(y+h)\,[\cosh k(b+h)\,(k\cos^2\vartheta+\nu)-\nu]}{k\cos^2\vartheta\cosh kh - \nu\sinh kh}\,\times\\
&\qquad\times\cos[k(x-a)\cos\vartheta]\cos[k(z-c)\sin\vartheta]\,dk -\\
&-4\int_{\vartheta_0}^{\frac12\pi} \frac{e^{-k_0 h}\operatorname{sech} k_0 h\cosh k_0(y+h)\,[\cosh k_0(b+h)\,(k_0\cos^2\vartheta+\nu)-\nu]}{\cos^2\vartheta - \nu h\operatorname{sech}^2 k_0 h}\,\times\\
&\qquad\times\sin[k_0(x-a)\cos\vartheta]\cos[k_0(z-c)\sin\vartheta]\,d\vartheta,
\end{aligned}\right\}\tag{13.37}$$

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where $k_0 = k_0(\vartheta)$ is the real positive root of

$$k_0 - \nu \sec^2\vartheta \tanh k_0 h = 0, \qquad \vartheta_0 < \vartheta < \tfrac{1}{2}\pi,$$

where $\vartheta_0 = \operatorname{arc\,cos} \sqrt{\nu h}$ if $\nu h = g h/u_0^2 < 1$, $\vartheta_0 = 0$ if $\nu h \geqq 1$. As before, $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ and $r_2^2 = (x-a)^2 + (y+2h+b)^2 + (z-c)^2$. We note that $k_0(\vartheta) < \nu \sec^2\vartheta$, $k_0(\vartheta) \to 0$ as $\vartheta \to \vartheta_0$, $k_0(\vartheta)/\nu \sec^2\vartheta \to 1$ as $\vartheta \to \frac{1}{2}\pi$ and $k_0(\vartheta) \to \nu \sec^2\vartheta$ as $h \to \infty$. In the double integral the principal value is necessary only for $\vartheta_0 < \vartheta < \frac{1}{2}\pi$, for the singularity does not occur in the denominator for $0 \leqq \vartheta < \vartheta_0$. The part of the double integral with $0 \leqq \vartheta < \vartheta_0$ approaches zero as $x \to \pm\infty$, so that no correction is necessary in order to satisfy condition 5. This is the explanation of the lower limit ϑ_0 in the second integral. In this integral the denominator vanishes only at $\vartheta = \vartheta_0$. One may verify that the integral is convergent by noting that

$$k_0'(\vartheta) = \frac{k_0 \sin 2\vartheta}{\cos^2\vartheta - \nu h \operatorname{sech}^2 k_0 h}$$

and rewriting it as an integral with respect to k_0. When $h \to \infty$, (13.37) reduces to a form of (13.36) in which r_1 is absorbed into the double integral.

For the stationary pulsating source the asymptotic form of the velocity potential for large R was found in the course of deriving the potential function. For the moving source of constant strength the asymptotic form is more difficult to compute. Since the form of the free surface, $\eta = g u_0^{-1} \varphi_x(x, 0, z)$ is of principal physical interest, we shall discuss the asymptotic form of φ_x instead of φ.

Introduce cylindrical coordinates $x - a = R\cos\alpha$, $z - c = R\sin\alpha$ into the x derivative of (13.36):

$$\left.\begin{aligned}\varphi_x(R,\alpha,y) &= \frac{-R\cos\alpha}{[R^2+(y-b)^2]^{\frac{3}{2}}} + \frac{R\cos\alpha}{[R^2+(y+b)^2]^{\frac{3}{2}}} + \\ &+ \frac{2\nu}{\pi}\int_0^{\frac{1}{2}\pi}\sec\vartheta\,d\vartheta\,\mathrm{PV}\int_0^\infty e^{k(y+b)}\frac{\sin[kR\cos(\vartheta-\alpha)]+\sin[kR\cos(\vartheta+\alpha)]}{k-\nu\sec^2\vartheta}\,k\,dk - \\ &- 2\nu^2\int_0^{\frac{1}{2}\pi} e^{\nu(y+b)\sec^2\vartheta}\{\cos[\nu R\sec^2\vartheta\cos(\vartheta-\alpha)] + \\ &\qquad + \cos[\nu R\sec^2\vartheta\cos(\vartheta+\alpha)]\sec^3\vartheta\,d\vartheta.\end{aligned}\right\}\quad(13.38)$$

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For large R the first two terms taken together are $O(R^{-3})$. Apply the theorem (13.16) to the integral with respect to k. This gives, after combining with the second integral,

$$\left.\begin{aligned}\varphi_x(R,\alpha,y) &= 2\nu^2\int_0^{\frac{1}{2}\pi}\sec^3\vartheta\, e^{\nu(y+b)\sec^2\vartheta}\times \\ &\times\{\cos[\nu R\sec^2\vartheta\cos(\vartheta-\alpha)][-1+\operatorname{sgn}\cos(\vartheta-\alpha)] + \\ &+\cos[\nu R\sec^2\vartheta\cos(\vartheta+\alpha)][-1+\operatorname{sgn}\cos(\vartheta+\alpha)]\}\,d\vartheta + O(R^{-1}).\end{aligned}\right\}\quad(13.39)$$

Since φ_x is symmetric in α, we consider only $0 \leqq \alpha \leqq \pi$. We have for $0 \leqq \alpha \leqq \frac{1}{2}\pi$

$$\varphi_x(R,\alpha,y) = -4\nu^2\int_{\frac{1}{2}\pi-\alpha}^{\frac{1}{2}\pi}\sec^3\vartheta\, e^{\nu(y+b)\sec^2\vartheta}\times \cos[\nu R\sec^2\vartheta\cos(\vartheta+\alpha)]\,d\vartheta + O(R^{-1}),$$

and for $\frac{1}{2}\pi < \alpha \leqq \pi$

$$\left.\begin{aligned}\varphi_x(R,\alpha,y) &= -4\nu^2\int_0^{\frac{1}{2}\pi}\sec^3\vartheta\, e^{\nu(y+b)\sec^2\vartheta}\cos[\nu R\sec^2\vartheta\cos(\vartheta+\alpha)]\,d\vartheta - \\ &- 4\nu^2\int_0^{\alpha-\frac{1}{2}\pi}\sec^3\vartheta\, e^{\nu(y+b)\sec^2\vartheta}\cos[\nu R\sec^2\vartheta\cos(\vartheta-\alpha)]\,d\vartheta + O(R^{-1}).\end{aligned}\right\}\quad(13.40)$$

Consider the two integrals containing $\cos(\vartheta+\alpha)$ and let

$$\lambda = \sec^2\vartheta \cos(\vartheta+\alpha).$$

Then for $0 \leqq \alpha \leqq \frac{1}{2}\pi$ the integral becomes

$$-4\nu^2 \int_0^{-\infty} \frac{2\,e^{\nu(y+b)\sec^2\vartheta}}{\sin(2\vartheta+\alpha) - 3\sin\alpha} \cos\nu R\lambda\, d\lambda.$$

If $\frac{1}{2}\pi < \alpha < \pi$, the lower limit is $\cos\alpha$. In either case one may show that the coefficient of $\cos\nu R\lambda$ is single-valued, continuous, absolutely integrable and monotonically decreasing as a function of λ. By integration by parts one may then establish the following estimates as $R \to \infty$ (cf. S. BOCHNER, Vorlesungen über Fouriersche Integrale, Leipzig, 1932, § 3):

for $0 \leqq \alpha \leqq \frac{1}{2}\pi$

$$O(R^{-2});$$

for $\frac{1}{2}\pi < \alpha < \pi$

$$-4\nu \frac{e^{\nu(y+b)}}{R\sin\alpha} \sin(\nu R\cos\alpha) + O(R^{-2}).$$

If $\alpha = \pi$, the two integrals in (13.40) combine to give

$$8\nu^2 \int_1^{\infty} e^{\nu(y+b)\lambda^2} \frac{\lambda^2}{\sqrt{\lambda^2-1}} \cos\nu R\lambda\, d\lambda$$

$$= 4\nu^2 \sqrt{2\pi} \frac{e^{\nu(y+b)}}{\sqrt{\nu R}} \cos\left(\nu R - \frac{3}{4}\pi\right) + O(R^{-1}).$$

Consider now the remaining integral in (13.40), and let

$$\mu(\vartheta) = \sec^2\vartheta \cos(\vartheta-\alpha).$$

The integral takes the form

$$-8\nu^2 \int_{\cos\alpha}^{0} e^{\nu(y+b)\sec^2\vartheta} \frac{\cos\nu R\mu}{\sin(2\vartheta-\alpha) + 3\sin\alpha} d\mu.$$

The denominator now becomes zero when

$$\tan\vartheta = -\tfrac{1}{4}\cot\alpha\left[1 \pm \sqrt{1 - 8\tan^2\alpha}\right], \tag{13.41}$$

an equation which has real roots when $\tan^2\alpha \leqq \frac{1}{8}$, i.e. when

$$180° - 19°28' \ldots < \alpha < 180°.$$

When $\frac{1}{2}\pi < \alpha < \pi - \arcsin\frac{1}{3} = \alpha_c$, the Fourier-integral estimate used for the other integral may be applied to give

$$4\nu\, e^{\nu(y+b)} \frac{\sin(\nu R\cos\alpha)}{R\sin\alpha} + O(R^{-2}).$$

When $\alpha_2 < \alpha < \pi$, ϑ is a two-branched function of μ and the resulting two integrals each have singularities at one of the limits. Thus the elementary method of analysis used above can no longer be applied. However, a modification of the method above can be carried through[1]; the classical treatment is by the method of stationary phase which is well discussed in STOKER (1957, Chap. 8).

[1] See, e.g., A. ERDÉLYI: Asymptotic expansions, pp. 46—56. Dover, New York 1956.

The estimates already derived are of the same order as the remainder term in (13.39). Analysis of this term produces terms which cancel the terms in $1/R \sin\alpha$ already derived, thus removing an apparent singular behavior near the x-axis

The asymptotic form for the surface $\eta(R, \alpha)$ above a source of strength m (i.e. $-m/r$) is as follows:

for $0 \leqq \alpha < \pi - \arcsin\frac{1}{3} = \alpha_c$

$$\eta(R, \alpha) = O\big((\nu R)^{-2}\big); \tag{13.42}$$

for $\alpha = \alpha_c$

$$\eta(R, \alpha_c) = 4 \cdot 3^{\frac{5}{6}} \Gamma\left(\frac{1}{3}\right) \frac{m\nu}{u_0} (\nu R)^{-\frac{1}{3}} e^{\frac{3}{2}\nu b} \cos\left(\frac{\sqrt{3}}{2} \nu R\right) + O\big((\nu R)^{-\frac{2}{3}}\big);$$

for $\alpha_c < \alpha < \pi$

$$\eta(R, \alpha) = 4\sqrt{2\pi}\, \frac{m\nu}{u_0}\, \frac{(\nu R)^{-\frac{1}{2}}}{[1 - 9\sin^2\alpha]^{\frac{1}{4}}} \times$$

$$\times \{\sec^{\frac{3}{2}}\vartheta_1 e^{\nu b \sec^2\vartheta_1} \cos(\nu R \mu_1 - \tfrac{1}{4}\pi) + \sec^{\frac{3}{2}}\vartheta_2 e^{\nu b \sec^2\vartheta_2} \cos(\nu R \mu_2 + \tfrac{1}{4}\pi)\} + O\big((\nu R)^{-1}\big);$$

for $\alpha = \pi$

$$\eta(R, \pi) = -4\sqrt{2\pi}\, \frac{m\nu}{u_0} (\nu R)^{-\frac{1}{2}} e^{\nu b} \cos(\nu R - \tfrac{3}{4}\pi) + O\big((\nu R)^{-1}\big).$$

Here ϑ_1 and $\vartheta_2 > \vartheta_1$ are the two roots (13.41) and $\mu_1 (<0)$ and $\mu_2 < \mu_1$ the corresponding values of $\sec^2\vartheta \cos(\vartheta - \alpha)$. As $\alpha \to \alpha_c$, $\vartheta_i \to \arctan\frac{1}{2}\sqrt{2}$, $\mu_i \to -\frac{1}{2}\sqrt{3}$; as $\alpha \to \pi$, $\vartheta_1 \to 0$, $\mu_1 \to -1$, $\vartheta_2 \to \frac{1}{2}\pi$, $\mu_2 \to -\infty$. In order to have some idea of the form of the free surface far behind the source, one may graph the curves

$$\nu R \mu_1(\alpha) - \tfrac{1}{4}\pi = -2n\pi, \quad \nu R \mu_2(\alpha) + \tfrac{1}{4}\pi = -2n\pi, \quad n > 0,$$

showing the traces of the wave crests in the region

$$\pi - \arcsin\tfrac{1}{3} < \alpha < \pi + \arcsin\tfrac{1}{3}.$$

This gives the well known pattern shown in Fig. 1a. The first equation gives the transverse waves, the second one the diverging waves. The wavelength along $\alpha = \pi$ is $2\pi/\nu$ and along the boundary lines $4\pi\sqrt{3}/3\nu$. The expansion is not suitable in the region near the boundary lines $\alpha = \alpha_c$. As $\alpha \to \alpha_c$, $\alpha > \alpha_c$, the term $[1 - 9\sin^2\alpha]^{\frac{1}{4}} \to 0$ and the amplitudes become infinite. A special investigation of the region near $\alpha = \alpha_c$ is necessary and shows $(\nu R)^{-\frac{1}{3}}$ as leading term; η may be expressed in terms of Airy functions [cf. URSELL (1960)].

Essentially the same pattern is produced by a moving concentrated pressure on the free surface; it was first analysed by KELVIN (1906 = Papers, Vol. IV, pp. 407—413). The asymptotic behavior for moving pressure distributions has been extensively studied [e.g., HOGNER (1923), TETURÔ INUI (1936), PETERS (1949), BARTELS and DOWNING (1955)]. LAMB (1926) has given the asymptotic form of the surface over a moving submerged dipole. The form of the surface near the moving dipole has been investigated by HAVELOCK (1928), who gives traces of the profile on planes α = const for several values of α between $\frac{1}{2}\pi$ and π (the radial lines of Figs. 1b and c) and for $|b\nu| = \frac{1}{2}$ and 4. HAVELOCK's computations were later used by WIGLEY (1930) to produce the contour curves shown in Figs. 1b and c.

A similar analysis may be made for (13.37), a source moving in fluid of finite depth. For a moving pressure distribution the problem has been treated by HAVELOCK (1908) and TETURÔ INUI (1936). The pattern is modified as follows.

If $\nu h > 1$, the pattern is qualitatively like that for $h = \infty$. However, the wedge within which the disturbance is chiefly contained has a wider aperture and as $\nu h \to 1$ the aperture approaches $\frac{1}{2}\pi$ radians on each side of the line of motion.

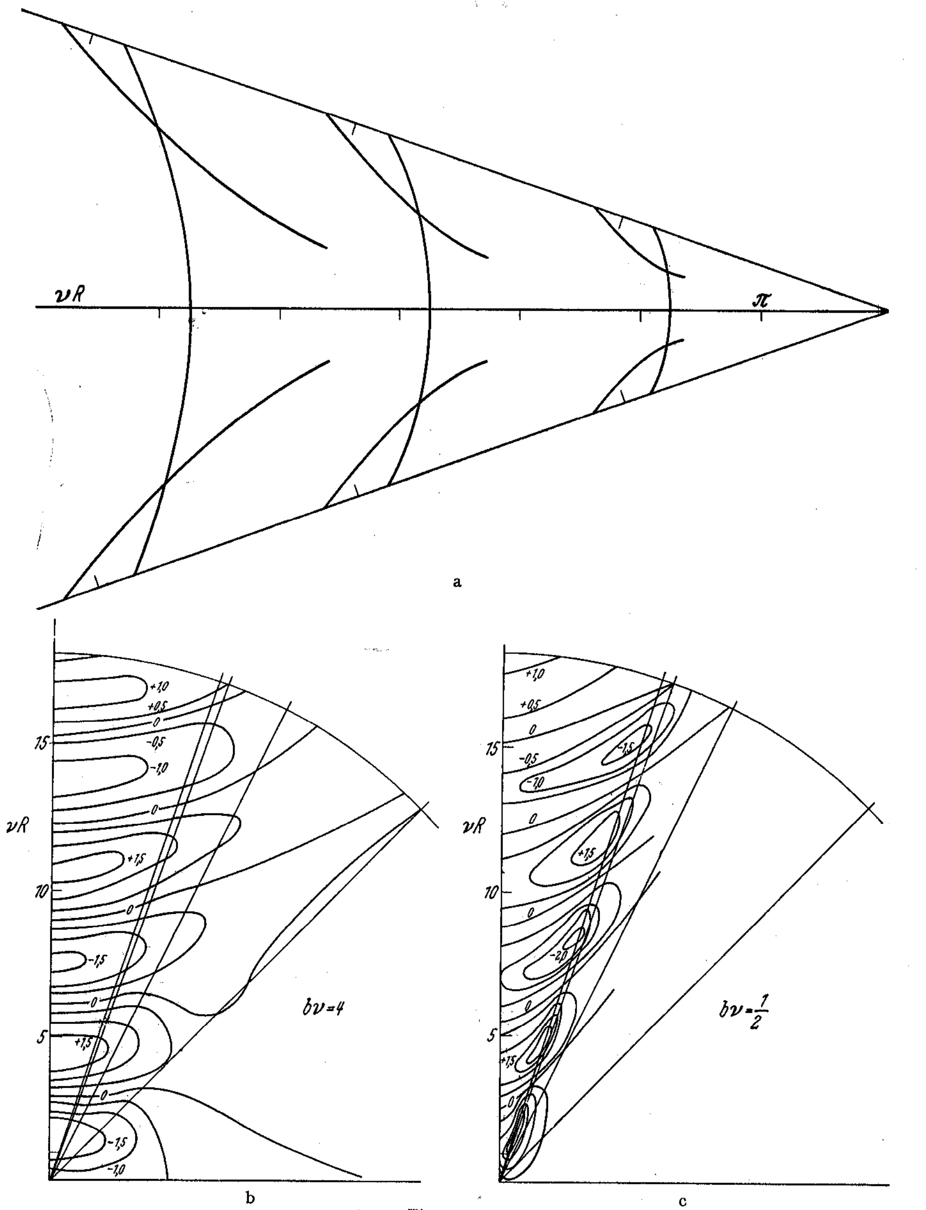

Fig. 1 a—c.

In addition, the wave length of the transverse wave system increases and approaches infinity as $\nu h \to 1$. When $\nu h \geqq 1$, the transverse wave system is missing completely, but diverging waves still occur in a wedge of aperture varying from π to 0 as $\nu h \to 0$. [See also EKMAN (1906), who has considered the free surface over a dipole on a flat bottom.]

Fig. 2 from HAVELOCK (1908) shows the half-angle of the aperture.

KOCHIN (1938c) has gone further in this type of problem. He has derived the potential for a source situated in a fluid of density ϱ_1 and depth h, bounded below by a plane, over which is lying another fluid of density $\varrho_2 < \varrho_1$, extending infinitely far upwards. The lower fluid moves with velocity c_1, the upper with velocity c_2 in the same direction. He also finds the asymptotic behavior of the solution.

Singularities of constant strength in uniform motion: two dimensions. For submerged sources and vortices in two-dimensional motion the complex-variable method used for the pulsating source may again be applied. For infinitely deep fluid, the computation has been carried out in this way by KELDYSH and LAVRENT'EV (1937) and KOCHIN (1937); a detailed exposition is given in the textbook of KOCHIN, KIBEL' and ROZE (1948, Chap. VIII, § 19). HAVELOCK (1927) and SRETENSKII (1938) have treated the problem by different methods. The complex velocity potential for a combined source of strength Q and vortex of intensity Γ at $c = a + ib$, $b < 0$, is given by

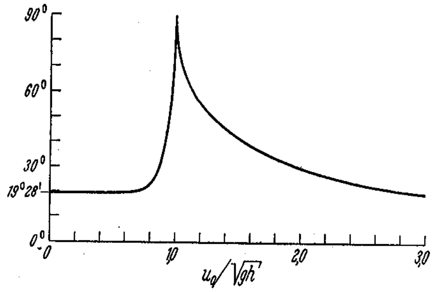

Fig. 2.

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$$\left.\begin{aligned} f(z) &= \frac{\Gamma + iQ}{2\pi i}\log(z-c) - \frac{\Gamma - iQ}{2\pi i}\log(z-\bar{c}) + \frac{\Gamma - iQ}{\pi i}\,\mathrm{e}^{-i\nu z}\int_{\infty}^{z}\frac{\mathrm{e}^{i\nu u}}{u-\bar{c}}\,du,\\ &= \frac{\Gamma + iQ}{2\pi i}\log(z-c) - \frac{\Gamma - iQ}{2\pi i}\log(z-\bar{c}) - 2(\Gamma - iQ)\,\mathrm{e}^{-i\nu(z-\bar{c})} + \\ &\qquad + \frac{\Gamma - iQ}{\pi i}\,\mathrm{e}^{-i\nu z}\int_{-\infty}^{z}\frac{\mathrm{e}^{i\nu u}}{u-\bar{c}}\,du,\\ &= \frac{\Gamma + iQ}{2\pi i}\log(z-c) - \frac{\Gamma - iQ}{2\pi i}\log(z-\bar{c}) - \frac{\Gamma - iQ}{\pi i}\,\mathrm{PV}\int_{0}^{\infty}\frac{\mathrm{e}^{-ik(z-\bar{c})}}{k-\nu}\,dk - \\ &\qquad - (\Gamma - iQ)\,\mathrm{e}^{-i(z-\bar{c})}. \end{aligned}\right\} \quad (13.43)$$

The real velocity potential for, say, a submerged source can be obtained from any of these equations. The last one gives a form analogous to (13.36):

$$\left.\begin{aligned} \varphi(x,y) &= \frac{Q}{2\pi}\log r + \frac{Q}{2\pi}\log r_1 + \\ &\quad + \frac{Q}{\pi}\,\mathrm{PV}\int_{0}^{\infty}\frac{\mathrm{e}^{k(y+b)}\cos k(x-a)}{k-\nu}\,dk + Q\,\mathrm{e}^{\nu(y+b)}\sin\nu(x-a). \end{aligned}\right\} \quad (13.44)$$

Higher-order singularities can be obtained by differentiating (13.43). The complex velocity potential for a dipole of moment M and axis in the direction $\mathrm{e}^{i\alpha}$ is given by

$$\left.\begin{aligned} f(z) &= -\frac{M}{2\pi}\,\frac{\mathrm{e}^{i\alpha}}{z-c} + \frac{M}{2\pi}\,\frac{\mathrm{e}^{-i\alpha}}{z-\bar{c}} - \frac{iM\nu}{\pi}\,\mathrm{e}^{-i\alpha}\,\mathrm{e}^{-i\nu z}\int_{\infty}^{z}\frac{\mathrm{e}^{i\nu u}}{u-\bar{c}}\,du,\\ &= -\frac{M}{2\pi}\,\frac{\mathrm{e}^{i\alpha}}{z-c} + \frac{M}{2\pi}\,\frac{\mathrm{e}^{-i\alpha}}{z-\bar{c}} + \frac{iM\nu}{\pi}\,\mathrm{e}^{-i\alpha}\,\mathrm{PV}\int_{0}^{\infty}\frac{\mathrm{e}^{-ik(z-\bar{c})}}{k-\nu}\,dk - \\ &\qquad - M\nu\,\mathrm{e}^{-i\alpha}\,\mathrm{e}^{-i\nu(z-\bar{c})}. \end{aligned}\right\} \quad (13.45)$$

If in the last term of the first equation of either (13.43) or (13.45), one makes use of the identity

$$e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\bar c}\,du=\int_{-\infty}^{0}\frac{-e^{-i\nu u}}{z-u-\bar c}\,du,$$

it is not difficult to see that this last term is equivalent to a distribution of dipoles on the ray from $\bar c$ to $-\infty$ parallel to the x-axis. The moment density and axis can be determined for the three cases, source, vortex and dipole, by comparison of the integrand with the first term of (13.45).

For the case of finite depth the complex velocity potential has been calculated by TIKHONOV (1940) and is also given by HASKIND (1945a) for both source and vortex. We give separately source, vortex and dipole:

source:

$$\left.\begin{aligned} f(z)&=\frac{Q}{2\pi}\log(z-c)+\frac{Q}{2\pi}\log(z-c_2)+\\ &+\frac{2Q}{\pi}\,\mathrm{PV}\int_0^\infty\frac{k+\nu}{k}\,e^{-kh}\,\frac{\cosh k(b+h)}{\nu\sinh kh-k\cosh kh}\sin^2\tfrac{1}{2}k(z-a+ih)\,dk-\\ &-\frac{Q\nu}{k_0}\,\frac{\cosh k_0(b+h)}{\nu h-\cosh^2 k_0 h}\sin k_0(z-a+ih);\end{aligned}\right\}\quad(13.46)$$

vortex:

$$\left.\begin{aligned} f(z)&=\frac{\Gamma}{2\pi i}\log(z-c)-\frac{\Gamma}{2\pi i}\log(z-c_2)-\\ &-\frac{\Gamma}{\pi}\,\mathrm{PV}\int_0^\infty\frac{k+\nu}{k}\,e^{-kh}\,\frac{\sinh k(b+h)}{\nu\sinh kh-k\cosh kh}\sin k(z-a+ih)\,dk+\\ &+\frac{\Gamma\nu}{k_0}\,\frac{\sinh k_0(b+h)}{\nu h-\cosh^2 k_0 h}\cos k_0(z-a+ih);\end{aligned}\right\}\quad(13.47)$$

dipole:

$$\left.\begin{aligned} f(z)&=-\frac{M}{2\pi}\,\frac{e^{i\alpha}}{z-c}-\frac{M}{2\pi}\,\frac{e^{-i\alpha}}{z-c_2}-\\ &-\frac{M}{2\pi}\,\mathrm{PV}\int_0^\infty(k+\nu)\,e^{-kh}\,\frac{e^{i\alpha}\sin k(z-c)+e^{-i\alpha}\sin k(z-c_2)}{\nu\sinh kh-k\cosh kh}\,dk+\\ &+\frac{\nu M}{2}\,\frac{e^{i\alpha}\cos k_0(z-c)+e^{-i\alpha}\cos k_0(z-c_2)}{\nu h-\cosh^2 k_0 h}.\end{aligned}\right\}\quad(13.48)$$

Here $c_2=a-ib+2ih$ and the last summand in each of (13.46) to (13.48) is to be deleted if $\nu h=gh/c^2\leqq 1$; k_0 is the positive real root of $\nu\sinh kh-k\cosh kh=0$, which exists only if $\nu h>1$.

Asymptotic form of these functions as $x\to-\infty$ is easily seen to be given by double the last term in each expression. When $\nu h<1$, the disturbance is only local, a fact which corresponds to the absence of transverse waves behind the three-dimensional source for $\nu h<1$.

KOCHIN (1937a, b) has derived the complex velocity potential when fluid of density ϱ_2 overlies the fluid of density $\varrho_1>\varrho_2$ containing the singularity. The lower fluid may be of infinite or finite depth; the upper one is taken infinitely deep. Their velocities may be different.

Source of variable strength, starting from rest and following an arbitrary path. Consider now a source whose position and strength at time

$t \geqq 0$ are given by $(a(t), b(t), c(t))$ and $m(t)$, where $b(t) < 0$. Let $m(t) = 0$ for $t < 0$. The conditions to be satisfied by the velocity potential $\Phi(x, y, z, t)$ are

1. $\Delta \Phi = 0, \quad y < 0, \quad (x, y, z) \neq (a(t), b(t), c(t))$,
2. $\Phi_{tt}(x, 0, z, t) + g\, \Phi_y(x, 0, z, t) = 0$,
3. $\Phi(x, y, z, t) = m(t)\, r^{-1} + \Phi_0(x, y, z, t)$, Φ_0 harmonic everywhere in $y < 0$,
4. $\lim_{y \to -\infty} \operatorname{grad} \Phi = 0$ for all x, z and t,
5. $\lim_{R \to \infty} \operatorname{grad} \Phi = 0$ for all t,
6. $\Phi(x, 0, z, 0) = \Phi_t(x, 0, z, 0) = 0$.

Here $r^2 = (x - a(t))^2 + (y - b(t))^2 + (z - c(t))^2$, $R^2 = (x - a(t))^2 + (z - c(t))^2$.

If one assumes a solution in the form

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$$\Phi = m\, r^{-1} - m_1^{-1}\, r + \Phi_1$$

where $r_1^2 = (x - a)^2 + (y + b)^2 + (z - c)^2$, then Φ_1 must be harmonic in $y < 0$ and satisfy 4., 5., 6. and

$$\Phi_{1tt}(x, 0, z, t) + g\, \Phi_{1y}(x, 0, z, t) = -2g\, m(t)\, b(t) \left[(x-a)^2 + b^2 + (z-c)^2\right]^{-\frac{3}{2}}, \quad t \geqq 0.$$

It follows from the conditions that, for $t < 0$, $\Phi_1 = \text{const}$, which we may take as zero. Let $\bar{\Phi}_1$ be the Laplace transform of Φ_1:

$$\bar{\Phi}_1(x, y, z, s) = \int_0^\infty \mathrm{e}^{-st}\, \Phi_1(x, y, z, t)\, dt.$$

Then $\bar{\Phi}_1$ is a harmonic function in $y < 0$ satisfying 4. and 5. for each s and also, after making use of 6., the condition

$$s^2 \bar{\Phi}_1(x, 0, z, s) + g\, \bar{\Phi}_{1y}(x, 0, z, s) = -2g \int_0^\infty \mathrm{e}^{-st}\, m(t)\, b(t) \left[(x-a)^2 + b^2 + (z-c)^2\right]^{-\frac{3}{2}} dt.$$

Since

$$s^2 \bar{\Phi}_1(x, y, z, s) + g\, \bar{\Phi}_{1y}(x, y, z, s) + \\ + 2g \int_0^\infty \mathrm{e}^{-st}\, m(t)\, (y + b(t)) \left[(x-a)^2 + (y+b)^2 + (z-c)^2\right]^{-\frac{3}{2}} dt$$

is a harmonic function in $y < 0$ vanishing on $y = 0$ and at infinity, it is identically zero. Making use of (13.12) differentiated with respect to y and changing the order of integration, one obtains

$$s^2 \bar{\Phi}_1(x, y, z, s) + g\, \bar{\Phi}_{1y}(x, y, z, s) \\ = \frac{g}{\pi} \int_0^\infty k\, dk \int_0^\infty dt\, \mathrm{e}^{-st} m(t)\, \mathrm{e}^{k(y+b)} \int_{-\pi}^{\pi} d\vartheta\, \mathrm{e}^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]} \\ = 2g \int_0^\infty k\, dk \int_0^\infty dt\, \mathrm{e}^{-st}\, m(t)\, \mathrm{e}^{k(y+b)}\, J_0\!\left(k\left[(x-a)^2 + (z-c)^2\right]^{\frac{1}{2}}\right).$$

The solution for $\bar{\Phi}_1$ is

$$\bar{\Phi}_1(x, y, z, s) = 2g \int_0^\infty dk\, \frac{k}{s^2 + g k} \int_0^\infty dt\, \mathrm{e}^{-st}\, m(t)\, \mathrm{e}^{k(y+b)}\, J_0(k\, R(t)).$$

Making use of the convolution theorem and the fact that $(s^2+gk)^{-1}$ is the transform of $(gk)^{-\frac{1}{2}}\sin(gk)^{\frac{1}{2}}t$, one may find the original function $\Phi_1(x,y,z,t)$:

$$\Phi_1(x,y,z,t)=2\int_0^\infty dk\,(g\,k)^{\frac{1}{2}}\int_0^t d\tau\sin[(g\,k)^{\frac{1}{2}}(t-\tau)]\,m(\tau)\,e^{k(y+b(\tau))}J_0(k\,R(\tau)).$$

For fixed t one may easily verify, using known properties of the Fourier-Bessel transform[1], that Φ_1 is $o(R^{-\frac{1}{2}})$ and hence that 5. is satisfied. One has then the result

$$\left.\begin{aligned}\Phi(x,y,z,t)&=\frac{m(t)}{r(t)}-\frac{m(t)}{r_1(t)}+2\int_0^\infty dk\,(g\,k)^{\frac{1}{2}}\int_0^t d\tau\,m(\tau)\sin[(g\,k)^{\frac{1}{2}}(t-\tau)]\times\\&\qquad\times e^{k(y+b)}J_0(k\,R(\tau))\\&=\frac{m(t)}{r(t)}-\frac{m(t)}{r_1(t)}+\frac{1}{\pi}\int_{-\pi}^{\pi}d\vartheta\int_0^\infty dk(g\,k)^{\frac{1}{2}}\int_0^t d\tau\,m(\tau)\sin[(g\,k)^{\frac{1}{2}}(t-\tau)]\times\\&\qquad\times e^{k[y+b(\tau)+i(x-a(\tau))\cos\vartheta+i(z-c(\tau))\sin\vartheta]}.\end{aligned}\right\}\quad(13.49)$$

see errata

By a more refined analysis of the behavior for large R [cf. STOKER (1957, pp. 190 to 191)] one may establish that Φ is $O(R^{-2})$ and Φ_R and Φ_y are $O(R^{-3})$ as $R\to\infty$.

For some time $t>t_0\geqq 0$, one may write Φ in the form

$$\begin{aligned}\Phi(x,y,z,t)&=2\int_0^\infty dk\,(g\,k)^{\frac{1}{2}}\int_0^{t_0} d\tau\,m(\tau)\sin[(g\,k)^{\frac{1}{2}}(t-\tau)]\,e^{k[y+b(\tau)]}J_0(k\,R(\tau))+\\&\quad+\frac{m(t)}{r(t)}-\frac{m(t)}{r_1(t)}+2\int_0^\infty dk\,(g\,k)^{\frac{1}{2}}\int_0^{t-t_0} d\tau\,m(\tau+t_0)\sin[(g\,k)^{\frac{1}{2}}(t-t_0-\tau)]\times\\&\quad\times e^{k[y+b(\tau+t_0)]}J_0(k\,R(\tau+t_0))=\Phi_2(x,y,z,t)+\Phi_3(x,y,z,t).\end{aligned}$$

Here the first summand Φ_2 represents the effect at time $t>t_0$ of the action of the source from $t=0$ to $t=t_0$. The remaining terms, Φ_3, are the same as (13.49) with t measured from $t_0(m(t)=m(t-t_0+t_0)$, etc.), and show the effect at time t of the action of the source from $t=t_0$ to $t=t$. (This is, of course, what one would expect from the linearity of the problem and the fact that the choice of $t=0$ is arbitrary.) When $t=t_0$, Φ_3 reduces to

$$\frac{m(t_0)}{r(t_0)}-\frac{m(t_0)}{r_1(t_0)}.$$

Thus

$$\Phi_3(x,0,z,t_0)=0.$$

This fact provides a basis for HAVELOCK'S procedure in similar problems, a procedure originating with KELVIN in the treatment of moving pressure distributions. The idea is roughly as follows. Divide the path of the source into small segments of time span Δt. If Δt is small enough, the effect of gravity upon the fluid motion produced by the source during this time interval will be negligible, and one may take the boundary condition at the free surface to be $\Phi=0$. The distortion of the surface resulting from the action of the source during this short interval is found and the future behavior of the distortion computed while taking account of gravity. Summing over all Δt and taking the limit leads to the potential function.

[1] Cf. G.N. WATSON: Bessel functions, § 14.41. Cambridge 1944.

The expression (13.49) has been essentially given by HASKIND (1946b) and BRARD (1948a). Special choices of $m(t)$ and of the motion of the source lead to cases similar to those treated earlier. Thus, if $m(t) = m \cos\sigma t$ and (a, b, c) is fixed, one has the potential function for a stationary source of oscillating strength, starting to oscillate at $t=0$. Carrying out the t integration and taking a limit by using, say, the Fourier Integral Theorem (13.16) allow one to derive (13.17). The radiation condition is automatically satisfied. The velocity potential for finite values of t may be written in the form

$$\left.\begin{aligned} \Phi(x,y,z,t) = {} & \frac{m\cos\sigma t}{r} - \frac{m\cos\sigma t}{r_1} + \\ & + 2m\cos\sigma t\,\mathrm{PV}\int_0^\infty \frac{k}{k-\sigma^2/g}\,\mathrm{e}^{k(y+b)} J_0(kR)\,dk - \\ & - 2m\,\mathrm{PV}\int_0^\infty \frac{k}{k-\sigma^2/g}\cos(gk)^{\frac{1}{2}}t\,\mathrm{e}^{k(y+b)} J_0(kR)\,dk. \end{aligned}\right\} \tag{13.50}$$

The leading term in the asymptotic expansion of the last summand gives the last summand of (13.17).

If one takes $m(t) = m =$ a constant, $a(t) = a_0 + u_0 t$, $b(t) = b_0$, $c(t) = c_0$, one obtains the velocity potential for a source suddenly brought into existence at $t=0$ and moving with constant velocity in the direction Ox [cf. LUNDE (1951, p. 18)]. A limit as $t\to\infty$ will give (13.36), the proper boundary conditions at infinity being again automatically satisfied. For finite t the velocity potential in a coordinate system moving with velocity u_0 in direction Ox $\big(\bar{x} = x - u_0 t$, so that $\Phi(x,y,z,t) = \varphi(\bar{x},y,z,t)\big)$ is given by

$$\left.\begin{aligned} \varphi(\bar{x},y,z,t) & = \frac{m}{r} - \frac{m}{r_1} + \\ & + \frac{m}{\pi}\int_{-\pi}^{\pi} d\vartheta \int_0^\infty dk\,(gk)^{\frac{1}{2}}\,\mathrm{e}^{k[y+b_0+i\omega(\vartheta)]}\int_0^t d\tau\,\sin\tau(gk)^{\frac{1}{2}}\,\mathrm{e}^{iku_0\tau\cos\vartheta}, \\ & \omega(\vartheta) = (\bar{x}-a_0)\cos\vartheta + (z-c_0)\sin\vartheta. \end{aligned}\right\} \tag{13.51}$$

The two cases just discussed may be combined by choosing $m(t) = m\cos\sigma t$ and $a(t) = a_0 + u_0 t$, $b(t) = b_0$, $c(t) = c_0$. The modification of (13.51) is simple: a factor $\cos\sigma t$ must be put with the first two terms and a factor $\cos\sigma(t-\tau)$ put at the end of the integral. The asymptotic form as $t\to\infty$ can again be found by use of the Fourier Integral Theorem (13.16) or simple modifications. However, if the resulting formula is written out as principal-value integrals plus other terms, the expression is very unwieldy; it may be found in HAVELOCK (1958). Use of complex integrals allows one to compress the formula. Let

$$\varphi(\bar{x},y,z,t) = m\cos\sigma t\left(\frac{1}{r} - \frac{1}{r_1}\right) + m\,\mathrm{Re}\,\mathrm{e}^{-i\sigma t}\varphi_0, \qquad \varphi_0 = \varphi_1 + i\varphi_2.$$

Then

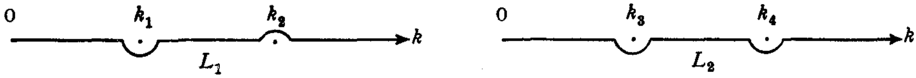
$$\varphi_0 = \frac{2g}{\pi}\int_0^{\gamma} d\vartheta\int_0^\infty dk\,F(\vartheta,k) + \frac{2g}{\pi}\int_{\gamma}^{\frac{1}{2}\pi} d\vartheta\int_{L_1} dk\,F(\vartheta,k) + \frac{2g}{\pi}\int_{\frac{1}{2}\pi}^{\pi} d\vartheta\int_{L_2} dk\,F(\vartheta,k),$$

$$F(\vartheta,k) = \frac{k\,\mathrm{e}^{k[y+b_0+i(\bar{x}-a_0)\cos\vartheta]}\cos[k(z-c_0)\sin\vartheta]}{gk-(\sigma+ku_0\cos\vartheta)^2}, \tag{13.52}$$

where

$$\tau = u_0 \sigma/g,$$

$$\gamma = \begin{cases} 0 & \text{if} \quad \tau < \frac{1}{4} \\ \arccos \frac{1}{4\tau} & \text{if} \quad \tau \geqq \frac{1}{4}, \end{cases}$$



$$\sqrt{g k_1}, \sqrt{g k_3} = \frac{1 - \sqrt{1 - 4\tau \cos\vartheta}}{2\tau \cos\vartheta} \sigma,$$

$$\sqrt{g k_2}, -\sqrt{g k_4} = \frac{1 + \sqrt{1 - 4\tau \cos\vartheta}}{2\tau \cos\vartheta} \sigma.$$

This potential has been derived by HASKIND (1946a), BRARD (1948a, b), HANAOKA (1953), SRETENSKII (1954), the last with an unfortunate mistake in sign

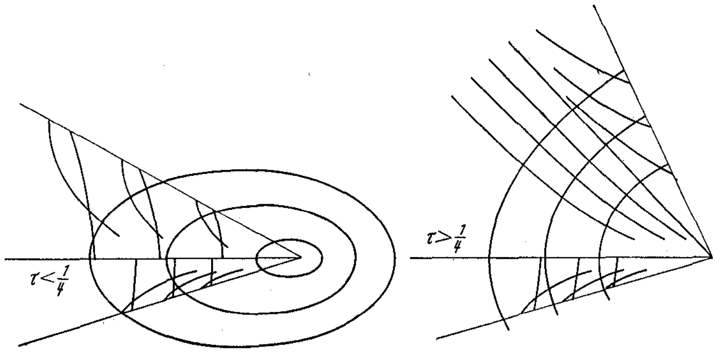


Fig. 3.

in one term, EGGERS (1957), and HAVELOCK (1958). HANAOKA, BRARD, EGGERS, and SRETENSKII have each considered the asymptotic form of the surface for large R. Fig. 3 shows qualitatively (cf. BECKER 1958) the curves of equal phase, say the crests, for the various systems of waves formed. The patterns must be completed by reflection in the x-axis.

Motion of a source on a circular path of radius D may be treated by taking $a = D\cos\sigma t$, $c = D\sin\sigma t$ in (13.49). For constant m this problem has been considered by SRETENSKII (1946a, b, 1957), HAVELOCK (1950), and STOKER (1957).

One may derive a formula analogous to (13.49) when the source moves in the presence of both a horizontal bottom at $y = -h$ and a free surface. The derivation may be carried out along lines similar to those used in deriving (13.49). The resulting velocity potential is [cf. LUNDE (1951, p. 32)]

$$\left.\begin{aligned} \Phi(x, y, z, t) &= \frac{m(t)}{r(t)} + \frac{m(t)}{r_2(t)} - 2m(t) \int_0^\infty e^{-kh} \frac{\cosh k(h + b(t))}{\cosh kh} \times \\ &\times \cosh k(y+h)\, J_0\big(k R(t)\big)\, dk + 2\int_0^\infty dk \sqrt{gk} \frac{\cosh k(y+h)}{\cosh^2 kh \sqrt{\tanh kh}} \times \\ &\times \int_0^t d\tau \sin\big[(t-\tau)\sqrt{gk \tanh kh}\big]\, m(\tau) \cosh k\big(h + b(\tau)\big)\, J_0\big(k R(\tau)\big), \end{aligned}\right\} \quad (13.53)$$

where
$$r_2^2 = (x - a(t))^2 + (y + 2h + b(t))^2 + (z - c(t))^2.$$

Two-dimensional formulas corresponding to (13.49) and (13.53) may also be derived. They are as follows, with the source and vortex separated for finite depth:

infinite depth:

see errata

$$\left.\begin{aligned} f(z,t) &= \frac{\Gamma(t) + i\,Q(t)}{2\pi i} \log\big(z - c(t)\big) + \frac{\Gamma(t) + i\,Q(t)}{2\pi i} \log\big(z - \bar{c}(t)\big) + \\ &+ \frac{g}{\pi i} \int_0^t [\Gamma(\tau) - i\,Q(\tau)]\,d\tau \int_0^\infty \frac{1}{\sqrt{g k}}\, e^{-i k (z - \bar{c}(\tau))} \sin\big[\sqrt{g k}\,(t-\tau)\big]\,dk; \end{aligned}\right\} \tag{13.54}$$

depth h, source:

$$\left.\begin{aligned} f(z,t) &= \frac{Q(t)}{2\pi} \log\big(z - c(t)\big) + \frac{Q(t)}{2\pi} \log\big(z - \bar{c}(t) + 2 i h\big) + \\ &+ \frac{Q(t)}{\pi} \int_0^\infty \frac{e^{-kh}}{k \cosh k h} \cosh k\big(b(t) + h\big) \cos k\big(z - a(t) + i h\big)\,dk - \\ &- \frac{g}{\pi} \int_0^\infty \frac{\operatorname{sech}^2 k h}{\sqrt{g k \tanh k h}}\,dk \int_0^t Q(\tau) \cosh k\big(b(\tau) + h\big) \cos k\big(z - a(\tau) + i h\big) \times \\ &\qquad \times \sin\big[\sqrt{g k \tanh k h}\,(t-\tau)\big]\,d\tau; \end{aligned}\right\} \tag{13.55}$$

depth h, vortex:

$$\left.\begin{aligned} f(z,t) &= \frac{\Gamma(t)}{2\pi i} \log\big(z - c(t)\big) - \frac{\Gamma(t)}{2\pi i} \log\big(z - \bar{c}(t) + 2 i h\big) + \\ &+ \frac{\Gamma(t)}{\pi} \int_0^\infty \frac{e^{-kh}}{k \cosh k h} \sinh k\big(b(t) + h\big) \sin k\big(z - a(t) + i h\big) - \\ &- \frac{g}{\pi} \int_0^\infty \frac{\operatorname{sech}^2 k h}{\sqrt{g k \tanh k h}}\,dk \int_0^t \Gamma(\tau) \sinh k\big(b(\tau) + h\big) \sin k\big(z - a(\tau) + i h\big) \times \\ &\qquad \times \sin\big[\sqrt{g k \tanh k h}\,(t-\tau)\big]\,dk. \end{aligned}\right\} \tag{13.56}$$

Higher-order singularities may be generated by taking derivatives with respect to z. One may transfer to moving coordinates, etc., just as in the three-dimensional case [see HAVELOCK (1949) for (13.54) in moving coordinates]. The velocity potential for a steadily moving source of pulsing strength in two dimensions has been given by HASKIND (1954, p. 23 ff.), who also gives the asymptotic expressions for large values of $\pm x$. When $\tau < \frac{1}{4}$, there exist one wave far ahead of the moving source propagating in the same direction and three far behind, one propagating in the same direction and two in the opposite direction; when $\tau > \frac{1}{4}$, there exist two waves far behind propagating in the opposite direction. The analysis for finite depth has been given by BECKER (1956).

14. Some simple physical solutions. In this section we consider periodic waves in an ocean of infinite horizontal extent, either infinitely deep or with a horizontal bottom, in canals, and at an interface. The linearizing parameter ε of Sect. 10α may be taken to be the ratio of amplitude to wave length.

α) *Standing waves in an infinite ocean.* It is appropriate to the physical problem to require that the motion remain bounded everywhere.

Consider first two-dimensional motion. Then, from Sect. 13β, the only solutions of the form $\Phi = \varphi \cos(\sigma t + \tau)$ are given by

$$\Phi(x, y, t) = a\, e^{\nu y} \cos(\nu x + \alpha) \cos(\sigma t + \tau), \qquad \nu = \sigma^2/g \tag{14.1}$$

for infinite depth, and

$$\left.\begin{aligned} \Phi(x, y, t) &= a \cosh m_0(y+h) \cos(m_0 x + \alpha) \cos(\sigma t + \tau), \\ m_0 \tanh m_0 h - \nu &= 0, \end{aligned}\right\} \qquad (14.2)$$

for finite depth.

The corresponding forms of the free surface are given by

see errata

$$\eta(x, t) = A \cos(\nu x + \alpha) \sin(\sigma t + \tau)$$

and

$$\eta(x, t) = A \cos(m_0 x + \alpha) \sin(\sigma t + \tau),$$

respectively. These represent standing waves according to our definition in Sect. 7. We recall that $m_0 > \nu$.

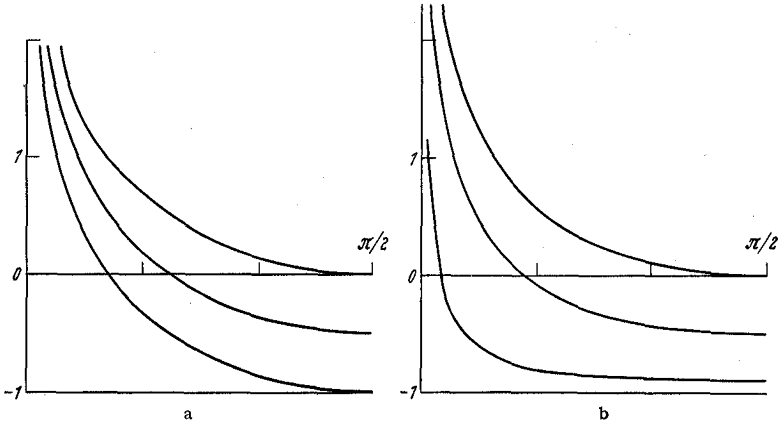


Fig. 4 a and b.

It is of interest to examine the streamlines and the paths of the individual fluid particles. The streamlines of the motion can be easily found from

$$\frac{dy}{dx} = \frac{\Phi_y}{\Phi_x} = \cot(\nu x + \alpha)$$

and

$$\frac{dy}{dx} = \frac{\Phi_y}{\Phi_x} = -\tanh m_0(y+h) \cot(m_0 x + \alpha),$$

respectively. The streamlines are then

$$e^{\nu(y-y_m)} |\sin(\nu x + \alpha)| = 1$$

and

$$\sinh m_0(y+h) |\sin(m_0 x + \alpha)| = \sinh m_0(y_m + h), \qquad 0 \geqq y_m \geqq h, \qquad (14.3)$$

for infinite and finite depth respectively; here y_m is the lowest point of the streamline. If the fluid is infinitely deep, the streamlines are all congruent. Fig. 4a shows three of them for a quarter wave length and $\alpha = 0$, $\nu = 1$. The vertical line $x = 0$ is also a streamline. If the fluid is of finite depth, the streamlines vary with depth. Fig. 4b shows streamlines corresponding to $y_m = 0, -0.5, -0.9$ for $\alpha = 0$, $h = 1$, $m_0 = 1$. The horizontal line $y = -1$ and the vertical line $x = 0$ are also streamlines.

Since the streamlines are time-independent, they also contain the curves for the trajectories of individual particles. However, the trajectory of an individual particle will be an oscillating motion of small amplitude along a segment of the streamline passing through the point. Thus, in Fig. 4b the particles on the bottom simply oscillate back and forth about an equilibrium position, those directly beneath a crest, i.e. at $x=0$, oscillate vertically, etc. In view of the infinitesimal-wave approximation used in this chapter the streamlines have physical significance for only a small distance above the equilibrium free surface, $y=0$.

In order to investigate, at least approximately, the behavior of the trajectories more fully, we may replace the actual trajectory by its tangent at an average position, say (x_0, y_0), an approximation consistent with the assumptions made in linearizing. Then the equations describing the trajectory become (setting $\alpha=\tau=0$)

$$\frac{dx}{dt} = -a\nu\, e^{\nu y_0} \sin \nu x_0 \cos \sigma t, \qquad \frac{dy}{dt} = a\nu\, e^{\nu y_0} \cos \nu x_0 \cos \sigma t$$

for infinite depth, and

$$\frac{dx}{dt} = -a m_0 \cosh m_0 (y_0 + h) \sin m_0 x_0 \cos \sigma t,$$
$$\frac{dy}{dt} = a m_0 \sinh m_0 (y_0 + h) \cos m_0 x \cos \sigma t$$

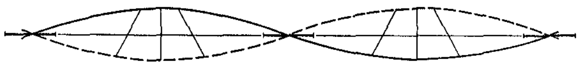


Fig. 5.

for finite depth. The approximate trajectories are then

$$x = x_0 - a\sigma^{-1}\nu\, e^{\nu y_0} \sin \nu x_0 \sin \sigma t, \qquad y = y_0 + a\sigma^{-1}\nu\, e^{\nu y_0} \cos \nu x_0 \sin \sigma t \quad (14.4)$$

for infinite depth, and

$$\left.\begin{aligned} x &= x_0 - a\sigma^{-1} m_0 \cosh m_0 (y_0 + h) \sin m_0 x_0 \sin \sigma t, \\ y &= y_0 + a\sigma^{-1} m_0 \sinh m_0 (y_0 + h) \cos m_0 x_0 \sin \sigma t \end{aligned}\right\} \quad (14.5)$$

for finite depth. For infinite depth, the amplitude of oscillation drops off very rapidly as depth of the equilibrium position increases, the ratio of the amplitude at depth y_0 to the amplitude at the free surface being $e^{\nu y_0}$. The same ratio for the case of finite depth is

$$\frac{\sinh^2 m_0 (y_0 + h) + \sin^2 m_0 x_0}{\sinh^2 m_0 h + \sin^2 m_0 x_0}.$$

Thus, on the bottom, when $y_0 = -h$, the amplitude is zero under the crests and maximum under the nodes. As is evident from the equations of the trajectories, the path lines of particles on the free surface are approximately as in Fig. 5. In order to explain the apparently inconsistent behavior at the nodes one mus. go to a higher approximation than the linearized theory used in this chaptert

see errata

Let us now consider three-dimensional solutions. The standing-wave solutions are of the form

see errata

$$\Phi(x, y, z, t) = e^{\nu y} \chi(x, y) \cos(\sigma t + \tau) \qquad \text{for finite depth},$$

or

$$\Phi(x, y, z, t) = \cosh m_0 (y + h)\, \chi(x, z) \cos(\sigma t + \tau) \qquad \text{for finite depth},$$

where $\chi(x, z)$ is a solution of

$$\Delta_2 \chi + \nu^2 \chi = 0 \quad \text{or} \quad \Delta_2 \chi + m_0 \chi = 0,$$

respectively, regular everywhere in $y \leqq 0$.

Two particular cases are of especial interest. The first corresponds to separation of variables in rectangular coordinates [see (13.5) and (13.6)]. The solutions are

$$\left.\begin{aligned} \Phi(x, y, z, t) &= a\, e^{\nu y} \cos(k_1 x + \alpha) \cos(k_2 z + \gamma) \cos(\sigma t + \tau), \\ k_1^2 + k_2^2 &= \nu^2 = \sigma^2/g, \end{aligned}\right\} \tag{14.6}$$

see errata

for infinite depth, and

$$\left.\begin{aligned} \Phi(x, y, z, t) &= a \cosh m_0(y+h) \cos(k_1 x+\alpha) \cos(k_2 z+\gamma) \cos(\sigma t+\tau), \\ k_1^2 + k_2^2 &= m_0^2, \qquad m_0 \tanh m_0 h - \nu = 0, \end{aligned}\right\} \tag{14.7}$$

for finite depth. The other solutions result from separating variables in polar coordinates [see (13.7) and (13.8)]. They are

$$\Phi(R, \alpha, y, t) = a\, e^{\nu y} J_n(\nu R) \cos(n\alpha + \delta) \cos(\sigma t + \tau), \qquad n = 0, 1, \ldots, \tag{14.8}$$

for infinite depth, and

$$\left.\begin{aligned} \Phi(R, \alpha, y, t) &= a \cosh m_0(y + h)\, J_n(m_0 R) \cos(n\alpha + \delta) \cos(\sigma t + \tau), \\ n &= 0, 1, \ldots, \end{aligned}\right\} \tag{14.9}$$

for finite depth. The form of the free surface may be found immediately from $\eta(x, z, t) = -\Phi_t(x, 0, z, t)/g$. These are all standing waves.

The streamlines and path lines may be found for these two cases with no special difficulty. For the first case for finite depth the streamlines are the intersections of the surfaces

$$\left.\begin{aligned} |\sin k_1 x|^{k_2^2} &= C_1 |\sin k_2 z|^{k_1^2}, \\ |\sin k_1 x \sin k_2 z| \sinh m_0(y + h) &= C_2. \end{aligned}\right\} \tag{14.10}$$

The vertical lines, $x = p\pi/k_1$, $z = q\pi/k_2$, passing through the maxima and minima are streamlines. The points on the vertical lines $x = (p + \frac{1}{2})\pi/k_1$, $z = (q + \frac{1}{2})\pi/k_2$ passing through the sattlepoints are all stagnation points. The projection on $y = 0$ of the field of streamlines is indicated qualitatively by Fig. 6. The behavior in a projection on a vertical plane is similar to that for two-dimensional motion.

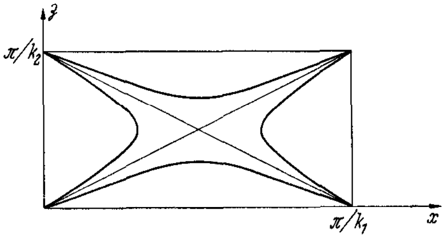


Fig. 6.

see errata

In the second case above one may easily visualize the streamlines for the case of pure ring waves, $n = 0$. For finite depht they are given in a plane $\alpha = \text{const}$ by

$$m_0 R\, J_1(m_0 R) \sinh m_0(y+h) = C_1 \tag{14.11}$$

together with the vertical lines at the zeros of $J_1(m_0 R)$. The behavior of the curves is qualitatively similar to that of the two-dimensional case.

In both cases approximations to the path lines can be found as in the two-dimensional case.

β) Progressive waves in an infinite ocean. By taking the proper linear combinations of the standing-wave solutions one may obtain progressive waves. Thus, adding

$$\Phi_1 = a\, e^{\nu y} \cos \nu x \cos \sigma t \quad \text{and} \quad \Phi_2 = a\, e^{\nu y} \sin \nu x \sin \sigma t,$$

one obtains

$$\Phi = a\, e^{\nu y} \cos(\nu x - \sigma t) \tag{14.12}$$

which represents a progressive wave moving to the right with velocity

$$c = \frac{\sigma}{\nu} = \frac{g}{\sigma} = \sqrt{\frac{g}{\nu}} = \sqrt{\frac{g\lambda}{2\pi}}, \tag{14.13}$$

where $\lambda = 2\pi/\nu$ is the wavelength. Subtracting yields a progressive wave moving to the left. If one takes the coefficient in Φ_1 as a_1 and in Φ_2 as a_2, the sum may be written

$$\Phi = \tfrac{1}{2} e^{\nu y} \left[(a_1 + a_2) \cos(\nu x - \sigma t) + (a_1 - a_2) \cos(\nu x + \sigma t)\right].$$

This is a superposition of two progressive waves of different amplitudes, one moving to the left and one to the right. If $a_1 = a_2$, a pure progressive wave is obtained; if $a_2 = 0$, one obtains again a standing wave, as a superposition of two progressive waves moving in opposite directions.

For water of finite depth h the corresponding expressions for Φ may be obtained by replacing $e^{\nu y}$ by $\cosh m_0(y+h)$ and ν by m_0. The phase velocity is given by

$$c = \frac{\sigma}{m_0} = \sqrt{\frac{g \tanh m_0 h}{m_0}} = \sqrt{\frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}}. \tag{14.14}$$

As $h \to \infty$, the velocity approaches that obtained above for deep water. In fact, if $h/\lambda > 0.2$, the velocity is already within 0.1 of the value for deep water. c is an increasing function of λ, but cannot increase indefinitely as in the case of infinitely deep water, for (14.14) implies

$$c < \sqrt{g h}. \tag{14.15}$$

The streamlines for the progressive wave moving to the right are given by

$$e^{\nu y} |\sin(\nu x - \sigma t)| = C \quad \text{and} \quad \sinh m_0 (y + h) |\sin(m_0 x - \sigma t)| = C \tag{14.16}$$

for infinite and finite depth, respectively. At a given instant t these have the same shape relative to a crest as the streamlines for a standing wave. However, since they are time-dependent, the path lines for particles do not lie on the streamlines. The path lines may be found approximately for a particle with equilibrium position (x_0, y_0) from the equations

$$\frac{dx}{dt} = \Phi_x(x_0, y_0, t), \qquad \frac{dy}{dt} = \Phi_y(x_0, y_0, t).$$

This approximation is consistent with the assumptions made in linearizing the boundary condition, as can be seen by assuming a solution in the form

$$x(t) = x_0 + \varepsilon x_1(t) + \cdots, \qquad y(t) = y_0 + \varepsilon y_1(t) + \cdots,$$

where $\varepsilon = a\sigma\nu/2\pi g$ for infinite depth and $\varepsilon = a\sigma m_0/2\pi g$ for finite depth, substituting in the exact path equations, and retaining only first-order terms.

For infinite depth the particle trajectories are given by

$$x = x_0 - a\nu\sigma^{-1} e^{\nu y_0} \cos(\nu x_0 - \sigma t), \quad y = y_0 - a\nu\sigma^{-1} e^{\nu y_0} \sin(\nu x_0 - \sigma t). \tag{14.17}$$

The particles follow circular orbits of radius $a\nu\sigma^{-1} e^{\nu y_0}$ about the equilibrium position (x_0, y_0); at the top of the orbit they are moving in the same direction as the wave. The orbital velocity is $a\nu\, e^{\nu y_0}$, so that the motion dies out quickly as $|y_0|$ increases; for example, at a depth of one wave length the velocity and orbit radius are only $\frac{1}{535}$ the value at the free surface. Although the particles at the crest of a wave are moving in the same direction as the wave, their velocity is not necessarily the same and is, in fact, much smaller in view of the assumed smallness of $\varepsilon = (a\nu/c)(\nu/2\pi)$.

For finite depth the orbits are elliptical with the major axis horizontal:

$$\left.\begin{aligned} x &= x_0 - a\, m_0\, \sigma^{-1} \cosh m_0 (y_0 + h) \cos (m_0\, x - \sigma t), \\ y &= y_0 - a\, m_0\, \sigma^{-1} \sinh m_0 (y_0 + h) \sin (m_0\, x - \sigma t). \end{aligned}\right\} \tag{14.18}$$

The particles again trace the orbit in a clockwise direction except that on the bottom they simply oscillate along a horizontal segment. Fig. 7 from RUELLAN and WALLET (1950) shows the path lines for a variety of cases of superposed waves. The topmost picture shows the orbits for a pure progressive wave moving to the right. The bottom picture is a superposition of progressive waves of equal amplitudes moving in opposite directions, i.e. a pure standing wave. The intermediate cases show superpositions with varying ratios of the amplitudes. The intermediate cases are instructive in that not only path lines, but also streamlines are visible.

Since the progressive-wave solutions are steady with respect to a coordinate system moving with the wave, it is clear that we could have obtained a steady-state solution as a small motion superposed upon a uniform flow. If we take a complex velocity potential in the form

$$F(z) = -c\,z + f(z). \tag{14.19}$$

Then [see Eq. (11.6)] f must satisfy

$$\operatorname{Re}\{i\,g\,f + c^2 f'\} = 0 \quad \text{for } y = 0$$

and either $|f'| \to 0$ as $y \to -\infty$ or $\operatorname{Im} f' = 0$ for $y = -h$. The solution for the first case, infinite depth, is given by

$$f = a\,e^{-i\nu z} = a\,e^{\nu y} [\cos \nu\, x - i \sin \nu\, x], \qquad \nu = g/c^2. \tag{14.20}$$

The solution for the finite-depth case is given by

$$\left.\begin{aligned} f &= a \cos m_0 (z + i\,h) \\ &= a\,[\cos m_0\, x \cosh m_0\, (y+h) - i \sin m_0\, x \sinh m_0\, (y+h)], \end{aligned}\right\} \tag{14.21}$$

where m_0 must satisfy

$$c^2\, m_0 - g \tanh m_0\, h = 0.$$

The same relation is found in (14.14). A real solution does not exist if $c^2/g\,h > 1$ and in this case there is no wave-like motion consistent with the linearized theory. The streamlines, identical here with the path lines, are obtained from

$$-c\,y + \psi(x, y) = 0.$$

One may replace this equation, consistently with the linearization assumptions [cf. (10.18)], by

$$-c\,y + \psi(x, y_0) = 0,$$

where y_0 is the mean height of the streamline. Thus, for finite depth, they are given by

$$y = -\frac{a}{c} \sinh m_0\, (y_0 + h) \sin m_0\, x, \tag{14.22}$$

an easily constructed family of curves. In the foregoing we have tacitly taken a to be real. However, it may be complex and thus include waves of different phase.

We note finally that (14.8) or (14.9) allow one to construct waves progressing like the spokes of a wheel. However, outwardly progressing waves can be constructed only when the solution involving Y_n is used, and this has a singularity at the origin.

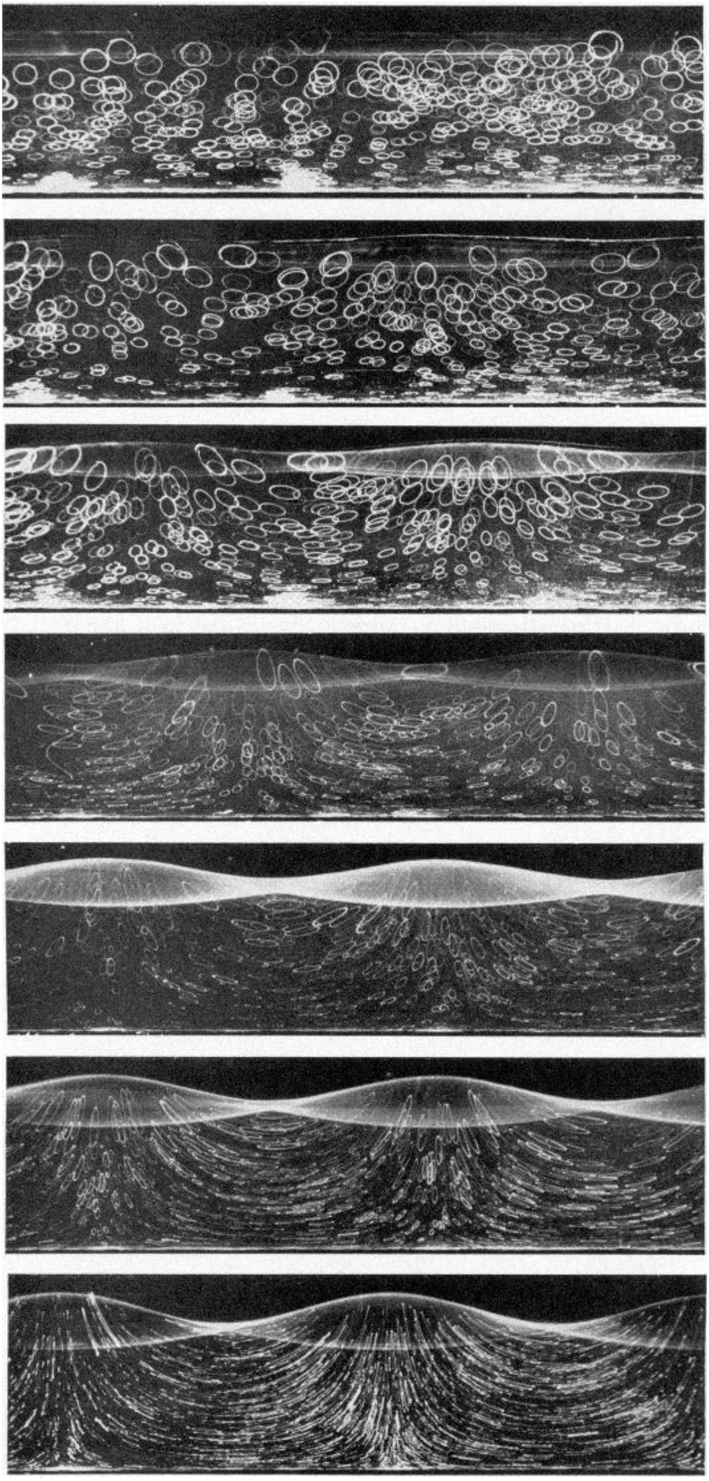


Fig. 7 a—g. Particle trajectories in progressive and standing waves.

γ) Periodic waves in rectangular canals. Let us suppose that the fluid is contained between the planes $z=0$ and $z=d$. Then the velocity potential must satisfy the additional conditions

$$\Phi_z(x, y, 0, t) = \Phi_z(x, y, d, t) = 0. \tag{14.23}$$

This condition is automatically satisfied by the two-dimensional waves discussed in 14α, so that they present no special interest here. However, condition (14.4) does restrict the three-dimensional solutions (14.6) and (14.7), for k_2 must now satisfy (taking $\gamma=0$).

$$k_2 = \frac{n\pi}{d}, \quad n = 1, 2, \ldots .$$

Since $k_1^2+k_2^2=\nu^2$ or m_0^2, there can be no solution periodic in x unless

$$n < \frac{\nu d}{\pi} \quad \text{or} \quad n < \frac{m_0 d}{\pi}, \tag{14.24}$$

respectively. Hence, for frequencies below a certain critical frequency σ_1, where

$$\sigma_1 = \sqrt{\frac{\pi g}{d}} \quad \text{or} \quad \sigma_1 = \sqrt{\frac{\pi g}{d}\tanh\frac{\pi h}{d}} \tag{14.25}$$

for infinite or finite depth respectively, there can exist no three-dimensional standing waves in a canal.

Let us form a three-dimensional progressive wave in a canal of finite depth by adding standing-wave solutions:

$$\Phi(x, y, z, t) = a\cosh m_0(y+h)\cos k_2 z\cos(k_1 x - \sigma t), \qquad k_2 = n\pi/d.$$

The velocity of the progressive wave is given by

$$c^2 = \frac{\sigma^2}{k_1^2} = g h\left(1 - \frac{n^2\pi^2}{m_0^2 d^2}\right)^{-1}\frac{\tanh m_0 h}{m_0 h} < g h\left(1 - \frac{n^2\pi^2}{m_0^2 d^2}\right)^{-1}. \tag{14.26}$$

As in the case treated above, there can exist no three-dimensional progressive waves unless $\sigma>\sigma_1$. However, if they exist, their velocity is higher than the velocity of two-dimensional waves of the same frequency.

One may define similarly a sequence of critical frequencies $\sigma_1, \sigma_2, \ldots$, where

$$\sigma_k = \sqrt{\frac{k\pi g}{d}} \quad \text{or} \quad \sigma_k = \sqrt{\frac{k\pi g}{d}\tanh\frac{k\pi h}{d}};$$

when $\sigma_k<\sigma<\sigma_{k+1}$, k types of three-dimensional waves are possible with $n=1, 2, \ldots, k$.

δ) Waves at an interface. Let us now suppose that two fluids are present, one lying over the other. Variables referring to the upper and lower fluids have subscripts 2 and 1 respectively. From (10.7) and (10.8) the linearized boundary conditions for a small disturbance are

$$\left.\begin{aligned} \Phi_{1y} &= \Phi_{2y}, \\ \varrho_1[\Phi_{1tt} + g\Phi_{1y}] &= \varrho_2[\Phi_{2tt} + g\Phi_{2y}], \end{aligned}\right\} \tag{14.27}$$

both equations to be satisfied at the equilibrium position of the interface. We shall consider several typical problems.

Let the upper fluid fill the region $y>0$, and the lower fluid the region $y<0$. We require of a solution that

$$|\mathrm{grad}\,\Phi_1| \to 0 \quad \text{as} \quad y \to -\infty \quad \text{and} \quad |\mathrm{grad}\,\Phi_2| \to 0 \quad \text{as} \quad y \to +\infty.$$

In looking for a standing-wave solution, one may, following Sect. 14α, take

$$\Phi_1 = a_1 e^{my} \varphi(x, z) \cos(\sigma t + \tau), \qquad \Phi_2 = a_2 e^{-my} \varphi(x, z) \cos(\sigma t + \tau),$$

where the relation between a_1 and a_2 and m and σ is to be determined by (14.27), and φ satisfies

$$\Delta_2 \varphi + m^2 \varphi = 0.$$

The first Eq. (14.27) gives immediately that

$$a_1 + a_2 = 0.$$

The second one gives the relation

$$\sigma^2 = \frac{\varrho_1 - \varrho_2}{\varrho_1 + \varrho_2} m g. \tag{14.28}$$

The equation of the interface may be obtained from (10.8):

$$\eta(x, z, t) = \frac{a_1 m}{\sigma} \varphi(x, z) \sin(\sigma t + \tau).$$

Since $a_1 = -a_2$, there is a discontinuity in u (and w if the motion is three-dimensional) as one crosses the interface.

The special choices of $\varphi(x, z)$ made in Sect. 14β may, of course, also be made here. In particular, one may make progressive and standing waves. If one forms two-dimensional progressive waves at the interface, one finds for the velocity

$$c^2 = \frac{\varrho_1 - \varrho_2}{\varrho_1 + \varrho_2} \frac{g}{m}. \tag{14.29}$$

If one assumes the fluids bounded above and below by planes $y = h_2$ and $y = -h_1$, respectively, a similar calculation shows

$$\sigma^2 = \frac{\varrho_1 - \varrho_2}{\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2} g m. \tag{14.30}$$

It is clear from (14.28) and (14.30) that these solutions exist only if $\varrho_2 < \varrho_1$. The case $\varrho_2 > \varrho_1$ will be discussed later.

A more complicated problem of this type is the following [cf. LAMB (1932, § 231), GREENHILL (1887)]. Suppose there is a solid horizontal bottom at $y = -h$, an interface at $y = -d$ and a free surface at $y = 0$. Then, in addition to (14.27) at $y = -d$, Φ_1 and Φ_2 must satisfy

$$\Phi_{2tt} + g \Phi_{2y} = 0 \quad \text{at} \quad y = 0, \qquad \Phi_{1y} = 0 \quad \text{at} \quad y = -h.$$

If one seeks solutions of the form

$$\Phi_2 = (a_2 \cosh m y + b_2 \sinh m y) \varphi(x, z) \cos(\sigma t + \tau),$$
$$\Phi_1 = a_1 \cosh m(y + h) \varphi(x, z) \cos(\sigma t + \tau),$$

substitution in the various boundary conditions yields the following relation between σ and m:

$$\left.\begin{aligned} &\left(\frac{\sigma^2}{g m}\right)^2 [\varrho_1 \coth m d \coth m(h - d) + \varrho_2] - \\ &\quad - \frac{\sigma^2}{g m} \varrho_1 [\coth m d + \coth m(h - d)] + (\varrho_1 - \varrho_2) = 0. \end{aligned}\right\} \tag{14.31}$$

If $\varrho_2 < \varrho_1$, one may establish that there exist two positive solutions for σ^2 for a given m, so that two possible frequencies are possible for a given wave pattern.

If the bottom fluid is taken infinitely deep, one replaces $\coth m(h-d)$ by 1 in (14.31) and the two solutions simplify to

$$\sigma_1^2 = g\,m, \qquad \sigma_2^2 = g\,m\,\frac{\varrho_1-\varrho_2}{\varrho_1\coth m\,d+\varrho_2} < \sigma_1^2. \tag{14.32}$$

The first solution, σ_1, is the same as would be obtained if the two fluids were identical (and there is no discontinuity in u and w at the interface); the second, σ_2, is interpreted below. The inequality $\sigma_2^2 < \sigma_1^2$ holds in general, and one may establish

$$\frac{\sigma_2^2}{g\,m} < \{\tanh m\,d,\ \tanh m\,(h-d)\} \leqq \frac{\sigma_1^2}{g\,m} \leqq \min\left\{1,\ \frac{\varrho_1}{\varrho_2}\tanh m\,h\right\}. \tag{14.33}$$

If one computes the ratio of the amplitude of the disturbance at the interface to that at the free surface, one finds, no matter whether h is finite or not,

$$\cosh m\,d - \frac{g\,m}{\sigma^2}\sinh m\,d. \tag{14.34}$$

An examination of the roots of (14.31) shows that the ratio (14.34) is negative for the smaller of the two roots and positive for the larger. Thus, in the solution associated with the smaller root, a maximum of the disturbance at the interface is associated with a minimum of that at the free surface, and vice versa. On the other hand, with the larger root the maxima and minima go together. For the values given in (14.32), the ratio becomes

$$e^{-m\,d} \quad\text{and}\quad -\frac{\varrho_1}{\varrho_1-\varrho_2}\,e^{m\,d}, \tag{14.35}$$

respectively. We note that, although the first ratio is <1, the second is in absolute value >1 if $\varrho_2(1+e^{m\,d}) > \varrho_1 > \varrho_2$, a condition satisfied if ϱ_1 is only slightly greater than ϱ_2. In fact, the ratio may become very large.

For a given wave length and amplitude of the wave at the free surface one may also compare the amplitudes of the two different modes of motion at the interface. If A_i is the amplitude associated with the frequency σ_i, then for the case $h=\infty$ one finds

$$\left|\frac{A_2}{A_1}\right| = \frac{\varrho_2}{\varrho_1-\varrho_2}\,\frac{1+\tanh m\,d}{1-\tanh m\,d},$$

which may be either less than or greater than 1.

see errata

It is of some interest to examine somewhat further the solution associated with the smaller root σ_2 of (14.31). Then, since $a_2/b_2 = g\,m/\sigma^2$, the inequality (14.39) implies that there exists an h_0 with $0<h_0<d$ such that

$$\frac{\sigma_2^2}{g\,m} = \frac{b_2}{a_2} = \tanh m\,h_0 < \tanh m\,d < 1$$

and that

$$\Phi_2 = \sqrt{a_2^2-b_2^2}\cosh m\,(y+h_0)\,\varphi(x,z)\cos(\sigma t+\tau).$$

Thus the part of the top fluid between $y=0$ and $y=-h_0$ behaves as if there were a solid boundary at $y=-h_0$; and, of course, the fluid between $y=-h_0$ and $y=-h$ as if it were between solid boundaries. If one has selected solutions for φ which can be combined to form a progressive plane wave, then one may conclude that the velocity $c_2=\sigma_2/m$ associated with this mode of motion has an upper bound:

$$c_2 = \sqrt{\frac{g}{m}\tanh m\,h_0} < \sqrt{g\,d}.$$

In fact, when $h=\infty$, one may verify immediately from (14.32) that

$$c_2 = \sqrt{\frac{g}{m}\tanh m\,d\,\frac{\varrho_1-\varrho_2}{\varrho_1+\varrho_2\tanh m\,d}} \leqq \sqrt{g\,d\,\frac{\varrho_1-\varrho_2}{\varrho_1}} = c_{2\max}\,.$$

Thus for $h=\infty$ a progressive wave travelling faster than $c_{2\max}$ will consist of only the one mode of motion, i.e. the one associated with σ_1. If $c<c_{2\max}$, there may be two modes of motion excited. This fact is associated with the phenomenon of "dead-water" resistance of ships [see LAMB (1916a), EKMAN (1904), SRETENSKII (1934)].

For superposed fluids one may also find solutions analogous to (14.20) and (14.21). Let us suppose that the first (upper) fluid flows to the left with mean velocity c_2 and the second with mean velocity c_1. We wish to find the possible steady periodic profiles of the interface, assuming as usual that the disturbance is small. The complex velocity potential for each fluid is taken in the form

$$F_1(z) = -c_1 z + f_1(z), \qquad F_2(z) = -c_2 z + f_2(z). \tag{14.36}$$

The conditions to be satisfied at the mean common boundary, $y=0$, are:

$$\left.\begin{aligned} c_1^{-1}\,\mathrm{Im}\,f_1 &= c_2^{-1}\,\mathrm{Im}\,f_2,\\ \varrho_1 c_1^{-1}\,\mathrm{Re}\{i\,g\,f_1 + c_1^2 f_1'\} &= \varrho_2 c_2^{-1}\,\mathrm{Re}\{i\,g\,f_2 + c_2^2 f_2'\}. \end{aligned}\right\} \tag{14.37}$$

If each fluid extends infinitely far vertically, then

$$f_1 = a_1\,\mathrm{e}^{-i m z}, \qquad f_2 = a_2\,\mathrm{e}^{i m z}$$

give a steady-state solution if

$$\frac{a_1}{c_1} = -\frac{\bar a_2}{c_2}$$

and

$$m = \frac{g(\varrho_1-\varrho_2)}{\varrho_1 c_1^2 + \varrho_2 c_2^2} > 0, \tag{14.38}$$

where $\bar a_2$ is the complex conjugate of a_2. If the upper fluid is bounded by $y=h_2$ and the lower by $y=-h_1$, then the solution is

$$f_1 = a_1\cos m\,(z+i\,h_1), \qquad f_2 = a_2\cos m\,(z-i\,h_2),$$

where, letting $a_k=\alpha_k+i\beta_k$, $k=1, 2$,

$$\frac{\alpha_1}{c_1}\sinh m\,h_1 = -\frac{\alpha_2}{c_2}\sinh m\,h_2, \qquad \frac{\beta_1}{c_1}\cosh m\,h_1 = \frac{\beta_2}{c_2}\cosh m\,h_2$$

and

$$m = \frac{g(\varrho_1-\varrho_2)}{\varrho_1 c_1^2\coth m\,h_1 + \varrho_2 c_2^2\coth m\,h_2}. \tag{14.39}$$

In either case the equation of the interface is given by

$$y = \frac{1}{c_k}\,\psi_k(x, 0).$$

SRETENSKII (1952b) has considered a three-dimensional analogue of the above problem in which the direction of flow of one of the fluids makes an angle ϑ with that of the other. Thus, take velocity potentials of the following form:

$$\left.\begin{aligned} \Phi_2(x, y, z) &= -c_2(x\cos\vartheta + z\sin\vartheta) + \varphi_2(x, y, z),\\ \Phi_1(x, y, z) &= -c_1 x + \varphi_1(x, y, z). \end{aligned}\right\} \tag{14.40}$$

The following are the boundary conditions at the interface $\eta(x, z)$ for small disturbance:

$$\left.\begin{aligned} &\varphi_{2y}(x, 0, z) + c_2(\eta_x \cos\vartheta + \eta_z \sin\vartheta) = 0, \qquad \varphi_{1y}(x, 0, z) + c_1 \eta_x = 0, \\ &g(\varrho_1 - \varrho_2)\,\eta = \varrho_1 c_1 \varphi_{1x}(x, 0, z) - \varrho_2 c_2 [\varphi_{2x}(x, 0, z)\cos\vartheta + \varphi_{2z}(x, 0, z)\sin\vartheta]. \end{aligned}\right\} \tag{14.41}$$

For a solution in the form

$$\varphi_1 = A_1 \mathrm{e}^{my}\cos(k_1 x + k_2 z), \qquad \varphi_2 = A_2 \mathrm{e}^{-my}\cos(k_1 x + k_2 z), \qquad k_1^2 + k_2^2 = m^2,$$

the following relations must hold

and

$$\left.\begin{aligned} &\frac{A_2}{A_1} = -\frac{c_2}{c_1}\,\frac{k_1\cos\vartheta + k_2\sin\vartheta}{k_1} \\ &\varrho_1 c_1^2 k_1^2 + \varrho_2 c_2^2 (k_1\cos\vartheta + k_2\sin\vartheta)^2 = g\,m(\varrho_1 - \varrho_2). \end{aligned}\right\} \tag{14.42}$$

These reduce to (14.38) for $\vartheta = 0$, $k_1 = m_1$ as they should. The equation for the interface is

$$y = -A_1 \frac{m}{k_1 c_1}\sin(k_1 x + k_2 z). \tag{14.43}$$

SRETENSKII studies the properties of the solution in more detail.

As a further extension of the preceding cases one may consider a time-dependent disturbance at the interface between two fluids flowing at different velocities. This will be treated in the section on stability of motion.

A natural generalization of the two-fluid system is the n-fluid system [see GREENHILL (1887)] and then the heterogeneous fluid with density given as a series

$$\varrho(x, y, z, t) = \varrho_0(y) + \varepsilon\,\varrho^{(1)}(x, y, z, t) + \varepsilon^2 \varrho^{(2)} + \cdots.$$

If one assumes a similar expansion for p and expansions for u, v, w, and η starting with ε, one may derive easily the linearized equations. These, discussion of some periodic solutions, and references to the literature may be found in LAMB (1932, § 235). GROEN (1948) has shown that the period for simple harmonic motion in the linearized problem is a monotonic increasing function of the wave length starting with the minimum $2\pi\sqrt{-\varrho_0(y)/g\,\varrho_0'(y)}$ for $\lambda = +0$. This theorem has been generalized by HEYNA and GROEN (1958) to allow a free upper surface. GROEN (1950) discusses properties of internal waves in an expository way and gives further references to the more recent literature. For some pertinent theorems about waves in heterogeneous fluids see Sect. 32β.

15. Group velocity and the propagation of disturbances and of energy. In the last section we considered periodic waves at a free surface or interface. In this section we wish to consider waves of a given but fairly general initial form and study the way in which they propagate. Although this will entail writing down the solution to a particular initial-value problem, this is of only incidental interest, the chief interest being in the history of the form of the free surface or interface. Initial-value problems as such will be treated in more detail later on. In fact, the remarks below apply equally well to other initial-value problems, for example, an initial distribution of velocity on the surface. What is essential is the resolution of the subsequent motion into a set of waves moving to the right and of ones moving to the left, as in (15.2).

The property of the fluid and its boundaries which is most important for this investigation is the functional relation between the frequency σ and the wave number k. The earlier parts of this chapter have shown that considerable variation is possible in the form of $\sigma(k)$. The two-fluid example with both free surface and interface gave a doubly valued function. A multiply valued function could have been obtained with more layers. However, each branch, or the branch, is a decreasing function of k, approaching zero as $k \to \infty$. When surface tension is taken into account (see Sect. 24), the form of $\sigma(k)$ for large k changes; it then becomes an increasing function, behaving like $k^{\frac{3}{2}}$. If h is large enough, $\sigma(k)$ decreases initially, i.e. for $k < k_m$, reaches a minimum at k_m and then increases; if h is small enough $\sigma(k)$ is everywhere increasing. It will be convenient to extend the definition of $\sigma(k)$ to negative k by setting $\sigma(-k) = -\sigma(k)$.

α) The propagation of an initial elevation. Let us suppose that at time $t=0$ the free surface is given by $y = \eta(x, 0)$ and that the fluid is at rest. How does the free surface behave subsequently? One may conveniently think of this as an initial humping up of the fluid near one point, but this is not essential. We shall also suppose that $\eta(x, 0)$ is sufficiently restricted to allow a Fourier-integral representation. In part of what follows we shall also assume it to be square integrable, i.e. the total available energy is finite, and on occasion that $x\eta$ is square integrable. Let

$$\left.\begin{aligned} \eta(x, 0) &= \int_0^\infty [C(k)\cos k x + S(k)\sin k x]\, dk \\ &= \int_{-\infty}^\infty e^{-ikx} E(k)\, dk = 2\,\mathrm{Re}\int_0^\infty e^{-ikx} E(k)\, dk, \end{aligned}\right\} \tag{15.1}$$

where

$$C(k) = \frac{1}{\pi}\int_{-\infty}^\infty \eta(x, 0)\cos k x\, dx, \qquad S(k) = \frac{1}{\pi}\int_{-\infty}^\infty \eta(x, 0)\sin k x\, dx,$$

$$E(k) = \frac{1}{2\pi}\int_{-\infty}^\infty \eta(x, 0)\, e^{ikx}\, dx = \frac{1}{2}[C(k) + i\, S(k)].$$

We shall call $E(k)$ the *spectrum* of $\eta(x, 0)$. Note that $E(-k) = E^*(k)$, the complex conjugate of $E(k)$ (we change notation temporarily in order to avoid conflict with the notation for averages introduced below).

A formal solution for Φ and $\eta(x, t)$ may be written down immediately:

$$\left.\begin{aligned} \Phi(x, y, t) &= -\int_0^\infty \frac{\sigma(k)}{k} Y(y)\,[C(k)\cos k x + S(k)\sin k x]\sin\sigma t\, dk \\ &= -\int_{-\infty}^\infty \frac{\sigma(k)}{k} Y(y)\, E(k)\, e^{-ikx}\sin\sigma t\, dk \\ &= \frac{1}{2} i \int_{-\infty}^\infty \frac{\sigma(k)}{k} Y(y)\, E(k)\,[e^{-i(kx-\sigma t)} - e^{-i(kx+\sigma t)}]\, dk, \\ \eta(x, t) &= \int_0^\infty [C(k)\cos k x + S(k)\sin k x]\cos\sigma t\, dk \\ &= \int_{-\infty}^\infty e^{-ikx} E(k)\cos\sigma t\, dk = \frac{1}{2}\int_{-\infty}^\infty E(k)\,[e^{-i(kx-\sigma t)} + e^{-i(kx+\sigma t)}]\, dk. \end{aligned}\right\} \tag{15.2}$$

Here $Y(y) = \cosh k(y+h)/\sinh kh$ for a single fluid of depth h, $Y(y) = e^{|k|y}\,\mathrm{sgn}\, k$ for infinite depth (the peculiar modification of Y for $h = \infty$ is necessary for

$k<0$). However, more general situations are allowable in which, for example, $\eta(x, t)$ describes an interface. The choice of an expression for Φ has been based upon the kinematic boundary condition $\Phi_y(x, 0, t) = \eta_t(x, t)$ in order not to exclude the possibility of surface tension. For simplicity we also restrict ourselves to single-valued σ's. For more complicated problems, such as the two-fluid problem with both free surface and interface discussed in Sect. 14δ, the freedom to fix both $\eta_1(x, 0)$ and $\eta_2(x, 0)$ independently requires the determination of two spectra for each surface with relations between them set by (14.34). The remarks below will still apply to motion resulting from each spectrum separately. Finally, we note that a statement concerning specific conditions to be satisfied by $\eta(x, 0)$ for the case of a single free surface may be found in a paper by Kampé de Fériet and Kotik (1953).

It is clear from (15.2) that one may express $\eta(x, t)$ as a sum of two functions, one, say $\eta_R(x, t)$, representing a superposition of waves moving to the right, the other, η_L, waves moving to the left. We consider only η_R since similar remarks apply to η_L with x replaced by $-x$. The spectrum of η_R is given by $\frac{1}{2} E(k)\, e^{i\sigma(k)t}$, so that clearly $\sigma(k)$ plays an important role in the change of shape of η_R. Since each harmonic component in η_R is moving to the right with velocity $\sigma(k)/k$, and since this is not a constant in the cases we have been considering, the different components will move with different velocities and we shall expect η_R to change its shape with time, even though moving as a whole to the right.

In order to get some idea of the overall motion it is reasonable to try to compute an average position of $\eta_R(x, t)$ and find how this moves. One must first decide how to define the average position. One possibility, which, as we shall see presently, is unsatisfactory is to use η_R itself as the weighting function, i.e. to define

$$\bar{x}_R(t) = \int_{-\infty}^{\infty} x\, \eta_R(x, t)\, dx \Big/ \int_{-\infty}^{\infty} \eta_R(x, t)\, dx$$

when this exists. An easy computation shows that

$$\bar{x}_R(t) = \bar{x}_R(0) + \sigma'(0)\, t,$$

i.e. the average motion is, on this definition, independent of the form of $\sigma(k)$ except near $k=0$. For deep-water gravity waves $\sigma'(0) = \infty$; for depth h, $\sigma'(0) = \sqrt{gh}$, the maximum velocity [see Eq. (14.15)]. In conformity with the above one may define the "spread" of the hump to be

$$\int_{-\infty}^{\infty} [x - \bar{x}_R(t)]^2\, \eta_R(x, t)\, dx \Big/ \int_{-\infty}^{\infty} \eta_R(x, t)\, dx.$$

A computation shows that this remains constant in time, when it exists. This definition of average is unsatisfactory, as could have been expected inasmuch as the weighting function can become negative. We note in passing that

$$\int_{-\infty}^{\infty} \eta_R(x, t)\, dx = \int_{-\infty}^{\infty} \eta_R(x, 0)\, dx,$$

an expression of conservation of mass.

Another possible weighting function without this shortcoming, but still allowing ease of computation, is $\eta_R^2(x, t)$. We note first that

$$\int_{-\infty}^{\infty} \eta_R^2(x, t)\, dx = \int_{-\infty}^{\infty} \eta_R^2(x, 0)\, dx = \tfrac{1}{2}\pi \int_{-\infty}^{\infty} E(k)\, E^*(k)\, dk.$$

Let us define two averages, one for functions of x:

$$\bar{f}(t) = \int_{-\infty}^{\infty} f(x)\, \eta_R^2(x,t)\, dx \Big/ \int_{-\infty}^{\infty} \eta_R^2(x,t)\, dx,$$

and one for functions of k:

$$\bar{\Phi} = \int_{-\infty}^{\infty} \Phi(k)\, E(k)\, E^*(k)\, dk \Big/ \int_{-\infty}^{\infty} E(k)\, E^*(k)\, dk.$$

Then, assuming that the various quantities in question exist, one finds, using well known theorems on Fourier transforms[1],

$$\bar{x}_R(t) = \bar{x}_R(0) + \bar{\sigma}'\, t \tag{15.3}$$

and

$$\left.\begin{aligned}\overline{[x-\bar{x}_R(t)]^2} = \overline{[x-\bar{x}_R(0)]^2} + t\left\{\overline{\sigma'\left[i\log\frac{E^*}{E}\right]'} - \bar{\sigma}'\,\overline{\left[i\log\frac{E^*}{E}\right]'}\right\} + \\ + t^2\left\{\overline{\sigma'^2} - \bar{\sigma}'^2\right\}.\end{aligned}\right\} \tag{15.4}$$

Thus, on this definition the average position of η_R moves to the right with constant velocity $\bar{\sigma}'$ and the hump spreads according to a quadratic law. We note that the coefficient of t^2 is positive except if σ' is a constant, when it vanishes. It may become infinite, and, in fact, does so for infinitely deep water if the gravest modes are present, i.e., if $\int \eta_R\, dx \neq 0$. The coefficient of t vanishes if σ' is constant or if $[i \log E^*/E]'$ is constant; the latter will occur if $\eta(x, 0)$ is either symmetric or antisymmetric about some point x_0, but this does not exhaust all possibilities. The sign of this term does not seem to be determined, so that the spread of the hump may conceivably decrease before starting to increase.

Investigations of the motion of the average position of the hump and of its spread give only a rather crude picture of its behavior. By other methods outlined below one may obtain further insight into the motion.

We begin by applying the analysis of the average motion to that part of η_R resulting from only a narrow band in its spectrum. Let

$$\eta_R(x, t; k_0, \varepsilon) = \mathrm{Re} \int_{k_0-\varepsilon}^{k_0+\varepsilon} \tfrac{1}{2} E(k)\, e^{-i(kx-\sigma(k)t)}\, dk. \tag{15.5}$$

We shall call this a *wave packet*. The average position satisfies

$$\bar{x}_R(t; k_0, \varepsilon) = \bar{x}_R(0; k_0, \varepsilon) + \bar{\sigma}'(k_0, \varepsilon)\, t,$$

where $\bar{\sigma}'(k, \varepsilon)$ is now the average of $\sigma'(k)$ over the narrow band $[k_0-\varepsilon, k_0+\varepsilon]$. The narrower the band, the closer $\bar{\sigma}'(k_0, \varepsilon)$ is to $\sigma'(k_0)$, assuming the latter continuous. As a limiting case we shall say that the wave packet resulting from an infinitesimal band about k_0 moves with velocity $\sigma'(k_0)$. It is customary to call $\sigma'(k)$ the *group velocity*. This is the same as the phase velocity $\sigma(k)/k$ only if $\sigma = a k$. A wave packet will spread with passage of time unless the two velocities are equal, for (15.4) is applicable to the wave packet with the restricted definition of average. As might be expected, the smaller the width of the band, the smaller the coefficient of t^2 and the smaller the rate of growth. However, as we shall see below, the initial spread may be wide for a narrow band.

The wave packet (15.5) may also be investigated by a different method. Let us expand $\sigma(k)$ in the first few terms of a Taylor series about k_0:

$$\sigma(k) = \sigma(k_0) + \sigma'(k_0)(k-k_0) + \tfrac{1}{2}\int_{k_0}^{k} \sigma''(\varkappa)(k-\varkappa)\, d\varkappa. \tag{15.6}$$

[1] See, e.g., S. Bochner and K. Chandrasekharan: Fourier transforms, Chap. IV, § 2. Princeton 1949.

We may then write

$$\left.\begin{aligned} \eta_R(x,t;k_0,\varepsilon) &= \operatorname{Re}\tfrac{1}{2}\,e^{-i[k_0x-\sigma(k_0)t]}\Bigg\{\int\limits_{k_0-\varepsilon}^{k_0+\varepsilon} E(k)\,e^{-i[x-\sigma'(k_0)t](k-k_0)}\,dk\;+ \\ &\quad+\int\limits_{k_0-\varepsilon}^{k_0+\varepsilon} E(k)\,e^{-i[x-\sigma'(k_0)t](k-k_0)}\left[\exp\Big(-\tfrac{1}{2}\,i\,t\int\limits_{k_0}^{k}\sigma''(\varkappa)\,(k-\varkappa)\,d\varkappa\Big)-1\right]dk\Bigg\} \\ &= \operatorname{Re}\tfrac{1}{2}\,e^{-i[k_0x-\sigma(k_0)t]}\,M\big(x-\sigma'(k_0)\,t;\,k_0,\varepsilon\big)+R. \end{aligned}\right\} \tag{15.7}$$

Using the inequality $|e^{iu}-1|\leqq|u|$, one finds

$$|R|\leqq\tfrac{1}{4}\,t\,\varepsilon^3\max_{|k-k_0|<\varepsilon}|E(k)|\cdot\max_{|k-k_0|<\varepsilon}|\sigma''(k)|. \tag{15.8}$$

The remainder can thus be made small by taking ε or t small enough. However, once ε is fixed, R will eventually become large as t increases. Let us suppose, however, that t and ε are small enough so that the first term determines the main features of the motion. The first factor represents a periodic wave of wave number k_0 moving with its phase velocity $\sigma(k_0)/k_0$. The second factor, determining the amplitude of the first, represents a profile being translated to the right with velocity $\sigma'(k_0)$. Thus one may say that the gross outline of the surface is moving to the right with the group velocity. One may see this more clearly if one assumes ε small enough so that we may take $E(k)$ as constant over the band width. Then

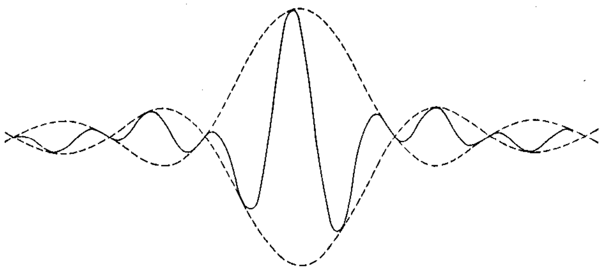


Fig. 8.

$$M\big(x-\sigma'(k_0)\,t;\,k_0,\varepsilon\big)=E(k_0)\,\frac{\sin\big(x-\sigma'(k_0)\,t\big)\,\varepsilon}{x-\sigma'(k_0)\,t},$$

and $\eta_R(x,t;k_0,\varepsilon)$ appears approximately as in Fig. 8. Here the dotted enveloping curves represent $\pm\frac{1}{2}M$ and move to the right with velocity $\sigma'(k_0)$, whereas the inscribed solid curves represent the first factor and move to the right with phase velocity $\sigma(k_0)/k_0$. The whole moves as a fixed pattern only if the two velocities are equal. Otherwise, assuming $\sigma'(k_0)<\sigma(k_0)/k_0$, the inscribed curves will progress through the wave packet, gradually disappearing at the right. For a very narrow band the packet will spread wide before its first zero on either side of the maximum.

A disadvantage of this last analysis is that it becomes less and less accurate as t becomes large. However, there exists another approximation to $\eta_R(x,t)$ for large values of t which helps to complete the picture. This ultimate behavior of η_R can to some extent be predicted from the analysis of the average motion of a wave band. If we think of η_R as made up of the contributions from a number of narrow wave bands, we know that each contribution is moving with the average group velocity of the band. Thus after some time we shall expect that these various contributions will have separated from one another, with the bands about the gravest modes, which travel fastest, having progressed the furthest. This prediction will be confirmed.

What is needed for this final approximation is an asymptotic expansion for large t. It is convenient to express η_R in the slightly altered form

$$\eta_R(x,t)=\tfrac{1}{2}\int\limits_{-\infty}^{\infty}E(k)\,e^{-i\left[k\frac{x}{t}-\sigma(k)\right]t}\,dk \tag{15.9}$$

and to consider it as depending upon the two parameters x/t and t. Then for each value of x/t we shall give an expansion for large values of t. For a derivation of the expansion we refer to STOKER (1957, § 6.8) or ERDÉLYI (1956, § 2.9).

Let the functions $k_r(x/t)$, $r=1, 2, \ldots, n$, be defined by

$$\sigma'(k_r) = x/t; \tag{15.10}$$

i.e. we allow the possibility of several roots. In the situation of interest to us there will be either one or two roots, or none. The asymptotic expression for η_R is then given by

$$\left.\begin{aligned} \eta_R(x,t) &= \mathrm{Re}\sum_r \frac{1}{2} E(k_r)\left[\frac{2\pi}{t\,|\sigma''(k_r)|}\right]^{\frac{1}{2}} \mathrm{e}^{-i[k_r x - \sigma(k_r)t - \frac{1}{4}\pi\,\mathrm{sgn}\,\sigma''(k_r)]} + \\ &\quad + \mathrm{Re}\sum_r \frac{1}{2} E(k_r)\frac{1}{\sqrt{3}}\,\Gamma\!\left(\frac{1}{3}\right)\left[\frac{6}{t\,|\sigma'''(k_r)|}\right]^{\frac{1}{3}} \mathrm{e}^{-i[k_r x - \sigma(k_r)t]} + O\left(t^{-\frac{2}{3}}\right), \end{aligned}\right\} \tag{15.11}$$

where the first summation is over all values of r for which $\sigma''(k_r) \neq 0$ and the second over all k_r for which $\sigma''(k_r)=0$ but $\sigma'''(k_r)\neq 0$; further terms would be necessary for values of r for which both vanish but this will not occur in our examples. If some $k_r=0$, then the corresponding term must be multiplied by $\frac{1}{2}$. For a value of x/t for which no solution to (15.10) exists, it is easy to show by a change of variables in (15.9), say $u=k\,x/t-\sigma(k)$, and integration by parts that $\eta_R(x,t)=O(t^{-1})$.

Let us examine in some detail the implications of one term of (15.11), say $r=1$, for the motion of η_R; if several terms are present for a given value of x/t one must superpose the resultant motions.

If x/t is held constant while t increases, then clearly one must set $x=\sigma'(k_1)\,t$, i.e. we are examining η_R from the standpoint of an observer moving with group velocity $\sigma'(k_1)$. Since the coefficient of the harmonic term is $t^{-\frac{1}{2}}$ times a function
see errata
of k_1, which is being held constant, the gross outline of η_R will appear constant in form, but decreasing in amplitude because of $t^{-\frac{1}{2}}$. However, just as in the analysis of (15.7), there is a harmonic of wave number k_1 moving through the gross outline with phase velocity $\sigma(k_1)/k_1$. The amplitude of the gross outline is proportional to $E(k_1)$, but also depends now upon $\sigma''(k_1)$, in contrast to the situation for small t according to (15.7).

If the value of x/t is such that $\sigma''(k_1)=0$, then one must examine a term from the second summation in (15.11). It is evident that the interpretation is the same except that σ''' occurs in place of σ'' and that the amplitude decreases more slowly because of the $t^{-\frac{1}{3}}$. This situation can happen, for example, in the case of gravity waves in water of depth h for $x=t\sqrt{gh}$. Then $k_1(\sqrt{gh})=0$, $\sigma''(0)=0$, and $\sigma'''(0)=-h^2\sqrt{gh}$. This also occurs for combined gravity-capillary waves when the curve $\sigma'(k)$ has a minimum.

The approximation (15.11) to η_R will obviously be very poor for a value x/t such that $\sigma''(k_r)$ is near to zero for some r unless t is extremely large. It is shown elsewhere[1] how an Airy function may be used to modify the relevant term in the second summand to give a useful asymptotic expansion for k_r near a zero of σ''.

If x/t is fixed at a value for which (15.10) has no solution, then for an observer moving with this velocity the disturbance of the surface is very small, for it has been dying out as t^{-1}. The first term of the expansion may, of course, be com-

[1] H. JEFFREYS and B. JEFFREYS: Methods of mathematical physics, 3rd ed., § 17.09. Cambridge 1956. — See also C. CHESTER, B. FRIEDMAN and F. URSELL: Proc. Cambridge Phil. Soc. 53, 599—611 (1957).

puted as indicated above. This situation will occur for a disturbance in water of depth h if $x/t > \sqrt{gh}$. It will also occur when surface tension is taken into account for $x/t < \sigma'_{\min}$.

The asymptotic expansion (15.11) may also be used in a different fashion. Let us fix our attention upon one value of x and let t increase. Then x/t will decrease and the value $k_1(x/t)$ associated with the point x at a given moment will also change; for pure gravity waves it will increase. The observer stationed at x will then observe waves of continually increasing wave number (decreasing wave length) moving by with phase velocities appropriate to their lengths. The amplitudes at a given instant will depend upon the first factor. The gross outline of the waves will pass the observer at the group velocity appropriate to the wave number present at the moment, and, of course, the amplitude is decreasing as $t^{-\frac{1}{2}}$. In the case of a disturbance on water of depth h, if the observer is initially far from the hump, then even for large enough values of t for the asymptotic expansion to be valid the value of x/t may be greater than $\sqrt{gh}$. Then the observer will see practically no disturbance until the gravest modes begin to reach him. We note again that he must anticipate the arrival of a given wave number by its group velocity, not phase velocity, for it is the former which controls the amplitude. In the case of combined gravity-capillary waves, when t is large enough one will have $x/t < \sigma'_{\min}$ and the disturbance will be negligible.

It is also possible to find an asymptotic expansion for $\eta_R(x, t)$ for x/t fixed and large x. It turns out to be the same as (15.11) with $O(t^{-\frac{3}{2}})$ replaced by $O(x^{-\frac{3}{2}})$. This expansion allows one, so to speak, to take snapshots of the right-hand end of η_R at different instants of time. If we fix t and let x increase, x/t increases also and $k_1(x/t)$ decreases for pure gravity waves. Thus the wave length increases as one moves to the right; the observed amplitude will depend upon the first factor. For gravity waves on water of depth h, if x is large enough, $x/t > \sqrt{gh}$ and the disturbance will be small of order x^{-1}.

Finally, we use the asymptotic expansion to investigate the motion of a particular phase of $\eta_R(x, t)$, say a zero, for large t. Such a point will be determined by

$$\alpha(x, t) \equiv k_1 x - \sigma(k_1)\, t = \text{const},$$

where, as usual, $k_1 = k_1(x/t)$; solving for x gives $x = x(t)$. One may find $\dot{x}(t)$ from

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$$\dot{x}(t) = -\frac{\alpha_t}{\alpha_x} = -\frac{-k_1' \frac{x}{t^2} - \sigma(k_1) + \sigma'(k_1)\frac{x}{t^2}\, t}{k_1 + k_1' \frac{x}{t} - \sigma'(k_1)\, k_1'} = \frac{\sigma(k_1)}{k_1}.$$

Thus a particular phase travels with the phase velocity of the harmonic component associated with it at the moment. However, if the group and phase velocities are different, it is then moving at a different velocity from a point just keeping pace with waves of a given wave number. In particular, for gravity waves it is moving faster, hence moves into region of lower wave number and higher velocity and is accelerating. Acomputation of $\ddot{x}$ bears this out:

$$\ddot{x}(t) = -\frac{1}{t\, k_1 \sigma''}\left[\frac{\sigma}{k_1} - \sigma'\right]^2,$$

for this is always positive for gravity waves. The right-hand side is, of course, a function of x and t. For deep-water gravity waves the function $x(t)$ may easily be found from the earlier equation:

$$x = \frac{\sigma(k_1)}{k_1} t - \frac{a}{k_1} = 2\sigma'(k_1)\, t - \frac{a}{k_1} = 2\frac{x}{t} t - \frac{4a}{g}\frac{x^2}{t^2}$$

or

$$x(t) = \frac{g\,t^2}{4a}.$$

Hence $\ddot{x} = g/2a$ and for large t the acceleration is constant. If the depth is finite, the computation is no longer simple, although it is possible to show that $x(t)$ varies from $x(t) = t\sqrt{g\,h}$ for a phase associated with $k = 0$ to $x(t) = A\,t^2$ for a phase associated with very large k.

Fig. 9 is taken from a paper of KELVIN'S (1907), and shows the computed values of $\eta(x, t)$ for an initial displacement given by

$$\eta(x, 0) = \frac{[1 + (1 + x^2)^{\frac{1}{2}}]^{\frac{1}{2}}}{2^{\frac{3}{2}}(1 + x^2)^{\frac{3}{2}}}\,[2 - (1 + x^2)^{\frac{1}{2}}]$$

and for $t/\pi^{\frac{1}{2}} = \frac{1}{2}$, 1, $\frac{3}{2}$, 4, 8 (the units have been chosen so that $g = 4$). The description of the behavior of $\eta_R(x, t)$ outlined in the preceding paragraphs can be easily

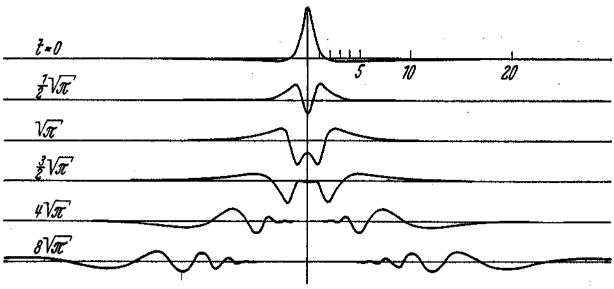

Fig. 9.

verified qualitatively by inspection of the successive snapshots of $\eta_R(x, t)$ GREEN (1909) has shown that if one estimates the wave length at any maximum as double the distance between the two including zeros, then the position is very close to that which would be estimated by using the group velocity (cf. HAVELOCK, 1914, p. 37).

Fig. 10 from a report by J.E. PRINS (1956; also 1958b) shows measured time histories taken at various distances from the center of an initial rectangular hump of length $2L$ and height Q in water of depth h for specific values shown in the figure. In general, the features of the motion described above were well verified by this experimental investigation.

We assemble here the expressions for $\sigma(k)$ and $k\sigma'/\sigma$ for a number of cases of water waves.

1. Deep-water gravity waves:

$$\sigma(k) = \sqrt{g\,k}, \qquad \frac{k\,\sigma'}{\sigma} = \frac{1}{2}.$$

2. Gravity waves at the interface of two fluids, each of infinite vertical extent:

$$\sigma(k) = \sqrt{\frac{\varrho_1 - \varrho_2}{\varrho_1 + \varrho_2}\, g\,k}, \qquad \frac{k\,\sigma'}{\sigma} = \frac{1}{2}.$$

3. Gravity waves in water of depth h:

$$\sigma(k) = \sqrt{g\,k \tanh k\,h}\,, \qquad \frac{k\,\sigma'}{\sigma} = \frac{1}{2}\left[1 + \frac{2k\,h}{\sinh 2k\,h}\right].$$

4. Gravity waves for a layer of thickness d of one fluid over a deep layer of a heavier one:

$$\sigma_1(k) = \sqrt{g\,k}\,, \qquad \frac{k\,\sigma_1'}{\sigma_1} = \frac{1}{2}\,,$$

$$\sigma_2(k) = \sqrt{\frac{\varrho_1 - \varrho_2}{\varrho_1 \coth k\,d + \varrho_2}\,g\,k}\,,$$

$$\frac{k\,\sigma_2'}{\sigma_2} = \frac{1}{2}\left[1 + \frac{2\varrho_1\,k\,d}{\varrho_1 \sinh 2k\,d + 2\varrho_2 \sinh^2 k\,d}\right].$$

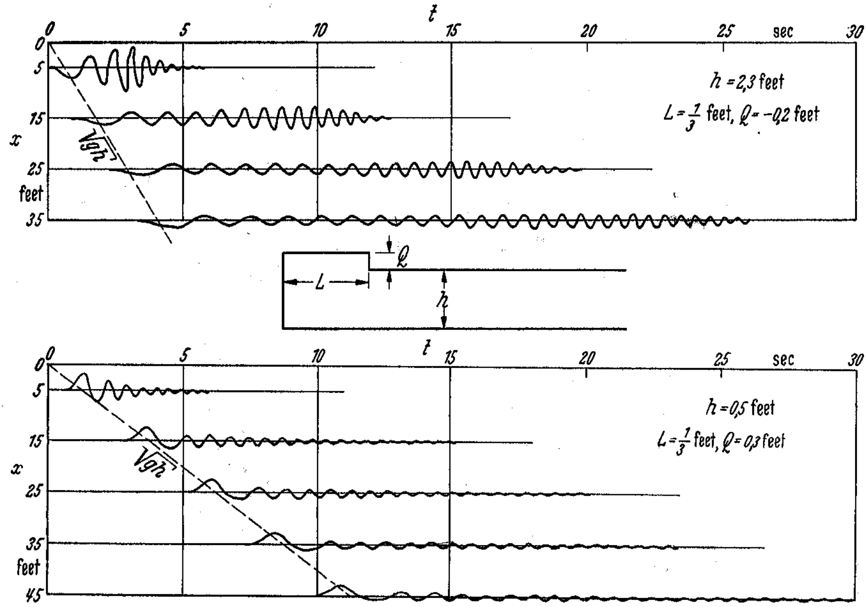

Fig. 10.

5. Waves at a free surface of a deep fluid with both gravity and surface tension acting:

$$\sigma(k) = \sqrt{g\,k + \frac{T\,k^3}{\varrho}}\,, \qquad \frac{k\,\sigma'}{\sigma} = \frac{1}{2}\,\frac{1 + 3\,T\,k^2/\varrho\,g}{1 + T\,k^2/\varrho\,g}.$$

6. Waves at a free surface of a fluid of depth h with both gravity and surface tension acting:

$$\sigma(k) = \sqrt{\left(g\,k + \frac{T\,k^3}{\varrho}\right)\tanh k\,h}\,,$$

$$\frac{k\,\sigma'}{\sigma} = \frac{1}{2}\left[1 + \frac{2h\,k}{\sinh 2h\,k} + 2\,\frac{T\,k^2/\varrho\,g}{1 + T\,k^2/\varrho\,g}\right].$$

In cases 1 to 4 σ'' is always negative if $k > 0$. In case 5 it crosses the k-axis at $k = [g\,\varrho\,T^{-1}\,\tfrac{1}{3}(2\sqrt{3} - 3)]^{\frac{1}{2}}$ and becomes positive. In cases 1 to 4 $\sigma' < \sigma/k$ for $k > 0$. In case 5 $\sigma' < \sigma/k$ for $0 < k < \sqrt{g\,\varrho/T}$; then σ' crosses σ/k at the minimum of the latter and thereafter remains larger. (Note that σ' always passes through a stationary value of σ/k, passing from beneath to above in going through a minimum, and the reverse at a maximum.) We shall not discuss 6 in detail. For

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$h > h_c = \sqrt{3T/2\varrho g}$, σ/k has a minimum for some k_0, $0 < k_0 < \sqrt{\varrho g/T}$ and σ' a minimum to the left of this. For $h \leqq h_c$, σ/k is an increasing function, starting at $\sqrt{gh}$ for $k=0$, and σ' is also increasing, $\sigma' > \sigma/k$ for $k > 0$, $\sigma'(0) = \sqrt{gh}$. Fig. 11 shows graphs of σ, σ/k and σ' for 1, 3, 5, and 6 (the scales were chosen for convenience).

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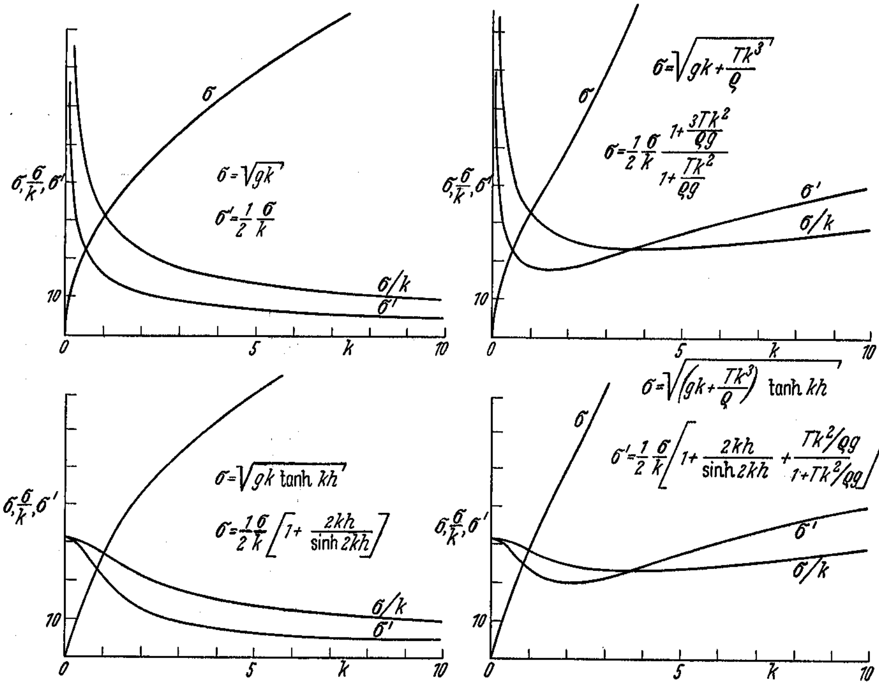

Fig. 11.

One may also take $\lambda = 2\pi/k$ as the independent variable, and then express the phase velocity c and group velocity U as functions of λ. An easy computation shows that

$$\lambda \frac{dc}{d\lambda} = c - U.$$

This equation has a simple interpretation in the geometry of the curve for $c(\lambda)$, as was shown by Lamb (1932, p. 382): For a given value of λ, U is the intercept on the vertical axis of the tangent to the $c(\lambda)$ curve at the point $(\lambda, c(\lambda))$. One value of U may correspond to more than one value of λ, as, for example, in the case of gravity-capillary waves. See Havelock (1914, § 11).

β) The propagation of energy. It seems intuitively clear that as long as the right-moving part of an initial hump keeps its integrity the energy associated with the motion will in some sense move with the hump. We wish to consider in what sense this is true. We limit ourselves in the following discussion to a single fluid of depth h, where h may become infinite. However, surface tension may act upon the free surface.

We first introduce the notion of energy density for a given value of x. It will be convenient to separate potential, kinetic and surface energy. Let

$$\left.\begin{aligned}
\mathscr{V}(x,t) &= \varrho g \int_0^{\eta_R(x,t)} y\,dy = \tfrac{1}{2}\varrho g \eta_R^2(x,t),\\
\mathscr{T}(x,t) &= \tfrac{1}{2}\varrho \int_{-h}^{0} (\Phi_x^2+\Phi_y^2)\,dy = \tfrac{1}{2}\varrho \int_{-h}^{0} (\Phi\,\Phi_x)_x\,dy + \tfrac{1}{2}\varrho\,\Phi\,\Phi_y(x,0,t),\\
\mathscr{S}(x,t) &= \tfrac{1}{2} T \eta_{Rx}^2(x,t)
\end{aligned}\right\} \tag{15.12}$$

be the densities of potential, kinetic and surface energies, respectively, where here Φ is the velocity potential corresponding to η_R.

These functions may now be treated in the same way as η_R was in Sect. 15α. We may ask for the average position of the distributions of the several densities. They are defined by

$$\left.\begin{aligned}
\bar{x}_V(t) &= \int_{-\infty}^{\infty} x\,\mathscr{V}(x,t)\,dx \Big/ \int_{-\infty}^{\infty} \mathscr{V}(x,t)\,dx,\\
\bar{x}_T(t) &= \int_{-\infty}^{\infty} x\,\mathscr{T}(x,t)\,dx \Big/ \int_{-\infty}^{\infty} \mathscr{T}(x,t)\,dx,\\
\bar{x}_S(t) &= \int_{-\infty}^{\infty} x\,\mathscr{S}(x,t)\,dx \Big/ \int_{-\infty}^{\infty} \mathscr{S}(x,t)\,dx,
\end{aligned}\right\} \tag{15.13}$$

respectively. Since all three densities are non-negative, one avoids the difficulty met with in defining the average position of η_R. In fact, it is obvious that the definitions of $\bar{x}_R$ and $\bar{x}_V$ coincide, so that the conclusions concerning $\bar{x}_R$ can be applied immediately to $\bar{x}_V(t)$. In particular,

$$\bar{x}_V(t) = \bar{x}_V(0) + \bar{\sigma}' t. \tag{15.14}$$

Consider now $\bar{x}_T(t)$. First we note that, from GREEN's Theorem,

$$\int_{-\infty}^{\infty} \mathscr{T}(x,t)\,dx = \tfrac{1}{2}\varrho \int_{-\infty}^{\infty} \Phi(x,0,t)\,\Phi_y(x,0,t)\,dx + \lim_{\substack{x_1\to-\infty\\ x_2\to+\infty}} \tfrac{1}{2}\varrho \int_{-h}^{0} [-\Phi(x_1,y,t)\,\Phi_x(x_1,y,t) + \Phi(x_2,y,t)\,\Phi_x(x_2,y,t]\,dy.$$

From the assumed square-integrability of η_R, the limit vanishes. Use of the identity $x(\Phi_x^2+\Phi_y^2) = (x\Phi)_x\,\Phi_x + (x\Phi)_y\,\Phi_y - \Phi\,\Phi_x$ and GREEN's Theorem gives

$$\begin{aligned}
\int_{-\infty}^{\infty} x\,\mathscr{T}(x,t)\,dx &= \tfrac{1}{2}\varrho \int_{-\infty}^{\infty} x\,\Phi(x,0,t)\,\Phi_y(x,0,t)\,dx - \\
&\quad - \lim_{\substack{x_1\to-\infty\\ x_2\to+\infty}} \tfrac{1}{4}\varrho \int_{-h}^{0} [\Phi^2(x_2,y,t) - \Phi^2(x_1,y,t)]\,dy + \\
&\quad + \lim_{\substack{x_1\to-\infty\\ x_2\to\infty}} \tfrac{1}{2}\varrho \int_{-h}^{0} [-x_1\,\Phi(x_1,y,t)\,\Phi_x(x_1,y,t) + x_2\,\Phi(x_2,y,t)\,\Phi_x(x_2,y,t)]\,dy,
\end{aligned}$$

where again the last two limits vanish. A similar computation shows

$$\begin{aligned}
\int_{-\infty}^{\infty} x^2\,\mathscr{T}(x,t)\,dx &= \tfrac{1}{2}\varrho \int_{-\infty}^{\infty} x^2\,\Phi(x,0,t)\,\Phi_y(x,0,t)\,dx + \tfrac{1}{2}\varrho \int_{-\infty}^{\infty}\int_{-h}^{0} \Phi^2(x,y,t)\,dx\,dy + \\
&\quad + \lim_{\substack{x_1\to-\infty\\ x_2\to\infty}} \tfrac{1}{2}\varrho \int_{-h}^{0} [-x_1^2\,\Phi(x_1,y,t)\,\Phi_x(x_1,y,t) + x_2^2\,\Phi(x_2,y,t)\,\Phi_x(x_2,y,t)]\,dy - \\
&\quad - \lim_{\substack{x_1\to-\infty\\ x_2\to\infty}} \tfrac{1}{2}\varrho \int_{-h}^{0} [-x_1\,\Phi^2(x_1,y,t) + x_2\,\Phi^2(x_2,y,t)]\,dy.
\end{aligned}$$

Collecting these results we have

$$\left.\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{T}(x,t)\,dx &= \tfrac{1}{2}\varrho \int_{-\infty}^{\infty} \Phi(x,0,t)\,\Phi_y(x,0,t)\,dx,\\
\int_{-\infty}^{\infty} x\,\mathcal{T}(x,t)\,dx &= \tfrac{1}{2}\varrho \int_{-\infty}^{\infty} x\,\Phi(x,0,t)\,\Phi_y(x,0,t)\,dx,\\
\int_{-\infty}^{\infty} x^2\,\mathcal{T}(x,t)\,dx &= \tfrac{1}{2}\varrho \int_{-\infty}^{\infty} x^2\,\Phi(x,0,t)\,\Phi_y(x,0,t)\,dx +\\
&\quad + \tfrac{1}{2}\varrho \int_{-\infty}^{\infty}\int_{-h}^{0} \Phi^2(x,y,t)\,dx\,dy.
\end{aligned}\right\} \tag{15.15}$$

Since from (15.2),

$$\eta_R(x,t) = \tfrac{1}{2}\int_{-\infty}^{\infty} E(k)\,\mathrm{e}^{-i(kx-\sigma t)}\,dk$$

and

$$\Phi(x,y,t) = \frac{1}{2}\int_{-\infty}^{\infty} i\,\frac{\sigma(k)}{k}\,Y(y)\,E(k)\,\mathrm{e}^{-i(kx-\sigma t)}\,dk, \tag{15.16}$$

one finds easily

$$\Phi(x,0,t) = \frac{1}{2}\int_{-\infty}^{\infty} i\,\frac{\sigma(k)}{k}\coth kh\,E(k)\,\mathrm{e}^{-i(kx-\sigma t)}\,dk,$$

$$\Phi_y(x,0,t) = \tfrac{1}{2}\int_{-\infty}^{\infty} i\,\sigma(k)\,E(k)\,\mathrm{e}^{-i(kx-\sigma t)}\,dk.$$

One may now apply again, as in Sect. 15a, theorems on Fourier transforms to obtain

$$\left.\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{T}(x,t)\,dx &= \frac{1}{4}\pi\varrho\int_{-\infty}^{\infty} E(k)\,E^*(k)\,\frac{\sigma^2}{k}\coth kh\,dk,\\
\int_{-\infty}^{\infty} x\,\mathcal{T}(x,t)\,dx &= \frac{1}{4}\pi\varrho\int_{-\infty}^{\infty} i\,E(k)\,E^{*\prime}(k)\,\frac{\sigma^2}{k}\coth kh\,dk +\\
&\quad + \frac{1}{4}\pi\varrho\,t\int_{-\infty}^{\infty} E(k)\,E^*(k)\,\sigma'(k)\,\frac{\sigma^2}{k}\coth kh\,dk,\\
\int_{-\infty}^{\infty} x^2\,\mathcal{T}(x,t)\,dx &= \int_{-\infty}^{\infty} x^2\,\mathcal{T}(x,0)\,dx +\\
&\quad + \frac{1}{2}\pi\varrho\,t\int_{-\infty}^{\infty} i\,E(k)\,E^{*\prime}(k)\,\sigma'\,\frac{\sigma^2}{k}\coth kh\,dk +\\
&\quad + \frac{1}{4}\pi\varrho\,t^2\int_{-\infty}^{\infty} E(k)\,E^*(k)\,\sigma'^2(k)\,\frac{\sigma^2}{k}\coth kh.
\end{aligned}\right\} \tag{15.17}$$

If one uses the definition introduced earlier for average of a function of k, one now finds

$$\bar{x}_T(t) = \bar{x}_T(0) + t\,\frac{\overline{\sigma'\,\sigma^2\,k^{-1}\coth kh}}{\overline{\sigma^2\,k^{-1}\coth kh}} \tag{15.18}$$

and a rather unwieldy expression for $\overline{[x-\bar{x}_T(t)]^2}$, similar in character to (15.4). We note that if we are dealing with pure gravity waves, so that $\sigma^2 = gk\tanh kh$, then formulas (15.17) simplify considerably and become identical with those for V. In this case the potential and kinetic energies are equal and propagate with the same velocities.

We may now carry out similar calculations for $S(x, t)$. The corresponding formulas follow

$$\left.\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{S}(x,t)\,dx &= \frac{1}{4}\pi T \int_{-\infty}^{\infty} k^2 E(k)\,E^*(k)\,dk,\\
\int_{-\infty}^{\infty} x\,\mathcal{S}(x,t)\,dx &= \frac{1}{4}\pi T \int_{-\infty}^{\infty} i\,k^2 E^{*\prime}(k)\,E(k)\,dk + {}\\
&\quad + \frac{1}{4}\pi T t \int_{-\infty}^{\infty} k^2 \sigma'(k)\,E(k)\,E^*(k)\,dk,\\
\int_{-\infty}^{\infty} x^2\,\mathcal{S}(x,t)\,dx &= \int_{-\infty}^{\infty} x^2\,\mathcal{S}(x,0)\,dx + \frac{1}{2}\pi T t \int_{-\infty}^{\infty} k^2 \sigma' E^{*\prime}(k)\,E(k)\,dk + {}\\
&\quad + \frac{1}{4}\pi T t^2 \int_{-\infty}^{\infty} k^2 \sigma'^2 E(k)\,E^*(k)\,dk,
\end{aligned}\right\} \tag{15.19}$$

and

$$\bar{x}_S(t) = \bar{x}_S(0) + t\,\frac{\overline{k^2\sigma'}}{\overline{k^2}} \tag{15.20}$$

and again a formula for $\overline{[x-\bar{x}_S(t)]^2}$ similar in character to (15.4).

One should note that the total potential, kinetic and surface energies associated with $\eta_R(x,t)$ each remain constant in time. If $T \neq 0$, then the mean positions of the three energy densities propagate with different velocities, each velocity being an average, in some sense, of σ'. If one considers a wave packet (15.5), then as the width 2ε of the band of wave numbers approaches zero the velocity of propagation of the individual energy densities will each approach $\sigma'(k_0)$, the group velocity.

Consider now the total energy density,

$$\mathcal{E}(x,t) = \mathcal{V}(x,t) + \mathcal{T}(x,t) + \mathcal{S}(x,t).$$

Making use of the form of $\sigma(k)$,

$$\sigma^2(k) = (g\,k + T\,k^3/\varrho)\tanh k\,h,$$

one finds

$$\left.\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{E}(x,t)\,dx &= \frac{1}{4}\pi\varrho \int_{-\infty}^{\infty} \left[g + \frac{\sigma^2}{k}\coth k\,h + \frac{T}{\varrho}k^2\right] E(k)\,E^*(k)\,dk\\
&= \frac{1}{2}\pi \int_{-\infty}^{\infty} [g\,\varrho + T\,k^2]\,E(k)\,E^*(k)\,dk,\\
\int_{-\infty}^{\infty} x\,\mathcal{E}(x,t)\,dx &= \frac{1}{2}\pi \int_{-\infty}^{\infty} [g\,\varrho + T k^2]\,i\,E^{*\prime}(k)\,E^*(k)\,dk + \frac{1}{2}\pi t \int_{-\infty}^{\infty} \sigma'(k)\,\times\\
&\quad \times [g\,\varrho + T\,k^2]\,E\,E^*\,dk,\\
\int_{-\infty}^{\infty} x^2\,\mathcal{E}(x,t)\,dx &= \int_{-\infty}^{\infty} x^2\,\mathcal{E}(x,0)\,dx + \pi t \int_{-\infty}^{\infty} \sigma'(g\,\varrho + T\,k^2]\,i\,E^{*\prime} E\,dk + {}\\
&\quad + \frac{1}{2}\pi t^2 \int_{-\infty}^{\infty} \sigma'^2 [g\,\varrho + T\,k^2]\,E\,E^*\,dk,
\end{aligned}\right\} \tag{15.21}$$

and

$$\bar{x}_E(t) = \bar{x}_E(0) + t\,\frac{\overline{\sigma'[\varrho g + T k^2]}}{\overline{\varrho g + T k^2}}\,. \tag{15.22}$$

At any instant t half of the total energy is kinetic energy and the other half is divided between potential and surface energy.

There is another way of considering the energy transported by surface waves which, at first glance, is different from the preceding treatment. Consider a fixed plane $x = \text{const}$. Then from the results in Sect. 8 one may compute the rate at which energy is being transported through this plane, the so-called *energy-flux*. Let us denote it by $\mathcal{F}(x, t)$. After appropriate linearization, formula (8.10) gives

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$$\mathcal{F}(x,t) = -\int_{-\infty}^{0} \varrho\, \Phi_t(x,y,t)\, \Phi_x(x,y,t)\, dy - T\,\eta_t(x,t)\,\eta_x(x,t)\,. \tag{15.23}$$

The expression for the flux has an advantage over the expressions for mean positions considered above in that no strong restrictions upon η are required for it to exist. In fact, it can be computed for a single harmonic wave

$$\eta = A \sin(k x - \sigma t)\,. \tag{15.24}$$

With

$$\Phi = -A\,\frac{\sigma}{k}\,\frac{\cosh k(y+h)}{\sinh k h}\cos(k x - \sigma t)\,,$$

one finds by a straightforward calculation

$$\mathcal{F}(x,t) = A^2 T k \sigma \cos^2(k x - \sigma t) + A^2 \varrho\,\frac{\sigma^3}{2k^2}\coth k h \left[1 + \frac{2 k h}{\sinh 2 k h}\right] \sin^2(k x - \sigma t)\,.$$

Averaging over a wavelength (or over a period, it makes no difference which), one finds

$$\left.\begin{aligned} \mathcal{F}_{\text{av}} &= A^2\,\frac{1}{4}\,\frac{\sigma}{k}\left\{2 T k^2 + \sigma^2 \varrho\,\frac{\coth k h}{k}\left[1 + \frac{2 k h}{\sinh 2 k h}\right]\right\} \\ &= \frac{1}{2} A^2 (g\varrho + T k^2)\,\sigma'(k)\,. \end{aligned}\right\} \tag{15.25}$$

Thus the group velocity enters again in connection with energy propagation, even though no "group" is present. The energy density and average energy per wave length for (15.24) are

$$\left.\begin{aligned} \mathcal{E}(x,t) &= A^2\left\{\frac{1}{2}\varrho g \sin^2(k x - \sigma t) + \frac{1}{2} T k^2 \cos^2(k x - \sigma t) + \right. \\ &\quad \left. + \frac{1}{4}\varrho\,\frac{\sigma^2}{k}\coth k h\left[1 - \frac{2 k h}{\sinh 2 k h}\cos 2(k x - \sigma t)\right]\right\}, \\ \mathcal{E}_{\text{av}} &= \tfrac{1}{2} A^2 (g \varrho + T k^2)\,. \end{aligned}\right\} \tag{15.26}$$

If one is dealing with a composite wave, averaging over a wave length is possible only if the resulting wave is periodic. However, even without this restriction, one may compute both the average flux and average energy per unit length from

$$\left.\begin{aligned} \mathcal{F}_{\text{av}} &= \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} \mathcal{F}(x,t)\, dx\,, \\ \mathcal{E}_{\text{av}} &= \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} \mathcal{E}(x,t)\, dx\,. \end{aligned}\right\} \tag{15.27}$$

Then if a composite wave propagating to the right is given by

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$$\eta(x,t) = \sum_{j=-\infty}^{\infty} E_j \, e^{-i(k_j x - \sigma t)} = \sum_{j=1}^{\infty} a_j \cos(k_j x - \sigma_j t) + b_j \sin(k_j x - \sigma_j t), \tag{15.28}$$

with

$$\Phi = \sum_{j=-\infty}^{\infty} i E_j \frac{\cosh k_j (y+h)}{\sinh k_j h} \frac{\sigma_j}{k_j} e^{-i(k_j x - \sigma_j t)},$$

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where $E_j = E^*_{-j} = \frac{1}{2}(a_j + b_j)$, $k_{-j} = -k_j$, $\sigma_j = \sigma(k_j) = -\sigma_{-j}$, one finds

$$\left.\begin{aligned} \mathscr{E}_{\text{av}} &= \frac{1}{2} \sum_{-\infty}^{\infty} |E_j|^2 \left[T k_j^2 + \varrho g + \varrho \frac{\sigma_j^2}{k_j} \coth k_j h\right] \\ &= \sum_{-\infty}^{\infty} |E_j|^2 [\varrho g + T k_j^2] = \frac{1}{2} \sum_{1}^{\infty} (a_j^2 + b_j^2) [\varrho g + T k_j^2] \end{aligned}\right\} \tag{15.29}$$

and

$$\left.\begin{aligned} \mathscr{F}_{\text{av}} &= \sum_{-\infty}^{\infty} |E_j|^2 \left\{T k_j \sigma_j + \varrho \frac{\sigma_j^3}{2k_j^2} \coth k_j h \left[1 + \frac{2k_j h}{\sinh 2k_j h}\right]\right\} \\ &= \sum_{-\infty}^{\infty} |E_j|^2 [\varrho g + T k_j^2] \, \sigma_j'. \end{aligned}\right\} \tag{15.30}$$

In order to obtain these relatively simple formulas in which the contributions from the individual harmonics are isolated, it is essential that the averages be taken. Otherwise, for $\mathscr{E}(x,t)$ or $\mathscr{F}(x,t)$ one obtains a complicated double summation, and the role of the group velocity is not apparent.

A similar analysis may be carried through for the right-moving initial hump (15.16). However, an average of either $\mathscr{F}$ or $\mathscr{E}$ computed according to (15.27) would vanish. Instead we take the total flux and total energy, respectively:

$$\mathscr{F}_{\text{total}} = \int_{-\infty}^{\infty} \mathscr{F}(x,t)\, dx, \qquad \mathscr{E}_{\text{total}} = \int_{-\infty}^{\infty} \mathscr{E}(x,t)\, dx. \tag{15.31}$$

The resulting formulas are analogous to (15.29) and (15.30):

$$\left.\begin{aligned} \mathscr{E}_{\text{total}} &= \tfrac{1}{2}\pi \int_{-\infty}^{\infty} [g\varrho + T k^2] \, E(k) \, E^*(k) \, dk, \\ \mathscr{F}_{\text{total}} &= \tfrac{1}{2}\pi \int_{-\infty}^{\infty} \sigma'(k) \, [g\varrho + T k^2] \, E(k) \, E^*(k) \, dk. \end{aligned}\right\} \tag{15.32}$$

If the last result is applied to a narrow wave band, such as (15.5), then one finds the limiting relationship

$$\lim_{\varepsilon \to 0} \frac{\mathscr{F}_{\text{total}}}{\mathscr{E}_{\text{total}}} = \sigma'(k_0).$$

In the first method of treating the propagation of energy, i.e. in terms of the motion of the mean position of the energy density, it was not surprising that σ' should appear, for it is a familiar property of Fourier transforms that taking the derivative of the transform is associated with multiplying the function by the variable. Thus, if

$$g(k) = \int_{-\infty}^{\infty} f(x) \, e^{ikx} \, dx,$$

then

$$g'(k) = \int_{-\infty}^{\infty} i x f(x) \, e^{ikx} \, dx,$$

In the cases considered above the transform contained $e^{i\sigma t}$ as a factor, and the derivative contained $\sigma' t$ in one summand. However, the appearance of σ' in the formulas for $\mathcal{F}_{\text{av}}$ or $\mathcal{F}_{\text{total}}$ seems in some ways coincidental: One makes a calculation, and after gathering and manipulating terms discovers that a certain combination of them indeed contains σ'. That this is not really coincidence is indicated by the following theorem for the case (15.21):

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} x\,\mathcal{E}(x,t)\,dx = \mathcal{F}_{\text{total}}. \tag{15.33}$$

It may be proved as follows. From the definition of $\mathcal{E}(x,t)$

$$\int_{-\infty}^{\infty} x\,\mathcal{E}(x,t)\,dx = \int_{-\infty}^{\infty} x\left[\tfrac{1}{2}\varrho g\eta^2 + \tfrac{1}{2}T\eta_x^2 + \tfrac{1}{2}\varrho\int_{-h}^{0}(\Phi_x^2+\Phi_y^2)\,dy\right]dx.$$

Hence

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$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} x\,\mathcal{E}(x,t)\,dx = \int_{-\infty}^{\infty} x\,[\varrho g\eta\eta_t + T\eta_x\eta_{xt} + \varrho\int_{-h}^{0}(\Phi_x\Phi_{xt}+\Phi_y\Phi_{yt})\,dy\,dx.$$

Integrating the second and third terms by parts and taking account of the assumed behavior of η and Φ at $\pm\infty$, one finds

$$\int_{-\infty}^{\infty} x\left[\varrho g\eta\eta_t - T\eta_{xx}\eta_t + \varrho\int_{-h}^{0}(\Phi_y\Phi_{yt} - \Phi_{xx}\Phi_t)\,dy\right]dx -$$
$$-\int_{-\infty}^{\infty}\left[T\eta_x\eta_t + \varrho\int_{-h}^{0}\Phi_x\Phi_t\,dy\right]dx.$$

Since $\Phi_{xx}+\Phi_{yy}=0$, one may express the third summand in the first integral as

$$\varrho\int_{-h}^{0}(\Phi_y\Phi_{yt}+\Phi_{yy}\Phi_t)\,dy = \varrho\int_{-h}^{0}(\Phi_y\Phi_t)_y\,dy = \varrho\,\Phi_y(x,0,t)\,\Phi_t(x,0,t) = \varrho\,\eta_t\,\Phi_t.$$

Hence the first integral may be written

$$\int_{-\infty}^{\infty} x\,\eta_t\,[\varrho g\eta - T\eta_{xx} + \varrho\,\Phi_t(x,0,t)]\,dx,$$

which vanishes, since the term in brackets is just the dynamical boundary condition at the free surface. The second integral above is just $\mathcal{F}_{\text{total}}$, so that (15.33) is proved.

A similar line of reasoning allows one to establish the following relation between $\mathcal{E}$ and $\mathcal{F}$:

$$\frac{\partial\mathcal{E}(x,t)}{\partial t} = -\frac{\partial\mathcal{F}(x,t)}{\partial x}, \tag{15.34}$$

essentially an expression of the conservation of energy. Eq. (15.33) may also be derived from (15.34) by writing the latter in the form

$$\frac{\partial(x\,\mathcal{E})}{\partial t} = -\frac{\partial(x\,\mathcal{F})}{\partial x} + \mathcal{F}$$

and integrating.

Although (15.33) may explain the presence of σ' in the energy flux for a continuous spectrum and finite total energy, one is still left with the apparently paradoxical situation that even for (15.24), when only one frequency is present, σ' enters into the expression for $\mathcal{F}_{\text{av}}$. One would expect the occurrence of σ' only if one were dealing not only with a specific value k but also with neighboring values. There is no useful analogue to (15.33) for the discrete spectrum, because there is no Fourier integral to connect in a natural way the mean position of a hump with σ'. However, if one approximates (15.24) or (15.28) by considering only the segment of η between $-L$ and L and taking $\eta = 0$ outside this segment, then one has approximated η by η_L, where the latter has a continuous spectrum and finite energy. For η_L it is reasonable that $\sigma'(k)$ should enter into the energy propagation. The definitions adopted for $\mathscr{F}_{\mathrm{av}}$ and $\mathscr{E}_{\mathrm{av}}$ in (15.27) reflect this approximation of η by η_L and then a passage to the limit in such a way as to keep these quantities finite. Thus it is perhaps not surprising after all that σ' has entered into the computation of $\mathscr{F}_{\mathrm{av}}$, for the method of averaging $\mathscr{F}$ and $\mathscr{E}$ is such that one replaces the discrete spectrum by a continuous one and then takes a limit. A different explanation of this paradoxical situation has been given by Rayleigh [*Theory of sound*, Vol. I p. 479]; generally it seems to be overlooked.

One should note that the definitions of velocity of propagation of mean positions of humps and energy distributions for finite total energy and of total or average energy flux all retain meaning even if the boundary condition at the free surface has not been linearized. The comparative simplicity of the formulas when the boundary condition is linearized and the occurrence in them of σ' both result from the special form of the spectrum, namely, $E(k,t) = E(k,0)\, e^{i\sigma(k)t}$, and the applicability of properties of Fourier transforms of convolutions.

For further information one may consult the monograph of Havelock (1914) already cited, papers by Bourgin (1936), Rossby (1945, 1947), Eckart (1948), Broer (1951), and Poincelot (1953, 1954), Jeffreys and Jeffreys, *Methods of mathematical physics* (3rd ed., Cambridge, 1956, pp. 511–518) and standard texts such as Lamb (1932, Sects. 236, 237, 240, 241) and Kochin, Kibel' and Roze (1948, Chap. 8, Sect. 8).

16. The solution of special boundary problems. In the next several sections we shall be considering a variety of problems, each associated with some special geometrical configuration.

In treating a particular boundary configuration one must first consider whether it is tractable at all by the theory of infinitesimal waves, i.e. whether it is possible to select a perturbation parameter ε satisfying the requirements mentioned in Sect. 10. On this basis, for example, it would appear unreasonable to try to apply infinitesimal-wave theory to the waves generated by a vertical circular cylinder moving with constant velocity, for the slope of free surfaces may be expected to become very large near the front of the cylinder. On the other hand, in certain similar situations, notably the theory of planing surfaces, it is possible to strain the theory to accomodate such a situation. The choice of parameter will be discussed in each individual case. We call attention to the fact that in many cases it is a consequence of the linearization procedure that the boundary condition on a solid boundary is no longer to be satisfied on the physical boundary, but instead on some neighboring surface. The same situation occurred earlier in linearizing the free-surface condition. This should not be considered as a further approximation, but rather as one consistent with the infinitesimal-wave approximation.

The methods for finding a solution to a boundary-value problem, once it has been properly formulated, seem to fall into two or possibly three groups. One method is a combination of separation of variables and expansion of the factors in Fourier-type series or integrals. This requires, of course, a geometric configuration related in a suitable way to the coordinate surfaces of a set of variables which allows separation and a complete set of associated elementary solutions to be used in the expansion. If a Fourier-series expansion is to be used, orthogonality of the elementary solutions is desirable.

If the motion is harmonic in time with frequency σ and if the fluid is of finite depth h, then the functions

$$\{\cosh m_0(y+h), \cos m_i(y+h)\} \tag{16.1}$$

occurring as factors in (13.2) and (13.4), in (13.6), and in (13.8) may be shown easily by direct computation to be orthogonal on the interval $0 \geqq y \geqq -h$. Completeness follows from known criteria[1]. However, both orthogonality and completeness are consequences of the general theory of Sturm-Liouville systems. The result may be used in the following way, for example. Suppose fluid occupies the region

$$x > 0, \quad 0 > y > -h, \quad 0 < z < l,$$

and that the boundary conditions on the walls and bottom are

$$\left.\begin{aligned} \Phi_x(0, y, z, t) &= F(y, z)\cos\sigma t, \\ \Phi_z(0, x, y, t) &= \Phi_z(l, x, y, t) = 0, \\ \Phi_x(x, y, -h, t) &= 0. \end{aligned}\right\} \tag{16.2}$$

Then, by expressing $F(y, z)$ as a double series

$$\left.\begin{aligned} F(y, z) = &\sum a_{0q}\cosh m_0(y+h)\cos\frac{\pi q}{l} z \\ &+ \sum\sum a_{pq}\cos m_p(y+h)\cos\frac{\pi q}{l} z \end{aligned}\right\} \tag{16.3}$$

(with appropriate restrictions upon F), one may construct a solution from the elementary solutions in (13.6). Further conditions relating to boundedness and behavior as $x \to \infty$ are necessary in order to ensure a unique solution, but will not be discussed here. The elementary solutions (13.8) can be used in a similar way for the region exterior to a vertical cylindrical boundary. Still other configurations are possible corresponding to the various coordinate systems allowing separation of $\Delta_2\varphi \pm m\varphi = 0$.

If the fluid is infinitely deep, it is possible to construct a Fourier-integral see errata expansion using the function.

$$\{e^{\nu y}, k\cos ky + \nu\sin ky\}, \quad \nu = \sigma^2/g, \quad 0 < k < \infty. \tag{16.4}$$

In fact, HAVELOCK (1929b) has remarked that the usual Fourier-integral representation of a function may be altered to give

$$\left.\begin{aligned} f(y) = &\frac{2}{\pi}\int_0^\infty\int_{-\infty}^0 f(\eta)\frac{(k\cos ky + \nu\sin ky)(k\cos k\eta + \nu\sin k\eta)}{k^2+\nu^2}\,d\eta\,dk \\ &+ 2\nu\, e^{\nu y}\int_{-\infty}^0 f(\eta)\, e^{\nu\eta}\,d\eta. \end{aligned}\right\} \tag{16.5}$$

If the problem is such that rectangular coordinates may be used conveniently, then (16.5) may be combined with a Fourier-series or Fourier-integral expansion in z and the elementary solutions (13.5) used to construct a solution analogous to (16.3). The necessary expressions in both rectangular and cylindrical coordinates can be found in the cited paper of HAVELOCK.

If the fluid is of bounded horizontal extent and is bounded by vertical surfaces which are constant-coordinate surfaces in one of the coordinate systems

[1] See, e.g., N. LEVINSON: Gap and density theorems. Amer. Math. Soc. Colloq. Publ. No. 27, Chap. I. New York 1940.

allowing separation of $\Delta_2\varphi \pm m\varphi = 0$, the various possible modes of motion of the fluid may be obtained as the solution of an eigenvalue problem of a classical type. If the container is of more general shape, it is more difficult to obtain explicit solutions. The problem will be discussed in Sect. 23.

The orthogonal functions (16.1) were associated with a single value of the frequency σ. It is possible to derive another result concerning orthogonality of solutions associated with different values of σ. Let $\varphi_1(x, y, z)\cos\sigma_1 t$ and $\varphi_2(x, y, z)\cos\sigma_2 t$, $\sigma_1 \neq \sigma_2$, be regular velocity potentials of harmonic oscillations of different frequencies. Furthermore, let any solid boundaries be fixed and, if the fluid is not bounded in extent, we suppose that $|\operatorname{grad}\varphi| = O(R^{-1-\varepsilon})$ as $R^2 = x^2 + z^2 \to \infty$. Consider the fluid contained within a large cylinder Ω_R of radius R and above the plane $y = -R$. The fluid will be bounded partly by free surface F_R, partly by solid boundaries S_R, partly by the horizontal plane B_R and partly by the cylinder Ω_R. Applying Green's theorem to the two potential function, one obtains

$$\left.\begin{aligned} 0 &= \iint\limits_{F_R + S_R + B_R + \Omega_R} (\varphi_1 \varphi_{2n} - \varphi_{1n}\varphi_2)\, d\sigma \\ &= \iint\limits_{F_R} (\varphi_1\varphi_{2y} - \varphi_{1y}\varphi_2)\, d\sigma + \iint\limits_{B_R + \Omega_R} (\varphi_1\varphi_{2n} - \varphi_{1n}\varphi_2)\, d\sigma. \end{aligned}\right\} \tag{16.6}$$

As $R \to \infty$, the integral over $\Omega_R + B_R \to 0$, and one has

$$\iint\limits_F (\varphi_1\varphi_{2y} - \varphi_{1y}\varphi_2)\, d\sigma = 0. \tag{16.7}$$

From the free-surface condition

$$\varphi_{iy}(x, 0, z) = -\frac{\sigma_i^2}{g}\varphi_i(x, 0, z), \qquad i = 1, 2, \tag{16.8}$$

and (16.7) becomes

$$\frac{\sigma_1^2 - \sigma_2^2}{g} \iint\limits_F \varphi_1(x, 0, z)\, \varphi_2(x, 0, z)\, d\sigma = 0, \tag{16.9}$$

or simply

$$\iint\limits_F \varphi_1(x, 0, z)\, \varphi_2(x, 0, z)\, d\sigma = 0. \tag{16.10}$$

Hence φ_1 and φ_2 are orthogonal over the free surface of the fluid. This theorem can be used for certain initial-value problems in a manner analogous to that in which the orthogonality of (16.1) can be used for boundary-value problems. This will be done in Sect. 23 α.

A second method for solving special problems is the method of Green's functions or source functions [cf. Volterra (1934)]. In this method one constructs first a potential function of the form

$$\left.\begin{aligned} G(x, y, z; \xi, \eta, \zeta) &= \frac{1}{r} + G_0(x, y, z; \xi, \eta, \zeta), \\ r^2 &= (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2, \end{aligned}\right\} \tag{16.11}$$

such that G_0 is regular in $y < 0$ and such that G satisfies the free-surface condition, conditions at infinity appropriate to the problem at hand, and, if the fluid is of finite depth, the boundary condition on the bottom. Such solutions are, of course, just the singular solutions derived in Sect. 13 γ. Next, if there are surfaces S in (or on) the fluid upon which certain further boundary conditions must be satisfied, we attempt to satisfy them by a distribution of the modified sources (16.11) over the surface(s) S:

$$\Phi(x, y, z, t) = \iint\limits_S \gamma(\xi, \eta, \zeta, t)\, G(x, y, z; \xi, \eta, \zeta; t)\, d\sigma. \tag{16.12}$$

Here γ is an unknown function which is to be determined from the boundary condition on S. In most problems this boundary condition consists in specifying Φ_n on S. Well known properties of surface distributions of sources then allow one to formulate an integral equation for γ:

$$\left.\begin{aligned} \Phi_n(x,y,z,t) = -2\pi\gamma(x,y,z,t) + \iint_S \gamma(\xi,\eta,\zeta,t)\,G_n(x,y,z;\xi,\eta,\zeta;t)\,d\sigma,\\ (x,y,z) \text{ on } S,\end{aligned}\right\} \quad (16.13)$$

where n is the exterior normal to the surface S (taken here as a closed surface). When it is convenient, one may also use distributions of dipoles.

It is also possible, and sometimes advantageous, to construct solutions satisfying given boundary conditions on a closed surface S by distributing the singular solutions on surfaces, lines or points completely inside S. Examples will occur later.

A third method of approach is to seek first, instead of $\Phi(x,y,z,t)$ or $f(z,t)$ the functions

$$\chi = \Phi_{tt} + g\,\Phi_y \quad \text{or} \quad F = f_{tt} + i g f'.$$

These functions satisfy a simpler condition on the plane $y=0$:

$$\chi(x,0,z,t) = 0 \quad \text{or} \quad \operatorname{Re} F(x - i0, t) = 0.$$

If the other boundary conditions are such that they can be formulated simply in terms of χ or F, the new problem may be simpler to solve. After finding χ or F, one must then solve a differential equation in order to obtain the desired solution Φ or f. This procedure is called the "reduction" method by WEINSTEIN (1949). It was apparently first introduced by LEVI-CIVITA and has since been much exploited by CISOTTI, KELDYSH,, KOCHIN, SEDOV, HASKIND, LEWY, STOKER and others. It has already been used in the derivation of (13.28) and will be applied in several other problems[1]. The solution of the reduced problem may, of course, be carried out by one of the two methods already described above, or any other one which is convenient.

The methods outlines above do not exhaust the possible ones for finding analytic solutions. However, they will occur frequently in the next several sections. Several of the special problems treated in the following sections can be solved by each of the three approaches. The choice of a particular one has been made either to illustrate a method or because it happens to be convenient. Techniques for finding numerical solutions will not be discussed.

17. Two-dimensional progressive and standing waves in unbounded regions with fixed boundaries. In this and the following section we shall consider situations in which the region occupied by fluid extends to infinity horizontally, the solid boundaries are fixed, but of more complicated shape than the simple flat bottom considered up to now, and the motion of the fluid at infinity is prescribed, or at least partly so. We shall assume that the velocity is bounded at all interior points of the fluid and also at the infinite limits of the fluid. The motion is taken to be periodic everywhere with period σ. Hence we shall assume (cf. Sect. 11) that

$$\Phi(x,y,t) = \varphi_1(x,y)\cos\sigma t + \varphi_2(x,y)\sin\sigma t = \operatorname{Re}\varphi\,e^{-i\sigma t}.$$

The restriction to standing or progressive waves can be properly applied only at $x=\pm\infty$. Thus, we shall look for solutions which at $x=\infty$ behave like

$$(A\cos m x + B\sin m x)\cos\sigma t$$

[1] The method is used also by MUSKHELISHVILI [Singular integral equations, Noordhoff, Groningen, 1953, § 74] to reduce a mixed boundary condition of more complicated type to a simple one.

or

$$A\cos(mx+\sigma t)+B\cos(mx-\sigma t),$$

and similarly at $x=-\infty$ if the fluid extends in that direction. As we shall see below, the coefficients cannot be chosen independently if φ remains bounded everywhere.

The parameter of linearization may be chosen as

$$\varepsilon=\max(Am, Bm).$$

If the solution φ is bounded everywhere, then as $\varepsilon\to 0$, $\varphi\to 0$ uniformly. However, if a singularity is allowed, then $\varphi\to 0$ uniformly only in a region excluding a neighborhood about the singularity. One may presume that the solution to the linearized problem loses physical significance within such a neighborhood. [It is assumed by STOKER (1947, p. 5) that singularities at the surface are associated with breaking of the waves.]

We shall discuss below two types of problems: obstacles in an infinite ocean and sloping beaches. For each type a special case will be discussed in some detail.

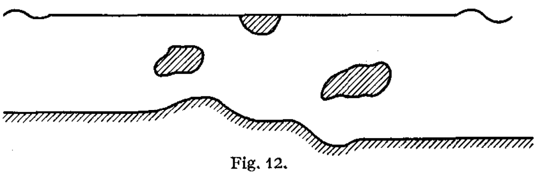

Fig. 12.

α) *Obstructions in an infinitely long canal.* Consider first the following situation. The fluid extends from $x=-\infty$ to $x=+\infty$; the bottom is given by $y=-h(x)$, where $h(x)=h_1>0$ for $x\geqq x_1$, $h(x)=h_2>0$ for $x\leqq x_2<x_1$; fixed obstacles may be present in the fluid or on the surface (see Fig. 12). The surface at $x=+\infty$ will be assumed to behave like

$$\eta=A_1\cos(m_1x+\sigma t+\alpha_1)+B_1\cos(m_1x-\sigma t+\beta_1)$$

and at $x=-\infty$ like

$$\eta=A_2\cos(m_2x+\sigma t+\alpha_2)+B_2\cos(m_2x-\sigma t+\beta_2).$$

A proof of the existence of a solution to this problem does not seem to exist for the general case. One would not expect a uniqueness theorem since no statement has been made about singularities or circulation. For infinite depth and a submerged body KOCHIN (1939) has proved the existence for sufficiently large values of m (the situation is slightly different, but the proof carries over). KREISEL (1949) has established the existence of a solution and its uniqueness in two cases. In the first case $h_1=h_2$, only obstacles on the bottom are allowed, Φ is assumed bounded, and a certain constant, defined in terms of the wavelength and the conformal mapping of the fluid region onto the strip $0<y<h_1$, must be less than 1. Included are theorems comparing the values of this constant for different types of obstructions. The second result allows a shallow obstruction in the surface, but requires a flat bottom and sufficiently long waves and again bounded Φ. ROSEAU (1952) has proved existence and uniqueness for no obstructions within the fluid, but for $h_1\neq h_2$; the curve joining the two ends is of a special sort. JOHN (1950, p. 78ff.) has proved uniqueness for a flat bottom and for a body in the free surface with the property that every vertical line intersects either the free surface or the body just once; certain regularity properties of Φ must also be assumed. If the body is convex and intersects the free surface perpendicularly, he is able to prove also existence of a solution.

Existence and uniqueness theorems have also been proved for several special configurations. In most of these cases explicit solutions are given. A vertical-line barrier extending from the free surface to a depth l in an infinite fluid has been considered by DEAN (1945), URSELL (1947) and HASKIND (1948). Both DEAN and URSELL, and also MARNYANSKII (1954), also consider a barrier extending from $-\infty$ to a distance l below the surface. JOHN (1948) has generalized both these problems to the case of a slanting barrier of slope $\pi/2n$, and obtained a more general solution even for the vertical barrier. DEAN (1948) and URSELL (1950) have also considered submerged circular cylinders in an infinitely deep fluid, and URSELL has established a uniqueness theorem for this case. A horizontal obstruction of finite width on the water surface (the "finite-dock problem") has been treated by RUBIN (1954), who proved existence of a solution by a variational method. Other references concerning the dock problem will be given below. BARTHOLOMEUSZ (1958) treats the long-wave approximation for reflection at a step in the bottom.

Reflection and transmission coefficients. If one assumes the existence of a solution to the general problem stated above, one may establish the form of the solution for $x > x_1$ and $x < x_2$ by using the completeness of the functions [cf. (16.1)]

$$\{\cosh m_0 (y+h), \qquad \cos m_n (y+h)\}$$

in the interval $-h \leqq y \leqq 0$ (cf. KREISEL 1949, pp. 26—29; JOHN 1948, p. 152). It is

$$\left.\begin{aligned}\Phi(x,y,t) &= [A_i \cos(m_0^{(i)} x + \sigma t + \alpha_i) + B_i \cos(m_0^{(i)} x - \sigma t + \beta_i)] \times \\ &\times \cosh m_0^{(i)} (y+h_i) + \sum_{n=1}^{\infty} (a_{in} \cos\sigma t + b_{in} \sin\sigma t) \exp(-m_n^{(i)} |x|) \cos m_n^{(i)} (y+h_i),\end{aligned}\right\} \tag{17.1}$$

where $i = 1, 2$ and $\sigma^2 = g m_0^{(i)} \tanh m_0^{(i)} h_i = -g m_n^{(i)} \tan m_n^{(i)} h_i$.

Let us now apply the formula for dE/dt in Eq. (8.2) to the region of fluid bounded by the planes $x = c_2 < x_2$, $x = c_1 > x_1$, the bottom and any other obstructions, which we take to be between these two planes. Then, if $\varphi_x^2 + \varphi_y^2$ is bounded in the region considered,

$$\frac{dE}{dt} = \int_{-h_1}^{0} \varrho\, \Phi_t\, \Phi_x(c_1, y, t)\, dy - \int_{-h_2}^{0} \varrho\, \Phi_t\, \Phi_x(c_2, y, t)\, dy,$$

since on the "physical" boundaries [cf. Eq. (8.3)] either $p = 0$ or $\Phi_n = 0$. Anticipating that we are interested only in the asymptotic values for $c_1 \to \infty$ and $c_2 \to -\infty$, we compute the above expression using only the first term in (17.1) and average over a period $2\pi/\sigma$:

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$$\left[\frac{dE}{dt}\right]_{\mathrm{av}} = \pi m_0^{(1)} h_1 \left[1 + \frac{\sinh 2 m_0^{(1)} h_1}{2 m_0^{(1)} h_1}\right] [A_1^2 + B_1^2] - \\ - \pi m_0^{(2)} h_2 \left[1 + \frac{\sinh 2 m_0^{(2)} h_2}{2 m_0^{(2)} h_2}\right] [A_2^2 - B_2^2].$$

Since the average energy in the region is constant,

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$$m_0^{(1)} h_1 \left[1 + \frac{\sinh 2 m_0^{(1)} h_1}{2 m_0^{(1)} h_1}\right] [A_1^2 - B_1^2] = m_0^2 h_2 \left[1 + \frac{\sinh 2 m_0^{(2)} h_2}{2 m_0^{(2)} h_2}\right] [A_2^2 - B_2^2]. \tag{17.2}$$

This is, of course, a statement of the conservation of energy. If A_1 is given $\neq 0$ and $B_2 = 0$, then A_2, B_1 are uniquely determined. For suppose two solutions

Φ and Φ' are possible, both with the same A_1 and $B_2=0$, but one with A_2, B_1, the other with A_2', B_1'. Apply (17.2) to the difference $\Phi-\Phi'$:

$$-m_0^{(1)} h_1\left[1+\frac{\sinh 2m_0^{(1)} h_1}{2m_0^{(1)} h_1}\right](B_1-B_1')^2 = m_0^{(2)} h_2\left[1+\frac{\sinh 2m_0^{(2)} h_2}{2m_0^{(2)} h_2}\right](A_2-A_2')^2.$$

Each side must be zero since they differ in sign and are equal. Hence $A_2=A_2'$ and $B_1=B_1'$. This does not, of course, imply the uniqueness of Φ itself.

If $h_1=h_2$, then (17.2) simplifies in an obvious way:

$$A_1^2-B_1^2=A_2^2-B_2^2. \tag{17.3}$$

Here h may also be infinite.

Setting $B_2=0$ and fixing A_1 as above corresponds physically to giving the amplitude of an incoming wave far to the right. B_1 is then the amplitude of the see errata reflected wave and A_2 of the transmitted wave. The theorem of the preceding paragraph states that A_1 fixes them uniquely. We define $|B_1/A_1|$ as the *reflection coefficient* R and $|A_2/A_1|$ as the *transmission coefficient* T. They are uniquely see errata determined and $R^2+T^2=1$. Properly one should define both left and right coefficients since the channel is not symmetric. However, the uniqueness theorem implies that both have the same value [see KREISEL (1949) or MEYER (1955)]. One can clearly arrange the phases so that A_1 and A_2 have the same sign. If this is done, $\alpha_2-\alpha_1$ will be the *phase shift* caused by the obstacles.

KREISEL (1949) has proved several general theorems concerning the reflection coefficient if $h_1=h_2$. In particular, if there are no obstacles within the fluid, he determines upper and lower bounds for the reflection coefficient in terms of the conformal mapping $z(\zeta)$ of the infinite strip $0>\eta>-h$ onto the region occupied by fluid, with infinities corresponding. His bounds become closer as the wavelength increases. He gives, for example, asymptotic expressions as $m_0\to 0$ for the reflection coefficient from a horizontal reef of width a and height ε and from a flat plate in the surface of beam b, namely,

$$\frac{\varepsilon}{h}\,\frac{2m_0 h\,|\sin 2m_0 a|}{\sinh 2m_0 h'(1+2m_0 h/\sinh 2m_0 h)}$$

and

$$\frac{m_0 b}{1+2m_0 h/\sinh 2m_0 h}.$$

Other general considerations will be found in BIESEL and LE MÉHAUTÉ (1955).

An interesting special result of DEAN (1947) [see also URSELL (1950)] is that the reflection coefficient from a submerged circular cylinder in infinitely deep water vanishes. The proof may be briefly sketched. Let a be the radius and let the center be at $(0,-b)$, $b>a$. Let the velocity potential be written as a sum of an incoming wave and a diverging wave:

$$\Phi = A\,\nu\,e^{\nu y}\cos(\nu x+\sigma t)+\Phi_0;$$

and suppose that Φ_0 can be expressed as a sum of multipoles (13.31), starting with dipoles:

$$\Phi_0=\sum a_n\Phi_n^{(s)}(x,y,t)+b_n\Phi_n^{(a)}(x,y,t)+c_n\Phi_n^{(s)}\left(x,y,t+\frac{\pi}{2\sigma}\right)+d_n\Phi_n^{(a)}\left(x,y,t+\frac{\pi}{2\sigma}\right),$$

where $\Phi_n^{(s)}$ is the potential for the symmetric potential of order n and strength $Q=1$, and $\Phi_n^{(a)}$ that for the antisymmetric one. The boundary condition on the cylinder [using the notation of (13.31)],

$$\left.\frac{\partial \Phi_0}{\partial r}\right|_{r=a} = A\nu\,\mathrm{e}^{-\nu b}\,\mathrm{e}^{\nu a\cos\vartheta}\left[\{\sin(\nu a\sin\vartheta)\sin\vartheta - \cos(\nu a\cos\vartheta)\cos\vartheta\}\cos\sigma t + {} \right.$$
$$\left. + \{\cos(\nu a\sin\vartheta)\sin\vartheta + \sin(\nu a\sin\vartheta)\cos\vartheta\}\sin\sigma t\right],$$

gives the relation $a_n = -d_n$, $b_n = c_n$. The reflected wave at $+\infty$ from the antisymmetric functions then just cancels that from the symmetric functions. They reinforce each other at $x=-\infty$. The phase change for $b/a = \frac{5}{4}$, $\sigma^2 a/g = \frac{4}{3}$ was computed numerically by both DEAN and URSELL and for this case was very close to 90°.

As mentioned above, straight-line barriers have been considered by DEAN (1945, 1946), URSELL (1947), HASKIND (1948), JOHN (1948), and LEVINE (1957). The last three authors use the reduction method, whereas the first two use a Fourier-integral method which leads to a singular integral equation. We shall treat this problem by the reduction method. DEAN and JOHN also treat barriers inclined at an angle $\pi/2n$. LEVINE and RODEMICH (1958) solve the vertical-barrier problem by several methods, including the cited ones, and then apply one of them to the problem of waves incident upon two parallel vertical barriers.

Vertical barrier. Let the barrier extend along the y-axis from $y=0$ to $y=-l$ and suppose an incoming wave is given at $x=+\infty$ as

$$\eta = A\cos(\nu x + \sigma t + \alpha), \qquad \sigma^2 = g\nu.$$

We shall look for a velocity potential Φ having the form

$$\Phi = -A\frac{g}{\sigma}\,\mathrm{e}^{\nu y}\sin(\nu x + \sigma t + \alpha) + \varphi_1\cos\sigma t + \varphi_2\sin\sigma t$$

and satisfying the following boundary conditions on the free surface and the barrier:

see errata

$$\Phi_{tt} + g\,\Phi_y(x,0,t), \qquad |x|>0 \quad\text{and}\quad \Phi_x(0,y,t)=0, \qquad 0>y>-l.$$

As $x\to\pm\infty$, $\varphi_1\cos\sigma t + \varphi_2\sin\sigma t$ must represent outgoing waves. In the neighborhood of $(0,-l)$ it will be assumed that

$$\lim\,[x^2 + (y+l)^2]\,(\Phi_x^2 + \Phi_y^2) = 0 \quad\text{as}\quad (x,y)\to(0,-l).$$

In the neighborhood of the intersection of the barrier and the surface (0, 0) as well as in the region of fluid bounded away from the barrier, we shall assume $\Phi_x^2+\Phi_y^2$ bounded. It should be noted, however, that this assumption excludes a large class of solutions of possible physical interest (cf. JOHN 1948).

If we introduce the stream functions Ψ, ψ_1, and ψ_2 corresponding to Φ, φ and φ_2 and the corresponding complex potentials F, f_1 and f_2, we have

$$F = \left(-\frac{Ag}{\sigma}\,i\,\mathrm{e}^{-i(\nu z+\alpha)} + f_1\right)\cos\sigma t + {}$$
$$+\left(-\frac{Ag}{\sigma}\,\mathrm{e}^{-i(\nu z+\alpha)} + f_2\right)\sin\sigma t = F_1\cos\sigma t + F_2\sin\sigma t$$

and the boundary conditions

$$\mathrm{Re}\{-\nu F_n + i F_n'\} = 0, \qquad y=0, \qquad |x|>0, \qquad n=1,2,$$

see errata

$$\mathrm{Re}\,F_n' = 0, \qquad x=0, \qquad 0>y>l, \qquad n=1,2.$$

After finding F_1 and F_2 satisfying these conditions, constants occurring in the solutions must be adjusted so that f_1 and f_2 satisfy the radiation conditions:

see errata

$$\lim_{x\to\pm\infty}(f_1'\pm\nu f_2)=0, \qquad \lim_{x\to\pm\infty}(f_2'\mp f_1)=0.$$

Consider the function

$$G_1=F_1'+i\nu F_1=\mathrm{e}^{-i\nu z}(\mathrm{e}^{i\nu z}F_1)'.$$

Then the boundary conditions imply that G_1 satisfies

$$\operatorname{Im} G_1=0 \quad \text{for} \quad y=0, \quad |x|>0,$$
$$\operatorname{Im} G_1'=0 \quad \text{for} \quad y=0, \quad |x|>0 \quad \text{and} \quad x=0, \quad 0>y>-l.$$

The function G_1 may be extended into the upper half-plane by defining $G_1(x+iy)=\overline{G}_1(x-iy)$ for $y>0$. Since we have assumed $|F'|\leqq B$ for $|z|>b>l$, we may conclude that $|G_1|<B+C|z|$ for $|z|>b$ and expand G_1 in a Laurent series

$$G_1(z)=cz+d+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots, \qquad |z|>b>l,$$

where all coefficients are real since $\operatorname{Im} G_1(x+i0)=0$. The condition $|F_1'|\to 0$ as $y\to-\infty$ implies $|G_1'|\to 0$ as $y\to-\infty$ and hence $c=0$. We may arbitrarily set $d=0$ by redefinition of Ψ_1. Further, we may show as follows that $a_1=0$. Consider a contour containing the obstruction and lying in the region $|z|>b$. Then

$$\oint G_1(z)\,dz=2\pi i\,a_1.$$

Let this contour be contracted onto the barrier. Then, from the assumed behavior of F_1 on the barrier, the integral vanishes; hence $a_1=0$. Thus

$$G_1(z)=\frac{a_2}{z^2}+\frac{a_3}{z^3}+\cdots.$$

Let us now exploit the boundary condition for G_1' by mapping the z-plane into a ζ-plane by the mapping

$$z=\sqrt{\zeta^2-l^2},$$

where the branch of the square root is chosen which makes $z\cong\zeta$ for large ζ. This maps the z-plane cut from $-il$ to il, i.e. along the barrier, onto the z-plane cut from $-l$ to $+l$, with infinities and upper and lower half-planes corresponding. Then $G_1'(z(\zeta))=H_1(\zeta)$ is analytic in the whole lower half-plane with a singularity only at $\zeta=0$, corresponding to $z=-il$, and $\operatorname{Im} H_1(\xi+i0)=0$. Since $H_1(\zeta)$ must agree with $G_1'(z)$ for large z, $H_1(\zeta)$ must have the form

$$H_1(\zeta)=\frac{b_3}{\zeta^3}+\frac{b_4}{\zeta^4}+\cdots, \qquad b_n \text{ real}.$$

The condition on Φ near the edge of the barrier, implies that $|z+il|\cdot|F_1'|\to 0$ as $z\to-il$, or $|\zeta^4 H_1(\zeta)|\to 0$ as $\zeta\to 0$ and hence that $b_n=0$, $n\geqq 4$. Thus

$$H_1(\zeta)=\frac{C_1}{\zeta^3}, \qquad C_1 \text{ real},$$

or

$$G_1'(z)=\frac{C_1}{(z^2+l^2)^{\frac{3}{2}}}.$$

Integrating, and writing $D_1=C_1/l^2$,

$$G_1(z)=D_1\frac{z-\sqrt{z^2+l^2}}{\sqrt{z^2+l^2}}=D_1\frac{z}{\sqrt{z^2+l^2}}-D_1,$$

where the constant of integration has been chosen so as to make $G_1(z)$ behave like z^{-2} for large z. Then

$$F_1(z) = E_1 e^{-i\nu z} + D_1 e^{-i\nu z} \int_{i\infty}^{z} \frac{z - \sqrt{z^2+l^2}}{\sqrt{z^2+l^2}} e^{i\nu z}\, dz,$$

where the path of integration will be taken around the right-hand side of the barrier. The boundary condition $\mathrm{Re}\, F_1'(0+iy) = 0$, $0<|y|<l$, relates E_1 and D_1 as follows. From $F_1(z)$:

$$F_1'(z) = e^{-i\nu z}\left[-i\nu E_1 + D_1 \frac{z}{\sqrt{z^2+l^2}} - i\nu D_1 \int_{i\infty}^{z} \frac{z\, e^{i\nu z}}{\sqrt{z^2+l^2}}\, dz\right].$$

Take the path of integration along the y-axis, so that the integral becomes

$$\int_{i\infty}^{z} = -i\int_{l}^{\infty} \frac{y\, e^{-\nu y}}{\sqrt{y^2-l^2}}\, dy \mp \int_{l}^{y} \frac{y\, e^{-\nu y}}{\sqrt{l^2-y^2}}\, dy$$

$$= -i\, l\, K_1(\nu l) \mp \int_{l}^{y} \frac{y\, e^{-\nu y}}{\sqrt{l^2-y^2}}\, dy, \qquad x = \pm 0.$$

Hence

$$\mathrm{Re}\, F_1'(0+iy) = e^{\nu y}\,[+\nu\, \mathrm{Im}\, E_1 - \nu\, l\, D_1\, K_1(\nu l)] = 0$$

or

$$\mathrm{Im}\, E_1 = +D_1\, l\, K_1(\nu l).$$

see errata Let $E_1 = e_1 + i l D K_1(\nu l)$. Then

$$F_1(z) = e^{-i\nu z}\left[e_1 + i\, l\, D_1\, K_1(\nu l) + D_1 \int_{i\infty}^{z} \frac{z - \sqrt{z^2+l^2}}{\sqrt{z^2+l^2}} e^{i\nu z}\, dz\right],$$

We now compute the asymptotic expressions for $F_1(z)$ for $x \to \pm\infty$. If the path of integration is taken on a large arc of radius R in the first quadrant and then to z, and if R is allowed to become infinite, it follows from JORDAN's lemma that the integral may also be written

$$\int_{\infty}^{z} \frac{z - \sqrt{z^2+l^2}}{\sqrt{z^2+l^2}} e^{i\nu z}\, dz.$$

Clearly,

$$F_1(z) \sim e^{-i\nu z}\,[e_1 + i\, l\, D_1\, K_1(\nu l)] \quad \text{as} \quad x \to +\infty.$$

As $x \to -\infty$,

$$F_1(z) \sim e^{-i\nu z}\left[e_1 + i\, l\, D_1\, K_1(\nu l) + D_1 \int_{\infty}^{-\infty} \frac{z - \sqrt{z^2+l^2}}{\sqrt{z^2+l^2}} e^{i\nu z}\, dz\right],$$

where the path of integration passes below the barrier. By completing this path by a large semicircle in the upper half-plane, which gives a zero contribution in the limit, and then contracting the contour about the barrier, one sees that

$$\int_{\infty}^{-\infty} \frac{z - \sqrt{z^2+l^2}}{\sqrt{z^2+l^2}} e^{i\nu z}\, dz = +2\int_{-l}^{l} \frac{y\, e^{-\nu y}}{\sqrt{l^2-y^2}}\, dy$$

$$= -2\nu \int_{-l}^{l} e^{-\nu y}\sqrt{l^2-y^2}\, dy = -2\pi\, l\, I_1(\nu l).$$

Hence

$$F_1(z) \sim e^{-i\nu z}\left[e_1 + i\,l\,D_1\,K_1(\nu l) - 2\pi l\,D_1\,I_1(\nu l)\right] \quad \text{as} \quad x \to -\infty.$$

Similar expressions hold for $F_2(z)$ with constants e_2 and D_2.

For f_1 and f_2 we have the asymptotic expressions:

$$\left.\begin{aligned} f_1(z) &\sim e^{-i\nu z}\left[\frac{A g}{\sigma}\, i\, e^{-i\alpha} + e_1 + i\,l\,K_1(\nu l)\,D_1\right], \\ f_2(z) &\sim e^{-i\nu z}\left[\frac{A g}{\sigma}\, e^{-i\alpha} + e_2 + i\,l\,K_1(\nu l)\,D_2\right] \end{aligned}\right\} \quad \text{as} \quad x \to +\infty,$$

$$\left.\begin{aligned} f_1(z) &\sim e^{-i\nu z}\left[\frac{A g}{\sigma}\, i\, e^{-i\alpha} + e_1 + (i\,K_1 - 2\pi I_1)\,l\,D_1\right], \\ f_2(z) &\sim e^{-i\nu z}\left[\frac{A g}{\sigma}\, e^{-i\alpha} + e_2 + (i\,K_1 - 2\pi I_1)\,l\,D_2\right] \end{aligned}\right\} \quad \text{as} \quad x \to -\infty.$$

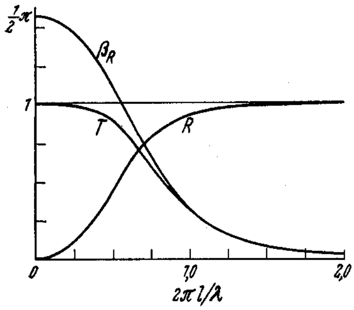

Fig. 13.

The radiation condition gives simultaneous equations for the determination of e_1, e_2, D_1 and D_2. The solution may be written

$$l(D_1 + i\,D_2) = -\frac{A g}{\sigma}\, i\, e^{-i\alpha}\,\frac{1}{\pi I_1 + i\,K_1},$$

$$e_1 + i\,e_2 = -\frac{A g}{\sigma}\, i\, e^{-i\alpha}\left(1 + \frac{\pi I_1}{\pi I_1 + i\,K_1}\right).$$

Substitution in the expressions for f_1 and f_2, and computation of $F_1 \cos\sigma t + F_2 \sin\sigma t$ give, after a somewhat tedious calculation, the following asymptotic expressions for Φ:

see errata

$$\left.\begin{aligned} \Phi &\sim \frac{A g}{\sigma}\, e^{\nu y}\left\{-\sin(\nu x + \sigma t + \alpha) + \frac{\pi I_1}{\sqrt{\pi I_1^2 + K_1^2}}\sin(\nu x - \sigma t - \alpha - \beta_R)\right\}, \\ &\qquad\qquad x \to +\infty, \\ \Phi &\sim \frac{-A g}{\sigma}\, e^{\nu y}\,\frac{K_1}{\sqrt{\pi^2 I_1^2 + K_1^2}}\sin(\nu x + \sigma t + \alpha + \beta_T), \quad x \to -\infty, \end{aligned}\right\} \tag{17.4}$$

where $\tan\beta_R = K_1/\pi I_1 = \cot\beta_T$, and $I_1 = I_1(\nu l)$, $K_1 = K_1(\nu l)$. Clearly the reflection and transmission coefficients are

$$R = \frac{\pi I_1}{\sqrt{\pi I_1^2 + K_1^2}}, \qquad T = \frac{K_1}{\sqrt{\pi I_1^2 + K_1^2}}. \tag{17.5}$$

R, T and $\beta_R = \frac{1}{2}\pi - \beta_T$ are shown in Fig. 13 as functions of $2\pi l/\lambda = \nu l$, The reflection coefficient is practically one if $l/\lambda \geqq \frac{1}{4}$.

One may now use the velocity potential to find the behavior of the fluid near the barrier, in particular, the water height and the pressure. The calculations will not be carried through, but may be found in HASKIND (1948). The elevation on either side of the barrier is given by

$$\eta(\pm 0, t) = A\left[\cos(\sigma t + \alpha) \mp \frac{1 + \nu l\, S(\nu l)}{\sqrt{\pi^2 I_1^2 + K_1^2}}\cos(\sigma t + \alpha + \beta_R)\right] \tag{17.6}$$

where

$$S(\nu l) = \frac{\pi}{2\nu l}\left[I_1(\nu l) + L_1(\nu l)\right] = \int_0^1 e^{\nu l y}\sqrt{1 - y^2}\,dy,$$

L_1 being a Struve function of imaginary argument[1]. Let the force and moment about the origin, per unit length of barrier, be denoted by X and M, the former being positive if directed along OX and the latter counterclockwise. Then

$$\left.\begin{aligned} X &= +2\varrho g A l X_0 \cos(\sigma t + \alpha + \beta_R), \\ M &= +2\varrho g A l^2 M_0 \cos(\sigma t + \alpha + \beta_R), \end{aligned}\right\} \tag{17.7}$$

where

$$X_0 = \frac{S}{\sqrt{\pi I_1^2 + K_1^2}}, \qquad M_0 = \frac{1}{\nu l \sqrt{\pi I_1^2 + K_1^2}}\left(S - \frac{\pi}{4}\right).$$

HASKIND also computes the average force and moment per unit length of the barrier. The results are:

$$\left.\begin{aligned} -X_{\text{av}} &= \frac{1}{2}\varrho g A^2 \frac{\pi^2 I_1^2}{\pi^2 I_1^2 + K_1^2} = \frac{1}{2}\varrho g A^2 R^2, \\ -M_{\text{av}} &= \frac{1}{2}\varrho g A^2 l\left[S(-\nu l) - T(-\nu l) - \frac{\pi I_1(\nu l)}{2\nu l}\right]\frac{\pi I_1}{\pi_1^2 I_1^2 + K_1^2}, \end{aligned}\right\} \tag{17.8}$$

where

$$\nu l\, S(-\nu l) = \tfrac{1}{2}\pi\left[I_1(\nu l) - L_1(\nu l)\right],$$

$$T(-\nu l) = \tfrac{1}{2}\pi\left[I_0(\nu l) - L_0(\nu l)\right].$$

Fig. 14 displays all four functions in dimensionless form.

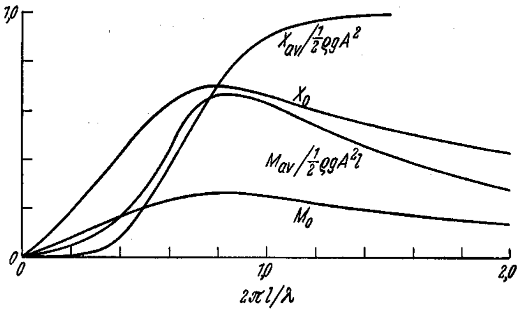

Fig. 14.

The method of integral equations. This method for finding solutions has been frequently used, especially by KOCHIN (1937, 1939, 1940) and his colleagues. One of its advantages is that approximate solutions to the integral

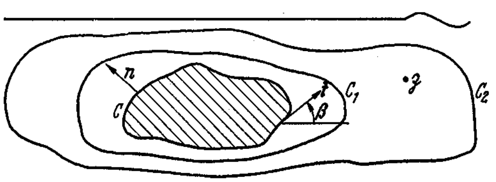

Fig. 15.

equation can frequently be found even when an explicit solution cannot be easily obtained. The following exposition follows approximately KOCHIN (1937) and KELDYSH and LAVRENT'EV (1937).

Consider a submerged obstacle whose contour C is given parametrically by $z = z(s)$ and is oriented counterclockwise. Let $\beta(s)$ be the angle between the tangent vector and the positive x-direction (see Fig. 15). We shall assume that

[1] G.N. WATSON: Bessel functions, p. 329; L_1 is tabulated in J. Math. Phys. **25**, 252—259 (1946).

as $x\to\infty$ the motion approximates to a standing wave:

$$\Phi(x, y, t) \sim A\,\frac{g}{\sigma}\,e^{\nu y}\cos(\nu x+\alpha)\cos(\sigma t+\tau), \qquad \nu=\frac{\sigma^2}{g}. \tag{17.9}$$

The other boundary conditions in terms of the complex potential $f(z)=\varphi(x, y)+i\,\psi(x, y)$ are

$$\left.\begin{aligned}\operatorname{Im}\{f'(x)+i\,f(x)\}&=0,\\ \operatorname{Im}\{f'(z(s))\,e^{i\beta(s)}\}&=0,\\ \lim_{y\to-\infty}|f'|&=0.\end{aligned}\right\} \tag{17.10}$$

Write $f(z)$ in the form

$$f(z)=f_1(z)+\frac{A\,g}{\sigma}\,e^{-i(\nu z+\alpha)}=f_1(z)+a\,e^{-i\nu z}. \tag{17.11}$$

Then $f_1(z)$ must satisfy

$$\left.\begin{aligned}\lim_{x\to\infty} f_1(z)&=0,\\ \operatorname{Im}\left[f_1'(z(s))-i\,a\,\nu\,e^{-i\nu z(s)}\right]e^{i\beta(s)}&=0,\end{aligned}\right\} \tag{17.12}$$

as well as the free surface condition and the condition as $y\to-\infty$.

We shall try to express $f_1(z)$ as a distribution of vortices over the contour C. However, the vortices are chosen so that the conditions on the free surface, at $x=\infty$ and at $y=-\infty$ are satisfied. As is apparent from the derivation of (13.28), the complex velocity potential for such vortices is given by

$$f_v(z; c)=\frac{\Gamma}{2\pi i}\left\{\log(z-c)(z-\bar c)-2\,e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\bar c}\,du\right\}. \tag{17.13}$$

We set $\Gamma=1$ and try to express $f_1(z)$ as follows:

$$f_1(z)=\int_C \gamma(s)\,f_v(z; z(s))\,ds, \tag{17.14}$$

where $\gamma(s)$ must be chosen so that the boundary condition on the body is satisfied.

In order to derive an integral equation for $\gamma(s)$, consider the following expression for $f_1'(z)$, a direct consequence of Cauchy's integral:

$$f_1'(z)=\frac{1}{2\pi i}\int_{C_1}\frac{f'(\zeta)}{z-\zeta}\,d\zeta-\frac{1}{2\pi i}\int_{C_2}\frac{f'(z)}{z-\zeta}\,d\zeta=g_1(z)+g_2(z).$$

The function $g_1(z)$ is regular everywhere outside C_1 and $g_2(z)$ is regular everywhere inside C_2. One may contract C_1 onto C and extend $g_2(z)$ analytically into the whole lower half-plane (or fluid strip if the depth is finite).

Consider now (for infinite depth; the finite-depth case is analogous) the following function:

$$g(z)=\frac{1}{2\pi i}\int_{C_1}\frac{f_1'(\zeta)}{z-\zeta}\,d\zeta+\frac{1}{2\pi i}\int_{C_2}\overline{f_1'(\zeta)}\left[\frac{1}{z-\bar\zeta}-2i\nu\,e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\bar\zeta}\,du\right]d\bar\zeta.$$

The first summand is identical with $g_1(z)$ and the second is also regular in the whole half-plane. $g(z)$ satisfies the same boundary conditions as $f_1'(z)$. Hence $f_1'(z)-g(z)$ is regular in the whole lower half-plane, satisfies the free-surface condition and vanishes as $y\to-\infty$ and $x\to+\infty$. The uniqueness argument used in the derivation of (13.28) shows that $f_1'(z) \equiv g(z)$. Thus we have

$$f_1'(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f_1'(\zeta)}{z-\zeta}\,d\zeta - \frac{1}{2\pi i}\int_{C_1} \overline{f_1'(\zeta)}\left[\frac{1}{z-\overline{\zeta}} - 2i\nu\,e^{-i\nu z}\int_{\infty}^{z} \frac{e^{i\nu u}}{u-\overline{\zeta}}\,du\right] d\overline{\zeta}. \tag{17.15}$$

Now contract C_1 to C. Then

$$[f_1'(\zeta) - i\,a\,\nu\,e^{-i\nu\zeta}]\,e^{i\beta} = v_t + i\,v_n = v_t$$

or

$$f_1'(\zeta) = v_t\,e^{-i\beta} + i\,a\,\nu\,e^{-i\nu\zeta}. \tag{17.16}$$

If one substitutes above, one finds that the contribution from the second summand in $f_1'(\zeta)$ vanishes and that, since $d\zeta/ds = e^{i\beta(s)}$,

$$\left.\begin{aligned} f_1'(z) &= \frac{1}{2\pi i}\int_C v_t(s)\left[\frac{1}{z-\zeta} - \frac{1}{z-\overline{\zeta}} + 2i\nu\,e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\overline{\zeta}}\,du\right] ds \\ &= \int_C v_t(s)\,f_v'(z;z(s))\,ds. \end{aligned}\right\} \tag{17.17}$$

This identifies $\gamma(s)$ as the tangential velocity $v_t(s)$ at a point of the contour.

Let us now consider the effect of letting $z \to z(s')$, a point of the contour C. Then, according to the Theorem of PLEMELJ-SOKHOTSKII,

$$\left.\begin{aligned} \int_C \gamma(s)\,f_v'(z;z(s))\,ds = \int_C \gamma(s)\,f_v'(z;z(s))\,e^{-i\beta(s)}\,dz(s) \to \tfrac{1}{2}\gamma(s')\,e^{-i\beta(s')} + \\ + \mathrm{PV}\int_C \gamma(s)\,f_v'(z(s');z(s))\,ds, \end{aligned}\right\} \tag{17.18}$$

whereas

$$f_1'(z) \to v_t(s')\,e^{-i\beta(s')} + i\,a\,\nu\,e^{-i\nu z(s')} = \gamma(s')\,e^{-i\beta(s')} + i\,a\,\nu\,e^{-i\nu z(s')}.$$

Hence we have the integral equation for $\gamma(s)$:

$$-\tfrac{1}{2}\gamma(s') + \mathrm{PV}\int_C \gamma(s)\,f_v'(z(s');z(s))\,e^{i\beta(s')}\,ds = i\,A\,\sigma\,e^{-i[\nu z(s') - \beta(s') + \alpha]}. \tag{17.19}$$

This is really two integral equations. The imaginary part gives a singular integral equation of the first kind:

$$\mathrm{PV}\int_C \gamma(s)\,K(s',s)\,ds = -\,2\pi A\,\sigma\,e^{\nu y(s')}\cos[\nu x(s') - \beta(s') + \alpha]. \tag{17.20}$$

The real part gives a Fredholm equation of the second kind with continuous kernel:

$$-\frac{1}{2}\gamma(s') + \frac{1}{2\pi}\int_C \gamma(s)\,L(s',s)\,ds = 2\pi A\,\sigma\,e^{\nu y(s')}\sin[\nu x(s') - \beta(s') + \alpha]. \tag{17.21}$$

Here

$$f_v'(z(s');z(s))\,e^{i\beta(s')} = \frac{1}{2\pi i}\,[K(s',s) + i\,L(s',s)]. \tag{17.22}$$

The kernel $K(s',s)$ is of the form

$$K(s',s) = \frac{1}{s'-s} + C(s',s), \tag{17.23}$$

where $C(s',s)$ is continuous; the first term comes from $e^{i\beta(s')}/[z(s') - z(s)]$. If the curve C is sufficiently smooth,

$$\lim_{s'\to s}\,\mathrm{Im}\,\frac{e^{i\beta(s')}}{z(s') - z(s)} = \frac{1}{\varrho(s)}, \tag{17.24}$$

where $\varrho(s)$ is the radius of curvature of C at $z(s)$.

If the obstacle consists of a smooth arc, an analogous argument leads to only the singular integral equation above, but with $\gamma(s)$ now identified with the jump in $v_t(s)$ as one goes from the left to the right side of the arc.

There does not seem to be a published proof that a solution to either integral equation exists for all ν. However, Kochin (1936, pp. 119—126) shows the existence of a solution for both sufficiently large and sufficiently small values of ν for the equation of the second kind when the body is completely submerged.

By adjusting the phases in (17.9) one may obtain two Φ's which may be added to give an outgoing progessive wave. The behavior as $x \to -\infty$ will then be a superposition of an incoming and an outgoing wave. However, one may also modify the preceding arguments in order to treat the pregressive-wave problem directly. One specifies, say, an incoming wave from the right, writes

$$\Phi(x, y, t) = \frac{A g}{\sigma} e^{\nu y} \cos(\nu x + \sigma t) + \Phi^*(x, y, t), \tag{17.25}$$

where Φ^* must satisfy the radiation condition, and tries to express the corresponding complex potential as a distribution of the vortices (13.28) since they already satisfy the radiation condition. We shall not dwell on the details except to remark that the problem leads to a pair of coupled integral equations since one needs a distribution not only of (13.20) as it stands, but also of the vortices obtained by replacing t by $t - \pi/2\sigma$. This method could have been applied, for example, to the problem of the vertical barrier considered above.

Dock problems. This term is generally applied to water-wave problems in which the obstruction is a horizontal plane of finite or semi-infinite extent, either submerged or lying on the surface. The solution for the semi-infinite dock in infinitely deep water was given by Friedrichs and Lewy (1948), and at about the same time the same problem in water of finite depth was treated by A. Heins (1948) who also allowed a restricted type of three-dimensional motion. The methods were quite different. Subsequently Heins (1950) and Greene and Heins (1953) extended the treatment to submerged docks in water of finite and infinite depth. As was remarked earlier, Rubin (1954) has shown the existence of a solution for the finite dock in infinitely deep water. Sparenberg (1957) has deduced an integral equation of the second kind for this problem.

As an example, consider a submerged dock at depth b and extending from $x = -a$ to $x = a$. The integral equation (17.20) then becomes

$$\mathrm{PV}\int_{-a}^{a} \gamma(\xi)\, K(x, \xi)\, d\xi = -2\pi A \sigma e^{-\nu b} \cos(\nu x + \alpha), \tag{17.26}$$

where $K(x, \xi) = K(x - \xi)$ with

$$\left.\begin{aligned} K(x) &= \mathrm{Re}\left\{\frac{1}{x} - \frac{1}{x - 2ib} + i\frac{2\nu}{\pi} e^{-i\nu x} \int_{\infty}^{x} \frac{e^{-i\nu u}}{u - 2ib}\, du\right\} \\ &= \frac{1}{x} - \frac{x}{x^2 + 4b^2} - \frac{2\nu}{\pi} \int_{\infty}^{x} \frac{u \sin\nu(u - x) + 2b \cos\nu(u - x)}{u^2 + 4b^2}\, du. \end{aligned}\right\} \tag{17.27}$$

Without actually establishing the existence of a solution to (17.26), Keldysh and Lavrent'ev (1937) in treating the flow about thin hydrofoils (see Sect. 20β) propose an approximate method of solution by expanding $\gamma(x)$ and $K(x)$ in a series in $\tau = a/2b$:

$$\gamma(x) = \gamma_0(x) + \gamma_1(x)\,\tau + \cdots,$$
$$K(x) = \frac{1}{x} + a \sum_n K_n \left(\frac{x}{a}\right)^n \tau^{n+1}$$

and determining the $\gamma_n(x)$ recursively. In the problem treated by them the total vorticity was fixed by the Kutta-Joukowski condition, in the present problem the corresponding condition is still to be determined.

If the dock extends from $-\infty$ to 0, one may modify the earlier arguments so as to apply to an unbounded body and derive the integral equation

$$\mathrm{PV}\int_{-\infty}^{0} \gamma(\xi)\, K(x-\xi)\, d\xi = -\,2\pi A\,\sigma\,\mathrm{e}^{-\nu b} \cos(\nu x + \alpha)\,. \tag{17.28}$$

An integral equation of this form is known as a Wiener-Hopf integral equation and in many cases can be solved by use of Fourier transforms. It does not seem possible to expound the method briefly, so we refer to the paper of GREENE and HEINS (1953) where this problem is treated, but with the kernel expressed differently.

When the semi-infinite dock is on the surface, the dock may be considered as a limiting case of a beach in which the angle between the bottom and the free surface is 180°. Although waves on beaches are discussed in the next section, the methods which allow extension of the angle to 180° are also difficult and will not be considered there. They may be found in STOKER'S *Water waves* (1957, § 5.4).

β) Waves on beaches. Let the fluid at rest be contained in the wedge defined by

see errata

$$\tan\gamma \leqq \frac{-y}{x} \leqq 1, \quad x > 0, \quad \gamma > 0,$$

i.e., the bottom is the plane $x\sin\alpha + y\cos\alpha = 0$. For such a body of fluid one may look for periodic waves which are either standing or progressive. The appropriate mathematical problem for standing wave is to find a velocity potential

$$\Phi(x, y, t) = \varphi(x, y)\cos(\sigma t + \tau) \tag{17.29}$$

satisfying

1. $\Delta\varphi = 0$,
2. $\varphi_y(x, 0) - \frac{\sigma^2}{g}\,\varphi(x, 0) = 0$,
3. $\varphi_x \sin\gamma + \varphi_y \cos\gamma = 0$ for $x\sin\gamma + y\cos\gamma = 0$,
4. $\lim\limits_{x^2+y^2\to\infty} \varphi_x^2 + \varphi_y^2 = 0$ for $x\sin\gamma + y\cos\gamma = 0$.

This problem, in both this form and the three-dimensional form to be considered in Sect. 18, has received intensive study in recent years (e.g., MICHE 1944, LEWY 1945, STOKER 1947, FRIEDRICHS 1948, ISAACSON 1948, 1950, WEINSTEIN 1949, PETERS 1950, 1952, ROSEAU 1952, LEHMAN 1954, BRILLOUËT 1957). In particular, the cited work of BRILLOUËT and Chap. 5 of STOKER'S *Water waves* (1957) contain a general exposition of the mathematical theory. We shall restrict the present treatment to simple cases.

KIRCHHOFF (1879) was apparently the first one to treat the two-dimensional case. The problem was taken up again by MACDONALD (1896), POCKLINGTON (1921), and by HANSON (1926), who considered both the two and three-dimensional cases. All these authors restricted the solution to be bounded everywhere. This has the effect of excluding a physically important class of solutions with singularities at the origin. One may see this easily if $\gamma = 90°$, i.e. when there is a vertical cliff. A bounded solution is obviously $\varphi(x, y) = A\ \mathrm{e}^{\nu y}\cos\nu x$, $\nu = \sigma^2/g$. This generates a standing wave behaving like $\cos \nu x$ at $x = \infty$. However, if we wish to construct a solution behaving, say, like an incoming wave at infinity we need also a standing-wave solution behaving like $\sin \nu x$ at infinity. No such solution exists which is bounded everywhere. However, as we shall see, it is possible to construct such a solution by allowing a singularity at the origin. If the two standing-wave solutions are used to construct an incoming progressive wave, the consequent loss of energy associated with the singularity is sometimes interpreted physically as representing loss of energy in breaking of the waves, at least when α is sufficiently small for this to happen. There is, of course, no a priori method of selecting the mathematical solution best representing the physical phenomena. The comparison between physical waves and mathematical solutions is discussed briefly in STOKER (1957, pp. 69—77).

KIRCHHOFF'S approach to the solution is interesting historically because of its similarity to the method used later by PETERS (1950) and ROSEAU (1951). His reasoning runs as follows, with a slight change in notation. Let $f(z) = \varphi + i\psi$ be the complex potential. Then

$$2\varphi(x, y) = f(x + iy) + \bar{f}(x - iy),$$
$$2i\psi(x, y) = f(x + iy) - \bar{f}(x - iy).$$

The free-surface condition becomes

$$i[f'(x) - \bar{f}'(x)] = \nu[f(x) + \bar{f}(x)], \qquad \nu = \sigma^2/g.$$

But then also

$$i[f'(z) - \bar{f}'(z)] = \nu[f(z) + \bar{f}(z)]. \tag{17.30}$$

The bottom must be a streamline. Hence

$$f(r\,e^{-i\gamma}) - \bar{f}(r\,e^{i\gamma}) = \text{const};$$

we may take this constant as 0. From this

$$\bar{f}(z) = f(z\,e^{-i2\gamma}). \tag{17.31}$$

Hence

$$\frac{d}{dz}[f(z) - f(z\,e^{-i2\gamma})] = -i\nu[f(z) + f(z\,e^{-i2\gamma})]. \tag{17.32}$$

This differential-difference equation must hold for all z for which $f(z)$ and $f(z\,e^{-i2\gamma})$ are both defined, namely for

$$-\gamma < \arg z < \gamma.$$

KIRCHHOFF'S formal arguments need to be supported in terms of analytic continuation by the reflection principle, but the essential idea is the same as that used more recently (cf., e.g., LEHMAN, 1954, § 3, or PETERS, 1950, § 3).

KIRCHHOFF proceeds to solve this equation in the special case $\gamma = m\pi/n$, m and n relatively prime integers, by assuming

$$f(z) = \sum_{k=0}^{n-1} A_k \exp(i\lambda\nu z\beta^k), \qquad \beta = e^{-i\frac{2m\pi}{n}}. \tag{17.33}$$

Substitution in (17.32) gives

$$A_k(\beta^k\lambda + 1) = A_{k-1}(\beta^k\lambda - 1), \qquad k = 0, \ldots, n-1, \tag{17.34}$$

with $A_{-1} \equiv A_{n-1}$. Multiplying all equations together and remembering that 1, $\beta, \ldots, \beta^{n-1}$ are all n-th roots of unity, one finds

$$\lambda^n - (-1)^n = \lambda^n - 1,$$

which can hold only if n is even, say $n=2q$ (hence m is odd). With $\lambda=-1=\beta^q$, the above equations determine successively $A_1, \ldots, A_{q-1}$ in terms of A_0, and $A_q=\cdots=A_{n-1}=0$:

$$A_k = i A_{k-1} \cot k\gamma = i^k A_0 \cot \gamma \cot 2\gamma \ldots \cot k\gamma. \tag{17.35}$$

Then

$$f(z) = \sum_{k=0}^{q-1} A_k \exp(-i\nu\beta^k z). \tag{17.36}$$

A_0 is still an arbitrary complex constant. The differential-difference equation is a necessary condition for $f(z)$, but not sufficient to ensure that all boundary conditions are satisfied. If one substitutes the above expression for $f(z)$ in (17.31), one finds after some computation that one must take

$$A_0 = B_0 e^{-i\pi(q-1)/4}, \tag{17.37}$$

where B_0 is pure imaginary (say $i B_0'$) if both $\frac{1}{2}(m+1)$ and q are even and otherwise is real. With this choice of A_0 one has

$$A_{q-k} = \bar{A}_{k-1}. \tag{17.38}$$

As Kirchhoff points out, the solution is physically acceptable for the problem at hand only if $m=1$; otherwise, φ does not remain bounded as $x \to +\infty$. If $m=1$, then for $y=0$, the dominant term as $x \to \infty$ is given by

$$\left.\begin{aligned} f(x) &\sim B_0 \exp\left(-i\nu x - i\pi\frac{q-1}{4}\right) \\ \text{or} \qquad \varphi(x,0) &\sim B_0 \cos\left(\nu x + \pi\frac{q-1}{4}\pi\right). \end{aligned}\right\} \tag{17.39}$$

Here are several easily computable special cases of (17.36):

$\gamma=90°$ $(m=1, q=1, \beta=-1)$:

$$f(z) = B_0 e^{-i\nu z} = B_0 e^{\nu y}(\cos\nu x - i\sin\nu x); \tag{17.40}$$

$\gamma=45°$ $(m=1, q=2, \beta=-i)$:

$$\left.\begin{aligned} f(z) &= B_0 e^{-i\frac{\pi}{4}}\left[e^{-i\nu z} + i e^{-\nu z}\right] \\ &= B_0\left[e^{\nu y}\cos\left(\nu x + \frac{\pi}{4}\right) + e^{-\nu x}\cos\left(\nu y - \frac{\pi}{4}\right)\right] - \\ &\quad - i B_0\left[e^{\nu y}\sin\left(\nu x + \frac{\pi}{4}\right) + e^{-\nu x}\sin\left(\nu y - \frac{\pi}{4}\right)\right]; \end{aligned}\right\} \tag{17.41}$$

$\gamma=30°$ $\left(m=1, q=3, \beta=\frac{1}{2}(\sqrt{3}-i)\right)$:

$$\begin{aligned} f(z) &= B_0 e^{-i\frac{\pi}{2}}\left[e^{-i\nu z} + i\sqrt{3}\, e^{-\frac{1}{2}(\sqrt{3}+i)\nu z} - e^{-\frac{1}{2}(\sqrt{3}-i)\nu z}\right] \\ &= B_0\{-e^{\nu y}\sin\nu x - e^{-\frac{1}{2}\nu(x\sqrt{3}+y)}\sin\tfrac{1}{2}\nu(x - y\sqrt{3}) + \\ &\quad + \sqrt{3}\, e^{-\frac{1}{2}\nu(x\sqrt{3}-y)}\cos\tfrac{1}{2}\nu(x + y\sqrt{3})\} + \\ &\quad + i B_0\{-e^{\nu y}\cos\nu x + e^{-\frac{1}{2}\nu(x\sqrt{3}+y)}\cos\tfrac{1}{2}\nu(x - y\sqrt{3}) - \\ &\quad - \sqrt{3}\, e^{-\frac{1}{2}\nu(x\sqrt{3}-y)}\sin\tfrac{1}{2}\nu(x + y\sqrt{3})\}. \end{aligned}$$

Numerical computations for $\varphi(x, y)$ for $\gamma=6°$ $(q=15)$ as well as for the above cases were carried out by STOKER (1947) and are presented graphically in his paper.

KIRCHHOFF's solution is limited to the special choice of angle noted above and furthermore presents only solutions which are bounded at the origin. The solution of the differential-difference equation (17.32) for arbitrary γ, $0<\gamma\leqq\pi$, see errata has been given by both PETERS (1950), ISAACSON (1950), and ROSEAU (1952, Chap. V). All use Laplace transforms. However, the method cannot be expounded briefly and we refer to either the original papers or STOKER's *Water waves* for the details.

The special case $\gamma=\pi/2q$ can be treated fairly simply by the reduction method used in the problem of the vertical barrier.

From (17.32) we have

$$f^{(k+1)}(z)+i\nu f^{(k)}(z)=\beta^{k+1}f^{(k+1)}(\beta z)-i\nu\beta^k f^{(k)}(\beta z),\quad k=0,1,\dots \tag{17.42}$$

The free surface condition [cf. (11.7)] implies

$$\operatorname{Im}\{f^{(k+1)}(x)+i\nu f^{(k)}(x)\}=0,\quad x>0. \tag{17.43}$$

Hence also

$$\operatorname{Im}\{\beta^{k+1}f^{(k+1)}(x\beta)-i\nu\beta^k f^{(k)}(x\beta)\}=0,\quad x>0.$$

This last equation can also be written

$$\operatorname{Im}\{\beta^{k+1}f^{(k+1)}(z)-i\nu\beta^k f^{(k)}(z)\}=0\quad\text{for}\quad z=r\,e^{-2i\gamma}. \tag{17.44}$$

If the numbers a_k and a_k' are real, (17.43) and (17.44) imply

$$\left.\begin{aligned}&\operatorname{Im}\left\{\sum_{k=0}^{s}a_k\left[f^{(k+1)}(x)+i\nu f^{(k)}(x)\right]\right\}=0,\\&\operatorname{Im}\left\{\sum_{k=0}^{s}a_k'\beta^k\left[\beta f^{(k+1)}(r\,e^{-2i\gamma})-i\nu f^{(k)}(r\,e^{-2i\gamma})\right]\right\}=0.\end{aligned}\right\} \tag{17.45}$$

We wish to find numbers $\{a_k\}$ and $\{a_k'\}$ such that

$$\sum_{k=0}^{s}a_k\left[f^{(k+1)}(z)+i\nu f^{(k)}(z)\right]\equiv\sum_{k=0}^{s}a_k'\beta^k\left[\beta f^{(k+1)}(z)-i\nu f^{(k)}(z)\right]. \tag{17.46}$$

Comparing coefficients of derivatives of the same order, one finds

$$\left.\begin{aligned}a_0&=-a_0',\\a_{k-1}+i\nu a_k&=\beta^k(a_{k-1}'-i\nu a_k'),\quad k=1,\dots,s,\\a_s&=\beta^{s+1}a_s'.\end{aligned}\right\} \tag{17.47}$$

These relations will be satisfied if one takes $s=q-1$ (for $\beta^q=-1$) and

$$\left.\begin{aligned}a_k&=-a_k'=a_{k-1}\frac{1}{i\nu}\,\frac{\beta^k+1}{\beta^k-1}=a_{k-1}\frac{1}{\nu}\cot k\gamma,\\&=\frac{a_0}{\nu^k}\cot\gamma\cot 2\gamma\dots\cot k\gamma,\quad k=1,\dots,s.\end{aligned}\right\} \tag{17.48}$$

We note that $\nu^{q-k}a_{q-k}=\nu^{k-1}a_{k-1}$. With this choice of the coefficients $\{a_k\}$, define

$$\left.\begin{aligned}g(z)&=\sum_{k=0}^{q-1}a_k\{f^{(k+1)}(z)+i\nu f^{(k)}(z)\}=P\left(\frac{d}{dz}\right)\left(\frac{d}{dz}+i\nu\right)f(z),\\&=-\sum_{k=0}^{q-1}a_k\beta^k\{\beta f^{(k+1)}(z)-i\nu f^{(k)}(z)\}=-P\left(\beta\frac{d}{dz}\right)\left(\beta\frac{d}{dz}-i\nu\right)f(z)\\&=\sum_{k=0}^{q-1}a_k\{f^{(k+1)}(\beta z)-i\nu f^{(k)}(\beta z)\}=P\left(\frac{d}{dz}\right)\left(\frac{d}{dz}-i\nu\right)f(\beta z)\end{aligned}\right\} \tag{17.49}$$

where the last equation follows from (17.42) and where

$$P(\lambda) = \sum_{k=0}^{q-1} a_k \lambda^k. \qquad (17.50)$$

From the assumptions originally made concerning $f(z)$ and from the method of selecting the $\{a_k\}$ it follows that $g(z)$ is regular everywhere in the wedge

$$-2\gamma \leqq \vartheta \leqq 0$$

except possibly at the origin, that

$$\operatorname{Im}\{g(z)\} = 0 \quad \text{for} \quad z = x > 0 \quad \text{and} \quad z = r\,e^{-2i\gamma},$$

and finally, from the last of Eqs. (17.24), that

$$g(\beta z) = -g(z).$$

Since $f(z)$ is assumed bounded as $x \to \infty$, this is true also of $g(z)$. These various conditions imply that $g(z)$ must have the form

$$g(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{(2n+1)q}}, \qquad b_n \text{ real}. \qquad (17.51)$$

We have thus shown that $f(z)$ satisfies the differential equation

$$P\left(\frac{d}{dz}\right)\left(\frac{d}{dz} + i\nu\right) f(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{(2n+1)q}}, \qquad b_n \text{ real}. \qquad (17.52)$$

From the definition of $P(\lambda)$ it follows that

$$P(\lambda)(\lambda + i\nu) \equiv P(\beta\lambda)(-\beta\lambda + i\nu).$$

see errata

Since the coefficients in $P(\lambda)$ are real, $\bar{\lambda}$ is a root or $P(\lambda) = 0$ if λ is a root. Furthermore, from the identity above also $\beta\lambda$ is a root providing $\beta\lambda \neq i\nu$. Since $\lambda = -i\nu$ is an obvious root of the left hand member, $-i\beta\nu$ is also a root and hence $-i\beta^2\nu$, $-i\beta^3\nu, \ldots$. Since $\beta^q = -1$, no new roots are added by going further than $-i\beta^{q-1}\nu$, and since $i\beta^{-k}\nu = -i\beta^{q-k}\nu$, a complete set of roots of $P(\lambda)(\lambda + i\nu)$ is

$$-i\nu, \quad -i\beta\nu, \quad -i\beta^2\nu, \ldots, \quad -i\beta^{q-1}\nu.$$

Thus the solution of the homogeneous equation can be expressed in the form

$$\sum_{k=0}^{q-1} A_k \exp(-i\nu\beta^k z). \qquad (17.53)$$

This is, of course, exactly the form of KIRCHHOFF's solution of (17.36). Since we have already determined the necessary form of the A_k in order to satisfy the boundary condition on the bottom, we need not pursue further the solution of the homogeneous equation.

The solution of the nonhomogeneous equation is straightforward. However, just as for the homogeneous equation, one must take care to satisfy the boundary condition on the bottom, i.e. $\operatorname{Im}\{e^{-i\gamma} f'(r\,e^{-i\gamma})\} = 0$. The detailed considerations may be found in the several cited papers; BRILLOUËT (1957) treats the matter thoroughly. If one considers (17.52) with the right-hand side replaced by only one of its summands, say $b_n z^{-(2n+1)q}$, then the complete solution can be put in the following form, as shown by BRILLOUËT:

$$f(z) = \sum_{k=0}^{q-1} A_k \exp(-i\nu\beta^k z)\left[c_n + \frac{1}{2}(-1)^{nq+q-1} \frac{b_n}{\sqrt{q}} \int_{\Gamma_k} \frac{e^t\,dt}{t^{(2n+1)q}}\right], \qquad (17.54)$$

where c_n is an arbitrary real constant, B_0 of (17.37) has been set equal to 1, and where Γ_k indicates that the integral is to be carried out over each of the paths Γ_k^+ and Γ_k^- shown in Fig. 16. However, one may obtain a variety of other forms for the solution.

An asymptotic expression as $x\to\infty$ and for $y=0$ is given by

$$\left.\begin{aligned} f(x) &\sim \left[c_n + i b_n \frac{(-1)^{nq+q-1}\pi}{(2nq+q-1)!\sqrt{q}}\right]\exp\left(-i\nu x - i\pi\frac{q-1}{4}\right) \\ &\text{or} \\ \varphi(x,0) &\sim c_n\cos\left(\nu x + \pi\frac{q-1}{4}\right) + b_n\frac{(-1)^{nq+q-1}\pi}{(2nq+q-1)!\sqrt{q}}\sin\left(\nu x+\pi\frac{q-1}{4}\right). \end{aligned}\right\}\tag{17.55}$$

In the neighborhood of $z=0$, $f(x)$ behaves like $\log z$ for $n=0$ and like z^{-2nq} for $n>0$.

It is not clear physically what type of singularity at $z=0$ most nearly describes the behavior of real waves. However, most writers have restricted their treatment to the weakest possible singularity, i.e., the logarithmic one.

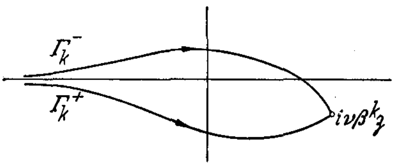

Fig. 16.

From the asymptotic expansion as $x\to\infty$ one sees that it is now possible to construct an incoming progressive wave by proper choice of the constants c_n and b_n. Thus, if we select

$$c_n = a\cos(\sigma t+\tau), \qquad b_n = -(-1)^{nq+q-1}\pi^{-1}(2nq+q-1)!\sqrt{q}\,a\sin(\sigma t+\tau),$$

then the resulting solution will behave like

$$a\cos(\nu x+\sigma t+\tau)$$

as $x\to\infty$ for $y=0$. In connection with (17.55) and the selection of b_n just made it is apparent that the formulas (17.54) and (17.55) will be more directly connected with parameters with a simple physical interpretation if we replace b_n by

$$d_n = b_n\frac{(-1)^{nq+q-1}\pi}{(2nq+q-1)\sqrt{q}}.$$

For $n=0$ companion singular solutions to the regular solutions (17.40) and (17.41) are not difficult to write out:

$\gamma=90°\ (q=1,\ n=0)$:

$$\varphi(x,y) = d_0 e^{\nu y}\sin\nu x - \frac{d_0}{\pi}\int_0^\infty e^{-\sigma x}\frac{\sigma\cos\sigma y+\nu\sin\sigma y}{\nu^2+\sigma^2}\,d\sigma;\tag{17.56}$$

$\gamma=45°\ (q=2,\ n=0)$:

$$\left.\begin{aligned}\varphi(x,y) = \frac{d_0}{\pi}e^{\nu y}\Big[\Big(\frac{\pi}{2}+\mathrm{Si}(\nu x)\Big)\sin\Big(\nu x+\frac{\pi}{4}\Big)+ \\ +\mathrm{Ci}(\nu x)\cos\Big(\nu x+\frac{\pi}{4}\Big)+\frac{1}{2}\sqrt{\nu}\,e^{-\nu x}\,\mathrm{Ei}(\nu x)\Big].\end{aligned}\right\}\tag{17.57}$$

Further formulas for $\gamma=30°$ and $\gamma=6°$ may be found in Brillouët (1957, p. 93 ff.).

18. Three-dimensional progressive and standing waves in unbounded regions with fixed boundaries. The general remarks at the beginning of Sect. 17 apply here also. Although most of the solvable problems in the present category are such that they can be reduced to two-dimensional ones (however, see the end of Sect. 19β), the methods of complex-function theory are no longer applicable to the same extent. The division of topics is the same as in the last section, namely, diffraction of waves by obstacles and waves on beaches.

α) *Diffraction of water waves.* In a horizontally unbounded ocean of uniform depth h assume that an incoming wave is specified by

$$\Phi_I(x, y, z, t) = \frac{A g}{\sigma} \cosh m (y + h) \cos (m x + \sigma t + \alpha) \tag{18.1}$$

and that it is scattered by one or more obstacles in the water. We wish to find the velocity potential for the motion of the water in the form

$$\Phi(x, y, z, t) = \Phi_I + \Phi_S, \tag{18.2}$$

where Φ_S is the scattered wave and satisfies the radiation condition if the body is of bounded extent.

As usual, we may write Φ in the form

$$\Phi(x, y, z, t) = \operatorname{Re} \varphi(x, y, z)\, e^{-i\sigma t}, \qquad \varphi = \varphi_1 + i\varphi_2, \tag{18.3}$$

where φ must be a potential function satisfying

see errata

$$\left.\begin{aligned} &\varphi_y(x, 0, z) - \nu\,\varphi(x, 0, z) = 0, \qquad \nu = \sigma^2/g,\\ &\varphi_n = \varphi_{I n} + \varphi_{S n} = 0 \quad \text{on the obstacles},\\ &\lim_{R\to\infty} \sqrt{R}\left(\frac{\partial \varphi_S}{\partial R} - i\,\nu\,\varphi_S\right) = 0, \quad \sqrt{R}\,\varphi_S = O(1) \quad \text{as } R \to \infty. \end{aligned}\right\} \tag{18.4}$$

General obstructions. Consider a single submerged obstacle bounded by the surface S. We shall try to express the scattered wave $\Phi_S = \operatorname{Re} \varphi_S\, e^{-i\sigma t}$ by a distribution of sources over S. However, in order to satisfy the various boundary conditions, we take sources in the complex form (13.18) or, in the case of infinite depth, in the form (13.17″):

$$\varphi_S(x, y, z) = \frac{1}{4\pi} \iint_S \gamma(\xi, \eta, \zeta)\, G(x, y, z; \xi, \eta, \zeta)\, dS, \tag{18.5}$$

where we have written $G = G_1 + i G_2$ for the complex form of (13.18). The boundary condition on the body now becomes

$$\left.\begin{aligned} 0 = \frac{\partial \varphi_I}{\partial n} + \frac{\partial \varphi_S}{\partial n} = \frac{\partial \varphi_I}{\partial n} - \frac{1}{2}\gamma(x, y, z) + &\\ + \frac{1}{4\pi} \iint_S \gamma(\xi, \eta, \zeta) \frac{\partial}{\partial n} G(x, y, z; \xi, \eta, \zeta)\, dS& \\ \text{or}\qquad \gamma(x, y, z) = 2\frac{\partial \varphi_I}{\partial n} + \frac{1}{2\pi} \iint \gamma(\xi, \eta, \zeta) \frac{\partial G}{\partial n}\, dS.& \end{aligned}\right\} \tag{18.6}$$

Since $\partial\varphi_I/\partial n$ is a known function, this is a Fredholm integral equation of the second kind for $\gamma(x, y, z)$. (We note in passing that if the motion of the surface S had been prescribed to be $\partial\varphi_I/\partial n$, then the same integral equation for γ would have been obtained.)

This equation has been considered by Kochin (1940) in the case of infinite depth, and he proves that a solution exists if $\nu = \sigma^2/g$ is large enough and the body is submerged. Iterative procedures for computing γ follow from the theory. Haskind (1946) has extended the argument to finite depth.

John (1950) has treated both the uniqueness and existence problem in great detail and has shown that a unique solution exists for a body whose surface intersects the free surface perpendicularly and which can be represented as a single-valued function over the area enclosed in the intersection. His result holds for all values of m (or ν if the depth is infinite). He also reduces the existence problem to solution of an integral equation.

Vertical cylinders. When the obstacle or obstacles are vertical cylinders extending from above the free surface to the bottom, it is possible to reduce the problem to one in diffraction of sound waves for which many special solutions are known [see, e.g., Havelock (1940)]. In this case we may separate the y variable in the manner shown in Sect. 13α:

$$\left.\begin{aligned} \varphi(x,y,z) &= \varphi(x,z)\,Y(y) \\ Y(y) &= \cosh m(y+h)\,\varphi(x,z) \end{aligned}\right\} \tag{18.7}$$

where

and

$$\varphi_{xx} + \varphi_{zz} + m^2\varphi = 0. \tag{18.8}$$

Here m must be the same as in (18.1) since the frequency is fixed by the incoming wave. $\varphi(x,z)$ must now satisfy (18.8) and the second two conditions of (18.4). This is exactly the same mathematical problem encountered in the diffraction of sound waves by a cylindrical body (in that case the air pressure replaces φ). Thus, any solutions known for sound diffraction by cylinders may be taken over immediately for water-wave diffraction. For example, if the obstacle is a vertical circular post of radius a, the velocity potential of the scattered wave is given by[1]

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$$\varphi_S(R,\vartheta,y) = \frac{-Ag}{\sigma}\cosh m(y+h)\,\Sigma(-i)^n\,\varepsilon_n\,e^{-i\gamma_n}\sin\gamma_n\cos\vartheta\,H_n^{(1)}(mR), \tag{18.9}$$

where

$$\tan\gamma_n = J_n'(ma)/Y_n'(ma)$$

and

$$\varepsilon_0 = 1, \qquad \varepsilon_n = 2 \quad \text{for} \quad n \geqq 1.$$

Various approximations for large and small values of ma are known. The maximum wave amplitude at any point is given by $\frac{\sigma}{g}|\varphi|$.

The diffraction of water waves by a vertical half-plane may also be treated by transferring known solutions due to Sommerfeld for sound and electromagnetic waves to the present context. This has been done by Haskind (1948) for normal incidence and by Penney and Price (1952a) for both normal and oblique incidence. Peters and Stoker (1954) [see also Stoker (1956) and (1957, pp. 109 to 133)] have also solved this problem by a new and rather easy method, following an investigation of boundary conditions which will ensure uniqueness. The solution has an obvious application in predicting the effect of breakwaters. Let the breakwater be the half-plane $z=0$, $x>0$ and the incoming wave be given by

$$\begin{aligned} \eta &= A\cos(m\,x\cos\alpha + m\,z\sin\alpha + \sigma t), \\ &= A\cos(m\,R\cos(\vartheta-\alpha) + \sigma t), \end{aligned}$$

where α is the angle between $-0x$ and the direction of propagation, measured clockwise. Then the solution given by Peters and Stoker is

$$\varphi(R,\vartheta,y) = \frac{Ag}{\sigma}\cosh m(y+h)\left[J_0(R) + 2\sum_1^\infty e^{in\pi/4}J_{n/2}(R)\cos\frac{n\alpha}{2}\cos\frac{n\vartheta}{2}\right]. \tag{18.10}$$

[1] See P.M. Morse: Vibration and sound, 2nd ed., pp. 347ff., 449. New York 1948.

The result can also be expressed by means of integrals. In the case of normal incidence these reduce to Fresnel integrals, for which tables exist. Graphical representations of the behavior of the wave amplitudes may be found in PENNEY and PRICE (1952a).

PENNEY and PRICE also apply this analysis to an approximate treatment of diffraction by a breakwater of finite length and through a gap. The results are presumably applicable if the wavelength is small compared to the length of the breakwater or the gap.

Periodic solutions for horizontal cylindrical obstacles. In two physical situations the dependence upon z may be precipitated out, leaving a two-dimensional problem which in many cases can be solved by methods analogous to those used for the two-dimensional problems of Sect. 17.

Let the obstruction be an infinitely long horizontal cylinder parallel to Oz. This might be, for example, a semi-infinite dock or submerged plane barrier, say $y=-b$, $x<0$, a finite horizontal barrier, say $y=-b$, $|x|<a$, a vertical
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barrier, $x=0$, $-b<Y\leqq 0$, a beach, $y=-x\tan\gamma$, etc. Let an incoming plane wave at infinity propagate at an angle α to the x axis:

$$\eta_I(x, y, z, t) = A\cos[m(x\cos\alpha + z\sin\alpha) + \sigma t]. \tag{18.11}$$

Although one will not expect the velocity potential Φ to be periodic in x, it seems reasonable to assume that it will be periodic in z. In fact, we shall assume that

$$\Phi(x, y, z, t) = \varphi(x, y)\, e^{-i(mz\sin\alpha+\sigma t)}, \tag{18.12}$$

where $\varphi(x, y)$ must now satisfy, with $k=m\sin\alpha$,

$$\varphi_{xx} + \varphi_{yy} - k^2\varphi = 0 \tag{18.13}$$

and the usual conditions on the free surface and rigid boundaries.

We should have come to the same conclusion if we had assumed an incoming wave at infinity of the form

$$\eta_I(x, y, z, t) = A\cos kz\cos(k_1 x + \sigma t), \qquad k^2 + k_1^2 = m^2, \tag{18.14}$$

a so-called short-crested wave (note that we assume $k^2<m^2$). That is, we shall now look for a solution in the form

$$\Phi(x, y, z, t) = \varphi(x, y)\cos kz\, e^{-i\sigma t} \tag{18.15}$$

satisfying Eq. (18.13) and the conditions on the free surface and rigid boundaries. Thus, a solution for one of these cases carries over easily to the other.

The problem is thus reduced to one almost identical with that of Sect. 17, with the exception that the two-dimensional Laplacian is replaced by (18.13). Many of the same methods may be carried over, e.g., the reduction method and the integral-equation method. HASKIND (1953) has considered some general aspects of the problem which will be outlined below, has derived the source solution of (18.13), and has treated the diffraction about a vertical barrier (an analogue of the problem treated in Sect. 17α) and a finite dock, all in infinitely deep water. MACCAMY (1957) has derived a source solution of (18.13) and treated the finite dock problem in water of finite depth. HEINS (1948, 1950, 1953) has given source solutions of (18.13) for finite depth and formulated and solved Wiener-Hopf integral equations for semi-infinite docks and submerged horizontal barriers. GREENE and HEINS (1953) treat the submerged barrier in water of infinite depth. The literature for beaches will be given in Sect. 18β.

Suppose the fluid infinitely deep and let a cross-section of the obstacle have contour C. We wish then to find a solution $\varphi(x, y) = \varphi_1 + i\varphi_2$ of (18.13) such that

$$\left.\begin{aligned} &\varphi_n = 0 \quad \text{on} \quad C, \\ &\varphi_y(x, 0) - \nu\,\varphi(x, 0) = 0 \qquad \text{on the free surface}, \\ &\varphi \sim \frac{A g}{\sigma}\, e^{\nu y} e^{-i k_1 x} + \frac{B^+ g}{\sigma}\, e^{\nu y} e^{i k_1 x} \quad \text{as} \quad x \to +\infty, \\ &\varphi \sim \frac{A g}{\sigma}\, e^{\nu y} e^{-i k_1 x} + \frac{B^- g}{\sigma}\, e^{\nu y} e^{-i k_1 x} \quad \text{as} \quad x \to -\infty, \end{aligned}\right\} \tag{18.16}$$

where $k_1^2 < \nu^2$. HASKIND (1953) applies the reduction method in the following manner (we follow his presentation closely). Introduce the function $f(x, y)$ by

$$\frac{\partial f}{\partial y} = \frac{\partial \varphi}{\partial y} - \nu\,\varphi. \tag{18.17}$$

Then f also satisfies (18.13) and

$$f_y(x, 0) = 0 \qquad \text{on the free surface}. \tag{18.18}$$

Consequently, f may be extended into the upper half-plane by defining $f(x, -y) = f(x, y)$ and f now satisfies (18.13) in the whole plane outside the contour C and its mirror image $\bar{C}$. Moreover, $|f| \to 0$ as $x^2 + y^2 \to \infty$. Assuming that f is known, one must now try to reconstruct φ from f in such a way that conditions (18.16) are satisfied. In order to do this, HASKIND differentiates (18.17) with respect to y, subtracts from

$$f_{xx} + f_{yy} - k^2 f = 0,$$

and after some easy manipulation obtains

$$\frac{\partial^2}{\partial x^2}(\varphi - f) + k_1^2(\varphi - f) = -\nu\left(\frac{\partial f}{\partial y} + \nu f\right). \tag{18.19}$$

Treating this as a differential equation for $\varphi - f$, he finds the following solution for φ:

$$\left.\begin{aligned} \varphi = f - \frac{\nu}{2 i k_1}\Big\{ & e^{i k_1 x}\int_{\infty}^{x} e^{-i k_1 \xi}(f_y + \nu f)\, d\xi - \\ & - e^{-i k_1 x}\int_{-\infty}^{x} e^{i k_1 \xi}(f_y + \nu f)\, d\xi\Big\} + \frac{A g}{\sigma}\, e^{\nu y} e^{-i k_1 x}, \end{aligned}\right\} \tag{18.20}$$

the integrals being taken along half-lines parallel to the x-axis and below C. One may verify without great difficulty that φ satisfies (18.13). The asymptotic form of φ as $x \to \pm\infty$ may be written down immediately, and gives

$$\frac{\nu}{2 i k_1}\, e^{\mp i k_1 x}\int_{-\infty}^{\infty} e^{\pm i k_1 \xi}(f_y + \nu f)\, d\xi + \frac{A g}{\sigma}\, e^{\nu y} e^{-i k_1 x}, \tag{18.21}$$

the path of integration being a line below the body. Consider now the region D bounded externally by this line and a large semicircle containing $C + \bar{C}$ and internally by $C + \bar{C}$. Application of GREEN's Theorem to f and $\chi = \exp(-\nu y + i k_1 x)$ shows that

$$e^{-\nu y}\int_{-\infty}^{\infty} e^{i k_1 \xi}(f_y + \nu f)\, d\xi = \int_{C+\bar{C}} (f \chi_n - \chi f_n)\, ds,$$

$$e^{-\nu y}\int_{-\infty}^{\infty} e^{-i k_1 \xi}(f_y + \nu f)\, d\xi = \int_{C+\bar{C}} (f \bar{\chi}_n - \bar{\chi} f_n)\, ds.$$

Hence, the asymptotic conditions are satisfied and, moreover,

$$\frac{B^+ g}{\sigma} = \frac{\nu}{2i k_1} \int\limits_{C+\bar{C}} (f \chi_n - \chi f_n)\, ds, \qquad \frac{B^- g}{\sigma} = \frac{\nu}{2i k_1} \int\limits_{C+\bar{C}} (f \bar{\chi}_n - \bar{\chi}_n f)\, ds. \tag{18.22}$$

By a similar application of GREEN's Theorem HASKIND shows that one may also write

$$\varphi = f + \nu\, e^{\nu y} \int\limits_{\infty}^{y} f\, e^{-\nu \eta}\, d\eta + \frac{B^{\pm} g}{\sigma}\, e^{\nu y \mp i k_1 x} + \frac{A g}{\sigma}\, e^{\nu y - i k_1 x}, \tag{18.23}$$

where the plus sign is used for points to the right of C and the minus sign for points to the left. It is easy to verify directly that φ satisfies (18.13) and (18.17); however, (18.20) allows one to investigate the asymptotic behavior more simply. If φ has no singularities, then (17.3) must also hold here, i.e., $(B^+)^2 + (B^-)^2 + 2A B^- = 0$.

This result may be used to find the source solutions giving outgoing waves at $\pm\infty$. For Eq. (18.13) the singular solutions for the whole plane are known to be the Bessel functions $K_n(kr)$, where $r^2 = (x-a)^2 + (y-b)^2$. To find the solution corresponding to (13.22), one assumes it may be expressed as

$$G(x, y; a, b) = \varphi_0 + K_0(k r) - K_0(k r_1),$$

with $r_1^2 = (x-a)^2 + (y+b^2)$, where φ_0 has no singularities for $y<0$. Then $f_{0y} = \varphi_{0y} - \nu \varphi_0$ may be extended as a regular solution of (18.13) to the whole plane. Also,

$$f_{0y}(x, 0) = 2 \frac{\partial}{\partial y} K_0(k r_1)\big|_{y=0}.$$

One may then show that this relation holds for all $y \leqq 0$:

$$f_{0y}(x, y) = 2 \frac{\partial}{\partial y} K_0(k r_1), \qquad y \leqq 0,$$

or

$$f_0(x, y) = 2 K_0(k r_1).$$

Substitution in (18.23) with $A=0$ and direct computation of $B^{\pm}$ from (18.22) by taking C as a small circle about the singularity gives

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$$G = K_0(k r) + K_0(k r_1) + 2\nu\, e^{\nu y} \int\limits_{\infty}^{-y} e^{-\nu y} K_0(k r_1)\, dy - 2\pi i \frac{\nu}{k_1} e^{\nu(y+b) \mp i k_1 (x-a)}. \tag{18.24}$$

For HASKIND's application of this method to the diffraction about a vertical and a horizontal barrier we refer to the original paper. Force and moment are obtained in terms of Mathieu functions. For the horizontal barrier in water of finite depth we refer to MACCAMY's paper (1957) where a formula analogous to (18.24) is derived.

β) Waves on beaches. Much of the immediately preceding discussion of diffraction of plane waves approaching at an angle or of short-crested waves approaching normally applies also to this case. One is led to the following boundary-value problem for $\varphi(x, y) = \varphi_1 + i\varphi_2$:

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$$\left.\begin{aligned}
&1.\quad \varphi_{xx} + \varphi_{yy} - k^2 \varphi = 0, \qquad k^2 < \nu^2,\\
&2.\quad \varphi_y(x, 0) - \nu\, \varphi(x, 0) = 0,\\
&3.\quad \varphi_x \sin\gamma + \varphi_y \cos\gamma = 0 \qquad \text{for } y + x\tan\gamma = 0,\\
&4.\quad \varphi \sim \frac{A g}{\sigma} e^{\nu y}\, e^{-i k_1 x} \quad \text{as} \quad x \to \infty, \qquad k = \nu^2 - k^2,\\
&5.\quad \varphi_x^2 + \varphi_y^2 \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \quad \text{along} \quad y + x\tan\gamma = 0.
\end{aligned}\right\} \tag{18.25}$$

Many of the authors cited in Sect. 17β considered this problem along with the two-dimensional one. In particular, we refer to HANSON (1926), MICHE (1944), STOKER (1947), WEINSTEIN (1949), ROSEAU (1952), and PETERS (1952). Both PETERS and ROSEAU solve the problem for arbitrary angle γ, $0<\gamma\leqq\pi$ [thus including the semi-infinite dock problem treated differently by HEINS (1948)]. The use of the reduction method limits one here, as in the two-dimensional case, to angles $\gamma=p\pi/2q$. We shall illustrate the procedure briefly for $\gamma=\pi/4$ and $\gamma=\pi/2$, following essentially WEINSTEIN'S (1949) treatment [see also BRILLOUËT (1957, Chaps. I, II)].

Since the boundary condition on the free surface and bottom is the same in the two- and three-dimensional cases, we may make use of the auxiliary function g constructed in (17.49) by using only the real part of the complex potential. Thus, for $\gamma=\pi/4$ one finds from (17.48) that $a_1=a_0/\nu$. Hence, from (17.50)

$$p(\lambda)=a_0(1+\lambda/\nu),$$

and

$$g(z)=\frac{a_0}{\nu}\left(\frac{d}{dz}+\nu\right)\left(\frac{d}{dz}+i\nu\right)(\varphi+i\psi),$$

$$\operatorname{Im} g(z)=\frac{a_0}{\nu}\left(\frac{\partial}{\partial x}+\nu\right)\left(-\frac{\partial}{\partial y}+\nu\right)\varphi.$$

Thus, the boundary conditions 2 and 3 of (18.25) imply that

$$h(x,y)\equiv\left(\frac{\partial}{\partial x}+\nu\right)\left(\frac{\partial}{\partial y}-\nu\right)\varphi(x,y)=0 \quad \left.\begin{array}{ll}\text{on} & y=0,\ x>0\\ \text{and} & x=0,\ y<0.\end{array}\right\} \tag{18.26}$$

We recall that the definition of $\varphi(x,y)$ has been extended from the original wedge by reflection in the bottom. One must now find a function $h(x,y)$ satisfying equation 1 of (18.25) and the boundary conditions (18.26) and which is regular everywhere in the extended wedge, $0\geqq\vartheta\geqq\frac{1}{2}\pi$, except possibly at the origin, bounded as $x^2+y^2\to\infty$, and symmetric about the line $y=-x$. It is known that the general solution of this problem is given by

$$h(x,y)=\left(\frac{\partial}{\partial x}+\nu\right)\left(\frac{\partial}{\partial y}-\nu\right)\varphi(x,y)=\sum_{n=0}^{\infty}A_n K_{2(2n+1)}(kr)\sin 2(2n+1)\vartheta. \tag{18.27}$$

A similar analysis for waves approaching a vertical cliff ($\gamma=\frac{1}{2}\pi$) leads to

$$h(x,y)\equiv\left(\frac{\partial}{\partial y}-\nu\right)\varphi(x,y)=\sum_{n=0}^{\infty}A_n K_{2n+1}(kr)\sin(2n+1)\vartheta. \tag{18.28}$$

Let us take the weakest possible singularity in each case, i.e., K_1 for the 90° cliff and K_2 for the 45° beach. Consider first the vertical cliff. Taking account of the relation $K_0'(u)=-K_1(u)$, we have

$$\left(\frac{\partial}{\partial y}-\nu\right)\varphi(x,y)=-\frac{A_0}{k}\frac{\partial}{\partial y}K_0(kr).$$

We may then identify $-A_0K_0/k$ with f and from (18.23), with $B^{\pm}=0$, we have

$$\varphi=-\frac{A_0}{k}K_0(kr)-A_0\frac{\nu}{k}e^{\nu y}\int_{\infty}^{y}e^{-\nu\eta}K_0\left(k\sqrt{x^2+y^2}\right)d\eta+\frac{Ag}{\gamma}e^{\nu y-ik_1x},$$

where A_0 must still be determined so that $\varphi_x(0, y) = 0$, $y < 0$. In computing φ_x as $x \to 0$, one must remember that $K_0(u) \sim \ln(2/u)$ as $u \to 0$. Hence, one finds

$$\varphi_x(0, y) = -\frac{A_0}{k} \nu\, e^{\nu y} \lim_{\varepsilon \to 0} \lim_{x \to 0} \int_{-\varepsilon}^{\varepsilon} \frac{-x}{x^2 + y^2}\, dy - i\,\frac{A g k_1}{\sigma}\, e^{\nu y}$$
$$= \frac{A_0}{k} \nu \pi\, e^{\nu y} - i\,\frac{A g k_1}{\sigma}\, e^{\nu y}.$$

Setting this equal to zero, one finds

$$\frac{A_0}{k} = \frac{A g}{\sigma} \cdot \frac{i k_1}{\pi \nu}.$$

Substituting above and separating the real and imaginary parts of $\varphi = \varphi_1 + i\varphi_2$, we obtain an everywhere regular solution φ_1 and a solution φ_2 with a singularity at the origin and $90°$ out of phase at $x = \infty$:

$$\left.\begin{aligned} \varphi_1(x, y) &= \frac{A g}{\sigma}\, e^{\nu y} \cos k_1 x, \\ \varphi_2(x, y) &= -\frac{A g}{\sigma} \frac{k_1}{\pi \nu} \left[K_0(k r) + \nu\, e^{\nu y} \int_{\infty}^{y} e^{-\nu \eta} K_0\!\left(k \sqrt{x^2 + \eta^2}\right) d\eta\right] + \\ &\quad + \frac{A g}{\sigma}\, e^{\nu y} \sin k_1 x. \end{aligned}\right\} \qquad (18.29)$$

The corresponding equation for (18.27) can be written in the form

$$\left(\frac{\partial}{\partial x} + \nu\right)\left(\frac{\partial}{\partial y} - \nu\right) \varphi(x, y) = A_0 K_2(k r) \sin 2\vartheta = \frac{2 A_0}{k^2} \frac{\partial^2}{\partial x\, \partial y} K_0(k r). \qquad (18.30)$$

One can find its integration discussed in ROSEAU (1952, Chap. IV). A solution for the next simplest case, $\gamma = 30°$, does not seem to have been published. For $\gamma = 45°$ the regular solution φ_1, and singular solution φ_2 as given by ROSEAU, but corrected according to personal communications from ROSEAU and LEHMAN, are

$$\left.\begin{aligned} \varphi_1 &= A_1 \{e^{\nu y} [k_1 \cos k_1 x - \nu \sin k_1 x] + e^{-\nu x} [k_1 \cos k_1 y + \nu \sin k_1 y]\}, \\ \varphi_2 &= A_2 \{e^{\nu y} [\nu \cos k_1 x + k_1 \sin k_1 x] + e^{-\nu x} [\nu \cos k_1 y - k_1 \sin k_1 y]\} + \\ &+ A_2 \frac{\nu^2 + k_1^2}{\pi \nu} \Big\{-K_0\!\left(k \sqrt{x^2 + y^2}\right) + \nu\, e^{-\nu x} \int_{-\infty}^{x} e^{\nu \xi} K_0\!\left(k \sqrt{\xi^2 + y^2}\right) d\xi + \\ &+ \nu\, e^{\nu y} \int_{y}^{\infty} e^{-\nu \eta} K_0\!\left(k \sqrt{x^2 + \eta^2}\right) d\eta - \nu^2 e^{\nu y} \int_{y}^{\infty} d\eta\, e^{-\nu \eta} \left(e^{-\nu x} \int_{-\infty}^{x} d\xi\, e^{\nu \xi} K_0\!\left(k \sqrt{\xi^2 + \eta^2}\right)\right)\Big\}. \end{aligned}\right\} \qquad (18.$$

In order to satisfy condition 4 of (18.25) one must take

$$A_1 = \frac{A g}{\sigma} \cdot \frac{k_1 + i \nu}{k_1^2 + \nu^2}, \qquad A_2 = \frac{A g}{\sigma} \cdot \frac{\nu - i k_1}{k_1^2 + \nu^2}, \qquad \varphi = \varphi_1 + \varphi_2.$$

Edge waves. In the investigation of diffraction of waves on horizontal cylindrical obstacles and of waves on beaches, it was specifically assumed that $k^2 < m^2$. This was automatically fulfilled for plane waves approaching at an angle, but needed to be assumed for short-crested waves. For the short-crested waves there also exist standing-wave solutions which can be exhibited in certain cases for $k^2 > m^2$. Such solutions were apparently first noticed by STOKES (1846, p. 7 = 1880, p. 167) in connection with the propagation of waves in a canal of non-rectangular cross-section. Certain peculiarities of these solutions have been pointed out by URSELL (1951, 1952).

Consider the first three conditions of (18.25) for waves on a sloping beach, but with $k^2 > \nu^2$. Then one may verify directly that

$$\varphi(x, y) = e^{k[y \sin\gamma - x\cos\gamma]}$$

is a solution. This gives a velocity potential for standing waves:

$$\Phi(x, y, z, t) = e^{k[y \sin\gamma - x\cos\gamma]} \cos(kz + \varepsilon)\cos(\sigma t + \tau), \tag{18.32}$$

where

$$k \sin\gamma = \sigma^2/g.$$

The wave amplitude is bounded at the origin and drops off very quickly as x increases. Clearly, one must have $\gamma < \frac{1}{2}\pi$. URSELL has pointed out other interesting aspects. For a given γ and σ there is only one allowable k, i.e., it is a discrete point of the spectrum. In the case discussed earlier with $k^2 < \nu^2$ all values of k between 0 and ν were allowable. In addition, the total energy per unit length in the z direction is finite for the Stokes edge wave.

From (18.29) one may construct a progressive wave moving in the direction Oz with velocity.

$$c = \frac{g \sin\gamma}{\sigma}.$$

There is evidence that such waves have been observed in nature (cf. MUNK, SNODGRASS and CARRIER 1956; DONN and EWING 1956).

URSELL (1952) has shown that (18.32) is only the first in a sequence of solutions of this nature for a sloping beach. He shows, in fact, that the following velocity potential also satisfies the condition:

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$$\left.\begin{aligned}\Phi(x, y, z, t) = \Big\{ & e^{-k[x\cos\gamma - y\sin\gamma]} + \sum_{m=1}^{n} A_{mn}\big[e^{-k[x\cos(2m-1)\gamma + y\sin(2m-1)\gamma} + \\ & + e^{-k[x\cos(2m+1)\gamma - y\sin(2m+1)\gamma]}\big]\Big\} \cos(kz + \varepsilon)\cos(\sigma t + \tau),\end{aligned}\right\} \tag{18.33}$$

where

$$A_{mn} = (-1)^m \prod_{r=1}^{n} \frac{\tan(n - r + 1)\gamma}{\tan(n + r)\gamma}, \qquad \sigma^2 = g k \sin(2n + 1)\gamma.$$

It follows from the last condition that one must have

$$(2n + 1)\gamma \leqq \frac{\pi}{2} \quad \text{or} \quad n < \frac{\pi}{4\gamma} + \frac{1}{2},$$

where $n = 0$ will be taken to indicate the Stokes edge wave. Thus, for fixed wave number k, the above formula gives one frequency σ if $\frac{1}{2}\pi > \gamma > \frac{1}{6}\pi$, two if $\frac{1}{6}\pi > \gamma > \frac{1}{10}\pi$, etc. An experiment carried out by URSELL confirms the existence of these other modes of motion. The solutions (18.33) for $\gamma = \pi/2(2n+1)$ have also been given by MACDONALD (1896). At these critical angles the solution (18.33) does not vanish as $x \to \infty$. MACDONALD apparently discarded the other solutions as being of little interest, not "being sensible at a distance from the edge". ROSEAU (1958) has recently carried through a systematic investigation of edge waves, including ones with singular behavior at the edge.

KELDYSH (1936) has stated without proof that for $\gamma = 45°$ the Stokes edge wave and the function φ_1 from (18.31) constitute a complete set of bounded solutions in the sense that for any absolutely integrable function $f(x, y)$, $x = 0$, the following Fourier-integral-like theorem holds [cf. formula (16.5)]:

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$$f(x, z) = \frac{2}{\pi^2} \int_0^\infty \int_0^\infty \frac{dk\, dk_1}{k^2 + 2k_1^2} \int_{-\infty}^{\infty} d\zeta \cos(z-\zeta) \int_0^\infty d\xi\, f(\xi, \zeta) \times$$
$$\times \{[k_1 e^{-\nu x} + k_1 \cos k_1 x - \nu \sin k_1 x]\, [k_1 e^{-\nu\xi} + k_1 \cos k_1 \xi - \nu \sin k_1 \xi] +$$
$$+ 2k^2 \exp(-k(x+\xi)/\sqrt{2})\}.$$

It is possible to construct other types of edge waves. First we rederive the Stokes wave from the third formula in (13.5) with $a = 0$. A surface satisfying $\Phi_n = 0$ is defined by

$$\frac{dy}{dx} = \frac{\Phi_y}{\Phi_x} = -\frac{\nu}{\sqrt{k^2 - \nu^2}},$$

or

$$y = -x \tan\gamma + C, \qquad \tan\gamma = \nu/\sqrt{k^2 - \nu^2},$$

where we may set $C = 0$ since it does not provide essentially different solutions for the bottom. This is just Stokes' solution.

One may expect to find a different type of solution by using the third equation of (13.6) with $a = 0$. Here the corresponding solution is

$$-\log \frac{\sinh m_0 (y+h)}{\sinh m_0 h} = \frac{m_0^2}{\sqrt{k^2 - m_0^2}}\, x, \tag{18.34}$$

where again we have dropped an added constant. This describes a bottom which starts as a sloping beach and approaches, as $x \to \infty$, a flat bottom at depth h. The initial slope of the beach is $\sigma^2/g \sqrt{k^2 - m_0^2}$. The velocity potential describes edge waves for such a configuration.

One may proceed in the same fashion with the last formulas of (13.5) and (13.6). They turn out to give identical bottoms:

$$\log \frac{\sin m_i (y+h)}{\sin m_i h} = \frac{m_i^2}{\sqrt{k^2 + m_i^2}}\, x. \tag{18.35}$$

This corresponds to edge waves along an overhanging cliff in water of finite depth. The initial backward slope of the cliff is $\sigma^2/g \sqrt{k^2 + m_i^2}$.

A particularly interesting sort of edge wave, although the name is now a misnomer since there is no edge, has been discovered by Ursell (1951). He has shown the existence of standing waves of the form

$$\varphi(x, y) \cos kz \cos \sigma t$$

in the neighborhood of a fixed submerged cylinder of radius a if ka is small enough. The waves are symmetric about the vertical plane through the axis of the cylinder and decay exponentially as $|x|$ increases. One can, of course, also construct waves progressing along the cylinder.

Ursell calls such modes of motion "trapping modes" since, if they occur in a canal with sides given by $z = 0$ and $z = n\pi/k$, no energy is radiated away, even though there is a path of escape. In fact, the motion is similar in this respect to standing waves in a basin of finite extent. The edge waves considered above also can be used to construct trapping modes.

γ) Waves in canals. The propagation of periodic waves along a canal leads to problems similar to those occurring in the propagation of waves parallel to a beach. Let the canal be parallel to Oz with cross-sectional contour C. We wish to find

$$\Phi(x, y, z, t) = \varphi(x, y) \cos(kz - \sigma t)$$

where $\varphi(x, y)$ satisfies

$$\varphi_{xx} + \varphi_{yy} - k^2 \varphi = 0, \tag{18.36}$$

$$\varphi_y(x, 0) - \nu \varphi(x, 0) = 0, \quad \nu = \sigma^2/g,$$

on the free surface,

$$\varphi_n = 0 \quad \text{on } C.$$

It will also be assumed that $\varphi_x^2 + \varphi_y^2$ is bounded.

Clearly the same equations arise in searching for standing-wave solutions in a horizontal cylindrical basin with cross-sectional contour C bounded at either end by vertical walls at a distance l apart. In this case k is restricted to the values $n\pi/l$. For progressive waves solutions with $k=0$ are, of course, of no interest.

The special case when C is a rectangle has already been discussed in Sect. 14γ. The configuration for C which seems to have attracted the next most attention is a triangular one in which the two sides are inclined at the same angle. KELLAND (1844) was apparently the first to consider this problem for infinitesimal waves, limiting his treatment to angles of 45°. The matter was treated systematically by MACDONALD (1894) who states that a solution with the properties of (18.36) exists only for angles $\gamma = 45°$ and $\gamma = 30°$. This does not exclude the possibility of the existence for other angles of a periodic progressive wave with a curved wave front, for these would not be described by the assumed form of Φ.

The solutions for $\gamma = 45°$ can be obtained from the fundamental solutions of (13.6), but it is nearly as easy to find them directly. In the third formula of (13.6) let $a = b = \frac{1}{2}A$, $k^2 = 2m_0^2$. This gives the velocity potential, after forming a progressive wave,

$$\Phi(x, y, z, t) = A \cosh \frac{k}{\sqrt{2}} (y + h) \cosh \frac{k}{\sqrt{2}} x \cos(kz - \sigma t). \tag{18.37}$$

Let the sides of the canal be given by $y = \pm x - h$. Then it is easy to verify that

$$\Phi_n|_{y=x-h} = -\Phi_x + \Phi_y|_{y=x-h} = 0, \qquad \Phi_n|_{y=-x-h} = \Phi_x + \Phi_y|_{y=-x-h} = 0,$$

so that the boundary conditions are all satisfied. Since

$$\sigma^2 = g m_0 \tanh m_0 h = \frac{1}{\sqrt{2}} g k \tanh \frac{1}{\sqrt{2}} k h,$$

the wave velocity is given by

$$c^2 = \frac{g}{k\sqrt{2}} \tanh \frac{kh}{\sqrt{2}}. \tag{18.38}$$

If $m_0^2 > m_i^2$ [in the notation of Eq. (13.6)], there will be i further symmetric modes. In (13.6), formula 4, set $a = b = \frac{1}{2}A$ and add this to formula 1 with $a = A$, $b = 0$. This gives

$$\Phi(x, y, z, t) = A\left[\cos m_i(y+h) \cosh \sqrt{k^2 + m_i^2}\, x + \right.$$
$$\left. + \cosh m_0(y+h) \cos \sqrt{m_0^2 - k^2}\, x\right] \cos(kz - \sigma t).$$

One may again verify easily that $\Phi_n = 0$ on the two sides of the canal if $k^2 = m_0^2 - m_i^2$. Hence this mode of motion will exist only if $m_0^2 > m_i^2$. For given σ there will be no modes of this sort if h is small enough, for then $m_0^2 < m_1^2$. The number gradually increases as h increases. If h and k are fixed and σ allowed to increase, there will be an infinite sequence $\sigma_1, \sigma_2, \ldots$ for which $k^2 = m_0^2 - m_i^2$ will be satisfied; $\sigma_n^2 h/g \to (n + \frac{3}{4})\,\pi$ as $n \to \infty$. The situation is easily visualized by plotting on one graph $\tanh mh$, $-\tan mh$ and $(\sigma^2 h/g)/mh$. One may write the potential function in the form

$$\Phi(x, y, z, t) = A\,[\cos m_i (y + h) \cosh m_0 x + \cosh m_0 (y + h) \cos m_i x] \cos(kz - \sigma t),$$

where

$$m_0 \tanh m_0 h = \nu, \qquad m_i \tan m_i h = -\nu; \qquad k^2 = m_0^2 - m_i^2. \tag{18.39}$$

The velocity is given by

$$c^2 = \frac{g\, m_0 \tanh m_0 h}{m_0^2 - m_i^2}. \tag{18.40}$$

Asymmetric modes of motion also exist, having first been noticed by GREENHILL (1886). These cannot be deduced from (13.6) but must be found directly. The velocity potential corresponding to (18.37) is

$$\Phi(x, y, z, t) = A \sinh \frac{k}{\sqrt{2}} (y + h) \sin \frac{k}{\sqrt{2}}\, x \cos(kz - \sigma t) \tag{18.41}$$

The wave velocity is

$$c^2 = \frac{g}{k\sqrt{2}} \coth \frac{kh}{\sqrt{2}}, \tag{18.42}$$

which approaches infinity as $kh \to 0$. In addition to this mode, other asymmetric modes may exist under conditions similar to those required for (18.39). The velocity potential for these modes is

$$\left.\begin{aligned}\Phi(x, y, z, t) = A\,[\sin n_i (y + h) \sinh n_0 x + \sinh n_0 (y + h) \sin n_i x] \times \\ \times \cos(kz - \sigma t),\end{aligned}\right\} \tag{18.43}$$

where

$$n_0 \coth n_0 h = \nu, \qquad n_i \cot n_i h = \nu, \qquad k^2 = n_0^2 - n_i^2.$$

The velocity of propagation is given by

$$c^2 = \frac{n_0 \coth n_0 h}{n_0^2 - n_i^2}. \tag{18.44}$$

The solution for the angle $\gamma = 30°$ will not be discussed here. It can be found in LAMB's Hydrodynamics (1932, p. 449) as well as in MACDONALD's paper cited above.

One may construct other possible contours for the canal cross-section by starting from one of the solutions (13.5) or (13.6) and finding surfaces for which $\Phi_n = 0$. Thus, from the third equation of (13.5) form

$$\Phi = A\, e^{\nu y} \sinh x \sqrt{k^2 - \nu^2} \cos(kz - \sigma t).$$

Solution of the differential equation $dy/dx = \Phi_y/\Phi_x$ leads easily to

$$y + h = \frac{\nu}{k^2 - \nu^2} \log \cosh x \sqrt{k^2 - \nu^2}$$

as an equation for the contour of a possible canal. The contour is reasonably shaped but varies with the choice of k. Also, the method is unsatisfactory in that it gives no information about other possible modes of motion.

19. Problems with steadily oscillating boundaries. Such problems include waves resulting from forced oscillation of a submerged body and the waves associated with steady oscillations of a freely floating body in oncoming waves.

In this section we shall assume the fluid of infinite extent. Waves in an oscillating bounded basin will be discussed later. The mathematical treatment has much in common with that of the last two sections, the scattered wave of those sections becoming the forced wave of this one.

α) Forced oscillations. Suppose that the surface of the oscillator in its equilibrium position is represented by $F(x, y, z) = 0$. Let us take it, for example, to be oscillating vertically with amplitude ε. Then the equation of the oscillating surface S may be written $F(x, y, z, t) = F(x, y + \varepsilon a \sin\sigma t, z) = 0$ where a is some length dimension of the oscillator. This ε will be taken as the expansion parameter in the perturbation procedure. In the perturbation theory of Sect. 10, we were concerned only with the functions $\Phi(x, y, z, t)$ and $\eta(x, y, t)$. However, we must similarly expand F before substituting it into the boundary condition satisfied on the surface of the oscillator, namely,

$$F_x \Phi_x + F_y \Phi_y + F_z \Phi_z + F_t = 0 \quad \text{on} \quad F(x, y, z, t) = 0. \tag{19.1}$$

The expansion for this case is

$$\left.\begin{aligned} &F(x, y + \varepsilon a \sin\sigma t, z) \\ &\quad = F(x, y, z) + \varepsilon a \sin\sigma t\, F_y(x, y, z) + \tfrac{1}{2}\varepsilon^2 a^2 \sin^2\sigma t\, F_{yy}(x, y, z) + \cdots. \end{aligned}\right\} \tag{19.2}$$

We don't wish to restrict ourselves to this one mode of motion for the oscillator, but an examination of the form of this and similar expansions indicates that we may assume in general that the surface of the oscillator can be represented by the series

$$\left.\begin{aligned} F(x, y, z, t) &= F^{(0)}(x, y, z) + \varepsilon\,[F_1^{(1)}(x, y, z) \cos\sigma t + F_2^{(1)}(x, y, z) \sin\sigma t] + \\ &\quad + \text{time-periodic terms in higher powers of } \varepsilon = 0, \end{aligned}\right\} \tag{19.3}$$

where $F^{(0)}(x, y, z) = 0$ is the equilibrium position of the oscillator. We may now assume either that Φ is periodic, i.e.,

$$\Phi(x, y, z, t) = \sum \varphi_{1n}(x, y, z) \cos n\sigma t + \varphi_{2n}(x, y, z) \sin n\sigma t \tag{19.4}$$

or, more simply, that it is simple harmonic,

$$\Phi(x, y, z, t) = \varphi_1(x, y, z) \cos\sigma t + \varphi_2(x, y, z) \sin\sigma t, \tag{19.5}$$

where each function φ_{in} or φ_i is still to be expanded in a perturbation series. The two assumptions are not quite equivalent, even for the first-order theory, but since under certain conditions (19.4) leads to the same first-order equations as (19.5), we shall assume the latter form, together with

$$\eta(x, z, t) = \eta_1(x, z) \cos\sigma t + \eta_2(x, z) \sin\sigma t. \tag{19.6}$$

Substitution of the perturbation series into the exact equations and boundary conditions, as in Sect. 10, then leads to the following first-order equation and boundary conditions:

$$\left.\begin{aligned} &1.\quad \Delta\varphi_k^{(1)} = 0, \qquad k = 1, 2, \\ &2.\quad \varphi_{ky}^{(1)}(x, 0, z) - \frac{\sigma^2}{g}\,\varphi_k^{(1)}(x, 0, z) = 0, \quad k = 1, 2, \\ &3.\quad \operatorname{grad} F^{(0)} \cdot \operatorname{grad} \varphi_1^{(1)} + \sigma F_2^{(1)} = 0 \quad \text{on} \quad F(x, y, z) = 0, \\ &4.\quad \operatorname{grad} F^{(0)} \cdot \operatorname{grad} \varphi_2^{(1)} - \sigma F_1^{(1)} = 0 \quad \text{on} \quad F(x, y, z) = 0. \end{aligned}\right\} \tag{19.7}$$

One should note that it is a natural consequence of the method that the boundary condition on the oscillator is to be satisfied at its equilibrium position.

If we let

$$A_1(x, y, z) = \frac{-\sigma F_2^{(1)}}{|\operatorname{grad} F^{(0)}|}, \quad A_2(x, y, z) = \frac{\sigma F_1^{(1)}}{|\operatorname{grad} F^{(0)}|} \quad \text{for } F^{(0)}(x, y, z) = 0, \tag{19.8}$$

then conditions 3. and 4. of (19.7) may be written

$$\varphi_n^{(1)} = A(x, y, z) \quad \text{on} \quad F^{(0)} = 0, \tag{19.9}$$

where

$$\varphi^{(1)} = \varphi_1^{(1)} + i\,\varphi_2^{(1)} \quad \text{and} \quad A = A_1 + i A_2.$$

We shall henceforth drop the superscripts and consider only the first-order equations. In addition to Eqs. (19.7) the functions φ_i must also satisfy the usual conditions on fixed solid boundaries, $\varphi_{in} = 0$, and, if the fluid is infinitely deep, $|\operatorname{grad} \varphi| \to 0$ as $y \to -\infty$. Finally, one needs a boundary condition to ensure only outgoing waves at infinity. As has been pointed out by URSELL (1951), the foregoing conditions are not always sufficient to guarantee uniqueness of solution.

KOCHIN (1939, 1940) has considered the general mathematical problem in water of infinite depth for both two and three dimensions. HASKIND (1942b, 1944, 1946) has extended KOCHIN'S methods to water of finite depth. The frequently-cited paper by JOHN (1950) treats the theoretical aspects of the problem in a thorough manner and includes many of the results of KOCHIN and HASKIND. Special problems have been considered by numerous authors. HAVELOCK (1929b) considers the waves generated by oscillation of a vertical plate extending to the bottom in water of infinite depth for both two and three dimensions, and in water of finite depth for two dimensions; he also considers waves generated by oscillations of a vertical cylinder. MACCAMY (1957) has treated the three-dimensional problem in water of finite depth. KENNARD (1949) has treated the two-dimensional problem as an initial-value problem. URSELL (1948) has considered waves generated by oscillation of a vertical strip with finite depth of immersion in water of infinite depth; the treatment is two dimensional. ALBAS (1958) treats a similar three-dimensional problem in which the motion is periodic along the length of the strip. In a later paper URSELL (1949b) considered the waves generated by the rolling of long cylinders of ship-like cross-section. In addition, URSELL has treated the waves generated by a heaving half-submerged circular cylinder (1949a, 1953c, 1954) and by a pulsing submerged cylinder (1950). HAVELOCK (1955) has treated the wave motion generated by a half-submerged sphere. Certain mathematical aspects of this last problem have been examined in more detail by MACCAMY (1954). Because of its interest in connection with the heaving motion of a ship there exist many papers attempting to compute approximately the force and moment on a heaving shiplike body resulting from wave formation. We mention particularly one by GRIM (1953). In the cited papers by KOCHIN and HASKIND certain special problems are solved approximately; by improving the approximation, LEVINE (1958) has clarified certain anomalous results of KOCHIN for an oscillating horizontal plate. In addition, HASKIND (1942, 1943b) has considered the motion resulting from forced oscillation of a plate, or a system of plates, on the surface. In a more recent paper HASKIND (1953a) has developed a method for finding solutions, and in particular the force and moment on the body, for a wide class of two-dimensional contours of ship-like cross-section. One should also consult a recent expository paper by MARUO (1957). A general survey of methods of generating waves in the laboratory, including some account of theoretical results, may be found in a recent paper by BIESEL and SUQUET (1951, 1952).

This brief summary of papers on forced water waves is by no means complete but lists many of the important papers and indicates the richness of the literature.

As was stated in the introductory remarks, the theory of forced water waves is mathematically almost identical with the diffraction theory. If one interprets the value of $-\partial\Phi_I/\partial n$ on the body as the function describing the motion of the oscillator, then it is clear that the problems are the same. Hence, the general remarks about existence of solutions, uniqueness, and special methods carry over directly and will not be repeated. However, we wish to consider here one further topic in the general theory.

KOCHIN'S H-function. The H-function was apparently first introduced by KOCHIN (1937) in connection with the theory of wave resistance. He later extended it (1939, 1940) to waves generated by oscillating bodies, and it has become a standard device among other Russian workers in this field, especially HASKIND, who has extended its definition to other situations.

Each potential function φ satisfying the boundary conditions has associated with it an H-function which is related to it much in the same way that the Fourier transform of a function is related to the function. One of its chief virtues is that it allows one to give compact formulas for force and moment on an oscillating body (in the present context) as well as certain other quantities. It is also sometimes helpful in suggesting approximate solutions.

Let us suppose that the surface S of a body of bounded extent is oscillating in some manner in fluid of infinite depth and let $\varphi = \varphi_1 + i\varphi_2$ be the solution to the potential-theory problem formulated earlier. Let S_1 and S_2 be two closed surfaces lying below $y=0$ with S_2 enclosing S_1 and S_1 enclosing S. Let us denote the source potential introduced in (13.17″) by $G(x, y, z; \xi, \eta, \zeta)$, where (ξ, η, ζ) are the coordinates of the singularity, and let us write it as a contour integral:

$$G(x, y, z; \xi, \eta, \zeta) = \frac{1}{r} + \frac{1}{2\pi}\int_{-\pi}^{\pi} d\vartheta \int_{0(L)}^{\infty} dk \frac{k+\nu}{k-\nu} e^{k(y+\eta-i(x-\xi)\cos\vartheta - i(z-\zeta)\sin\vartheta)}, \tag{19.10}$$

where the path L passes below the singularity at $k=\nu=\sigma^2/g$. [The residue at this point gives exactly the imaginary part of (13.17″).]

Now apply GREEN'S Theorem to the region between S_1 and S_2 (the following argument is very similar to a two-dimensional one used in Sect. 17α in discussing the integral-equation method):

see errata

$$\left.\begin{aligned} \varphi(x, y, z) &= -\frac{1}{4\pi}\iint_{S_1}\left[\frac{1}{r}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma + \frac{1}{4\pi}\iint_{S_2}\left[\frac{1}{r}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma \\ &= \varphi^{(1)} + \varphi^{(2)}. \end{aligned}\right\} \tag{19.11}$$

Then $\varphi^{(1)}$ may be extended to a function harmonic in the whole space exterior to S_1. $\varphi^{(2)}$ is harmonic in the whole interior of S_2, but since S_2 may be indefinitely enlarged as long as it remains below $y=0$, $\varphi^{(2)}$ may be extended to be harmonic in the whole half-space, $y \leqq 0$. Consider now the function

$$\psi(x, y, z) = -\frac{1}{4\pi}\iint_{S_1}\left[G\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial G}{\partial n}\right]d\sigma. \tag{19.12}$$

ψ satisfies the free-surface condition and the condition at infinity. Moreover, since $G = r^{-1} +$ a function harmonic in the lower half-space, $\varphi - \psi$ is harmonic in the lower half-space and satisfies the other boundary conditions. But then $\varphi - \psi \equiv 0$, as follows from a uniqueness theorem proved by KOCHIN (1940, Sect. 1).

Hence, we may write

$$\varphi(x, y, z) = \frac{1}{4\pi} \iint_{S_1} [\varphi(\xi, \eta, \zeta)\, G_n(x, y, z; \xi, \eta, \zeta) - G\varphi_n]\, d\sigma. \tag{19.13}$$

Now define

$$\left.\begin{aligned} H(k, \vartheta) &= \iint_{S_1} e^{k[\eta + i\xi\cos\vartheta + i\zeta\sin\vartheta)]} \{\varphi_n(\xi, \eta, \zeta) - \\ &\quad - k\varphi[\cos(n, \eta) + i\cos\vartheta\cos(n, \xi) + i\sin\vartheta\cos(n, \zeta)]\}\, d\sigma, \\ &= \iint_{S_1} e^{k[\eta + i\xi\cos\vartheta + i\zeta\sin\vartheta]} \{\varphi_n(\xi, \eta, \zeta) + \\ &\quad + i\cos\vartheta[\varphi_\eta\cos(n, \xi) - \varphi_\xi\cos(n, \eta)] + \\ &\quad + i\sin\vartheta[\varphi_\eta\cos(n, \zeta) - \varphi_\zeta\cos(n, \eta)]\}\, d\sigma. \end{aligned}\right\} \tag{19.14}$$

Then, after some manipulation with (19.13), one can show that

$$\left.\begin{aligned} \varphi(x, y, z) &= \frac{1}{4\pi} \iint_{S_1} \left[\varphi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \varphi}{\partial n}\right] d\sigma - \\ &\quad - \frac{1}{8\pi^2} \int_{-\pi}^{\pi} d\vartheta \int_{0\,(L)}^{\infty} dk\, \frac{k+\nu}{k-\nu}\, e^{k(y - ix\cos\vartheta + iz\sin\vartheta)}\, H(k, \vartheta). \end{aligned}\right\} \tag{19.15}$$

We give a few of KOCHIN's derived formulas. The asymptotic form of the free surface in a direction α is given by

$$\eta(R, \alpha, t) \cong \mathrm{Re}\left[\frac{\sigma}{g}\sqrt{\frac{\nu}{2\pi R}}\, \overline{H}(\nu, \alpha)\, e^{i\left(\nu R - \sigma t - \frac{\pi}{4}\right)}\right] \quad \text{as} \quad R \to \infty. \tag{19.16}$$

The rate at which energy is being carried off by the waves (and hence also the power input) is given by

$$N = \frac{1}{8\pi}\, \frac{\varrho\sigma^3}{g} \int_0^{2\pi} |H(\nu, \vartheta)|^2\, d\vartheta. \tag{19.17}$$

The force components on the oscillating body, averaged over a period, are given by

$$\left.\begin{aligned} X_{\text{av}} &= \frac{\varrho\nu^2}{8\pi} \int_{-\pi}^{\pi} |H(\nu, \vartheta)|^2 \cos\vartheta\, d\vartheta, \\ Y_{\text{av}} &= \varrho g V + \frac{\varrho}{16\pi^2} \int_{-\pi}^{\pi} \mathrm{PV} \int_0^{\infty} k\, \frac{k+\nu}{k-\nu}\, |H(k, \vartheta)|^2\, dk\, d\vartheta, \\ Z_{\text{av}} &= \frac{\varrho\nu^2}{8\pi} \int_{-\pi}^{\pi} |H(\nu, \vartheta)|^2 \sin\vartheta\, d\vartheta. \end{aligned}\right\} \tag{19.18}$$

The formulas can be derived from (8.4), (9.4), and asymptotic expressions for φ.

In formulas (19.14) and (19.15) the surface S_1 over which the integrals are taken may be contracted to S. This sometimes makes it possible to express H directly in terms of known boundary values. If φ can be expressed by means of a source distribution, say

$$\varphi(x, y, z) = \frac{1}{4\pi} \iint_S \gamma(\xi, \eta, \zeta)\, G(x, y, z; \xi, \eta, \zeta)\, d\sigma, \tag{19.19}$$

then one has
$$H(k,\vartheta) = -\iint_S \gamma(\xi,\eta,\zeta)\, e^{k[\eta+i\xi\cos\vartheta+i\zeta\sin\vartheta]}\, d\sigma. \tag{19.20}$$

In order to find approximate answers, Kochin frequently uses the distribution γ which would be proper in an infinite fluid without free surface, substitutes this in (19.20) and then uses the resulting approximation to H in (19.17) and (19.18) above. The procedure may be looked upon as the first two steps in an alternating type of approximation in which one first satisfies the boundary condition on the body, neglecting the free surface, next corrects this so as to satisfy the free-surface condition, but now disturbing the condition on the body, then corrects again to satisfy the condition on the body, etc. This method of approximation has frequently been used by HAVELOCK (e.g., 1929a).

KOCHIN (1939) has also defined the H-function for two-dimensional wave motion excited by an oscillating body. We simply reproduce the formulas. Let, as usual, $f(z,t) = f_1(z)\cos\sigma t + f_2(z)\sin\sigma t$ be the complex potential and let C_1 and C_2 be two contours in the lower half-plane containing C, C_1 inside C_2. Define
$$H_s(k) = \int_{C_1} e^{-ik\zeta} f_s'(\zeta)\, d\zeta, \qquad s = 1, 2. \tag{19.21}$$

Then
$$\left.\begin{aligned} f_s'(z) &= \frac{1}{2\pi i}\int_{C_1} \frac{f_s'(\zeta)}{z-\zeta}\, d\zeta - \frac{1}{2\pi}\int_0^\infty \overline{H}_s(k)\, e^{-ikz}\, dk - \\ &\quad - \frac{\nu}{\pi}\,\mathrm{PV}\int_0^\infty \frac{\overline{H}_s(k)}{k-\nu}\, e^{-ikz}\, dk + (-1)^{s+1} H_{s+1}(\nu)\, e^{-i\nu z}, \end{aligned}\right\} \tag{19.22}$$

where $H_3 \equiv H_1$. This follows immediately from a formula similar to (17.15). For the asymptotic form of the waves one gets
$$\left.\begin{aligned} \eta(x,t) &\cong \mathrm{Re}\,\frac{i\nu}{\sigma}\,[\overline{H}_1(\nu) - i\overline{H}_2(\nu)]\, e^{-i(\nu x-\sigma t)} \quad \text{as} \quad x\to+\infty, \\ \eta(x,t) &\cong \mathrm{Re}\,\frac{-i\nu}{\sigma}\,[\overline{H}_1(\nu) + i\overline{H}_2(\nu)]\, e^{-i(\nu x+\sigma t)} \quad \text{as} \quad x\to-\infty. \end{aligned}\right\} \tag{19.23}$$

The rate of dissipation of energy is

see errata
$$N = \tfrac{1}{2}\varrho\sigma[|H_1(\nu)|^2 + +|H_2(\nu)|^2]. \tag{19.24}$$

The mean values of the force and moment, averaged over a period, are
$$\left.\begin{aligned} X_{\mathrm{av}} &= \varrho\nu\,\mathrm{Im}\{\overline{H}_1(\nu)\, H_2(\nu)\}, \\ Y_{\mathrm{av}} &= \frac{\varrho}{4\pi}\,\mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\{|H_1(k)|^2 + |H_2(k)|^2\}\, dk, \\ M_{\mathrm{av}} &= -\frac{\varrho}{4\pi}\,\mathrm{Im}\left\{\mathrm{PV}\int_0^\infty \frac{k+\nu}{k-\nu}\,[H_1'\overline{H}_1 + H_2'\overline{H}_2]\, dk\right\} + \\ &\quad + \tfrac{1}{2}\nu\varrho\,\mathrm{Im}\{H_1'(\nu)\,\overline{H}_2(\nu) - H_2'(\nu)\,\overline{H}_1(\nu)\}. \end{aligned}\right\} \tag{19.25}$$

Roughly the same remarks apply to the use of the two-dimensional formulas as of the three-dimensional ones.

Waves from an oscillator in a wall. In order to illustrate the use of the H-function, we consider the following problem. Let the (y,z)-plane be a rigid wall expect for a certain bounded area S in which there is a membrane oscillating according to a given law

$$x = F(y, z) \sin \sigma t, \qquad (y, z) \text{ in } S. \tag{19.26}$$

The boundary condition which has to be satisfied on the plane $x=0$ is then

$$\varphi_x(0, y, z) = \begin{cases} \sigma F(y, z), & (y, z) \text{ in } S \\ 0, & (y, z) \text{ not in } S, \end{cases} \tag{19.27}$$

where we still have $\varphi = \varphi_1 + i\varphi_2$.

This boundary condition, as well as the ones at infinity, can be satisfied by distributing "modified" sources (13.17″) or (19.10) over S with density $-\sigma F(y, z)/2\pi$:

$$\varphi(x, y, z) = \frac{-\sigma}{2\pi} \iint_S F(\eta, \zeta)\, G(x, y, z; 0, \eta, \zeta)\, d\eta\, d\zeta. \tag{19.28}$$

In order to compute the H-function, we shall interpret the source distribution as representing a thin body making symmetric pulsations in an infinite fluid. Hence, we may assume that the wall is removed and the membrane replaced by a doubled one. (That the requisite motion is physically impossible doesn't invalidate the considerations; a more realistic model can easily be devised.) In (19.14) we take S_1 to be both sides of the thin body. Then, remembering that

$$\varphi_n(+0, \eta, \zeta) = \varphi_x(0, \eta, \zeta) = \sigma F, \qquad \varphi_n(-0, \eta, \xi) = -\varphi_x(0, \eta, \zeta) = \sigma F,$$

$$\cos(n, \xi) = 1 \quad \text{for } x > 0 \quad \text{and} \quad \cos(n, \xi) = -1 \quad \text{for } x < 0,$$

one finds easily that

$$H(k, \vartheta) = 2\sigma \iint_S F(\eta, \zeta)\, e^{k(\eta + i\zeta \sin\vartheta)}\, d\eta\, d\zeta. \tag{19.29}$$

From (19.17) one then finds immediately, after carrying out the ϑ integration, that the rate of dissipation of energy to one side is given by

$$N = \frac{\varrho\sigma^5}{4\pi g} \iint_S dy\, dz \iint_S d\eta\, d\zeta\, F(y, z)\, F(\eta, \zeta)\, e^{\nu(y+\eta)}\, J_0(\nu(z - \zeta)). \tag{19.30}$$

Expressions for Y_{av} and Z_{av} may also be written down. The result $X_{av}=0$ is not really significant because the integral is over both sides of the thin pulsing body.

The theory for generation of two-dimensional waves in a semi-infinite channel by a vertical wave maker in the end-wall is easily derived in the same way. If the motion of the wave-maker is described by

$$x = F(y) \sin \sigma t, \qquad a \leqq y \leqq b \leqq 0 \tag{19.31}$$

then

$$H_1(k) = \int_a^b e^{k\eta} F(\eta)\, d\eta, \qquad H_2(k) = 0, \tag{19.32}$$

and, for example, the rate of dissipation of energy is given by

$$N = \varrho\sigma^3 \left[\int_a^b e^{\nu y} F(y)\, dy\right]^2. \tag{19.33}$$

The generation of short-crested waves is subject to the limitations described in Sect. 14γ. Suppose, for example, that the water is of depth h, the channel of breadth b, and that the motion of the wave-maker is described by

$$x = F(y) \cos kz \sin \sigma t, \qquad k = n\pi/b, \qquad -h \leqq y \leqq 0. \tag{19.34}$$

Then, since $\cos m_i(y+h)$, $\cosh m_0(y+h$ form a complete set of functions in $-h \leqq y \leqq 0$, there is no difficulty in representing $F(y)$ by a series of the fundamental solutions (13.6), but if $k^2 > m_0^2$, no progressive waves will move down the tank (within the limits of applicability of the linearized theory, of course). The analysis of the filtering effect of the tank on more complicated wave-maker motions can easily be carried through by Fourier analysis.

Waves from an oscillator not in a wall. Let us now suppose that we have a two-dimensional oscillator in infinitely deep water moving according to the law

$$x = F(y) \sin \sigma t, \qquad a < y < b \leqq 0, \tag{19.35}$$

but with no wall present. This small change complicates the solution of the problem in a substantial way, the complication being associated with the now possible flow under (and over if $b<0$) the oscillator. In addition, in order to ensure a unique solution some further condition analogous to the Kutta-Joukowski condition in airfoil theory is required; here the last two conditions of (19.36) play this role. Then the boundary conditions to be satisfied on the oscillator by the velocity potential

$$\Phi(x, y, t) = \varphi_1 \cos \sigma t + \varphi_2 \sin \sigma t$$

are

$$\left.\begin{aligned} &\Phi_x(0, y, t) = \sigma F(y) \cos \sigma t, \quad a < y < b \leqq 0, \\ &\Phi_y(0, a, t) = 0, \\ &\Phi_y(0, b, t) = 0 \quad \text{if} \quad b < 0, \end{aligned}\right\} \tag{19.36}$$

The problem is clearly closely related to that of diffraction of plane waves by a vertical barrier and could be treated by a modification of the method used in Sect. 17α for that problem. It may also be solved by the integral-equation method discussed in Sect. 17α. A modification of this method has been used by Ursell (1948).

Introduce the complex potential

$$\left.\begin{aligned} \Phi + i\Psi &= \mathrm{Re}_j\{f(z)\, \mathrm{e}^{-j\sigma t}\}, \\ f(z) &= f_1(z) + j f_2(z) = (\varphi_1 + j\varphi_2) + i(\psi_1 + j\psi_2). \end{aligned}\right\} \tag{19.37}$$

where

We try to construct a solution by means of a distribution of vortices of the form (13.28)

$$\left.\begin{aligned} f_v(z;\zeta) &= \frac{1}{2\pi i} \log(z-\zeta)(z-\bar{\zeta}) + \frac{1}{\pi i} \mathrm{PV} \int_0^\infty \frac{\mathrm{e}^{-ik(z-\bar{\zeta})}}{k-\nu}\, dk - j i\, \mathrm{e}^{-i\nu(z-\bar{\zeta})} \\ &= f_{v1} + j f_{v2} \end{aligned}\right\} \tag{19.38}$$

with intensity

$$\gamma(\eta) = \gamma_1 + j\gamma_2, \qquad a < y < b, \tag{19.39}$$

along the oscillator:

$$f(z) = \int_a^b \gamma(\eta)\, f_v(z; i\eta)\, d\eta. \tag{19.40}$$

An analysis almost identical with that in Sect. 17α leads quickly to the integral equation

$$\mathrm{Re}_i \int_a^b \gamma(\eta)\, f_v'(iy; i\eta)\, d\eta = \sigma F(y) + j\cdot 0, \qquad a < y < b. \tag{19.41}$$

Separating γ_1 and γ_2 and noting that $f'_v(iy;i\eta)$ is real with respect to i, one finds

$$\left.\begin{aligned}\int_a^b [\gamma_1(\eta)\, f'_{v1}(iy;i\eta) - \gamma_2(\eta)\, f'_{v2}(iy;i\eta)]\, d\eta &= \sigma F(y),\\ \int_a^b [\gamma_1(\eta)\, f'_{v2}(iy;i\eta) + \gamma_2(\eta)\, f'_{v1}(iy;i\eta)]\, d\eta &= 0.\end{aligned}\right\} \tag{19.42}$$

The equations can be uncoupled by applying the operator $[\partial/\partial y - \nu]$ to each (so that the reduction method enters after all!). Introducing

$$\mu_k = \gamma'_k - \nu\gamma_k, \qquad G(y) = F' - \nu F, \tag{19.43}$$

one finally obtains the pair of equations

$$\left.\begin{aligned}\int_a^b \mu_1(\eta)\frac{d\eta}{y^2-\eta^2} &= \frac{\gamma_1(b)}{y^2-b^2} - \pi\sigma\frac{G(y)}{y},\\ \int_a^b \mu_2(\eta)\frac{d\eta}{y^2-\eta^2} &= \frac{\gamma_2(b)}{y^2-b^2},\end{aligned}\right\} \tag{19.44}$$

where we have taken advantage of the fact that $\varphi_y(\pm 0, y) = \mp\gamma(y)$ and hence $\gamma(a) = 0$; if $b<0$ also $\gamma(b) = 0$. The integral equations are easily reduced to a known type occurring in airfoil theory[1] by the transformation

$$r = y^2 - \tfrac{1}{2}(a^2+b^2), \qquad \varrho = \eta^2 - \tfrac{1}{2}(a^2+b^2).$$

The solution may be written in the form

$$\left.\begin{aligned}\mu_1(\eta) &= \frac{2\eta}{\pi\sqrt{(a^2-\eta^2)(\eta^2-b^2)}}\times\\ &\quad\times\left[\nu\int_a^b\gamma_1(y)\,dy + 2\varepsilon\sigma\,\mathrm{PV}\int_a^b G(y)\sqrt{(a^2-y^2)(y^2-b^2)}\frac{dy}{\eta^2-y^2}\right],\\ \mu_2(\eta) &= \frac{2\eta\nu}{\pi\sqrt{(a^2-\eta^2)(\eta^2-b^2)}}\int_a^b\gamma_2(y)\,dy.\end{aligned}\right\} \tag{19.45}$$

It is evident that the solution is not uniquely determined without some statement about the total circulation. Fixing the total circulation is equivalent to fixing $\gamma(b)$, as follows easily from the form of $\mu(\eta)$ and the relation

$$\gamma(\eta) = e^{\nu\eta}\int_a^\eta e^{-\nu s}\mu(s)\,ds. \tag{19.46}$$

It is possible to compute the H-function directly in terms of $\mu(s)$. First, we note that

$$\begin{aligned}H(\lambda) &= \oint_{C_1} e^{-i\lambda\zeta} f'(\zeta)\,d\zeta = \oint_{C_1} e^{-i\lambda\zeta}\,d\zeta\int_a^b\gamma(y)\,f'_v(\zeta;iy)\,dy\\ &= \int_a^b\gamma(y)\,dy\oint_{C_1} e^{-i\lambda\zeta} f'_v(\zeta;iy)\,d\zeta = \int_a^b\gamma(y)\,e^{\lambda y}\,dy.\end{aligned}$$

It then follows from (19.46) that

$$H(\lambda) = \frac{e^{\lambda b}}{\lambda+\nu}\gamma(b) - \frac{1}{\lambda+\nu}\int_a^b\mu(y)\,e^{\lambda y}\,dy. \tag{19.47}$$

[1] See, e.g., W. Schmeidler: Integralgleichungen ..., pp. 55–56. Leipzig: Akademische Verlagsgesellschaft 1950, or S. G. Mikhlin: Integral'nye uravneniya ..., pp. 149–154, Gostekhizdat, Moscow 1949.

One may now apply formulas (19.23) to (19.25) to find the quantities indicated there (note that $\overline{H}=H$).

One notes again that the function $H(\lambda)$ is determined only after $\gamma(b)$ is fixed. Taking $\gamma(b)\neq 0$ is equivalent to having a singularity at the end. If the oscillator is totally submerged, it seems reasonable to set $\gamma(b)=0$, as we assumed in (19.36), for then the vertical velocity is continuous at the end, i.e., $\varphi_y(+0, b)=\varphi_y(-0, b)$, as has already been assumed for the lower end at $y=a$. It is not clear what is the proper assumption if $b=0$, i.e., if the oscillator extends through the surface. In the similar problem of diffraction by a vertical plate, treated by the reduction method in Sect. 17α, the assumption of no singularity at the surface is equivalent to assuming $\gamma(0)=0$. We note that if $\gamma(b)=0$, then it follows from (19.46) and the form of μ_2 in (19.45) that $\mu_2\equiv 0$, and hence that $\gamma_2\equiv 0$. This is not true, of course, for γ_1.

Waves generated by a heaving hemisphere. We describe briefly a procedure used by HAVELOCK (1955) and MACCAMY (1954), and before them also by URSELL (1949a) for an analogous two-dimensional problem. Let a hemisphere of radius a have its center on the free surface in its undisturbed position and let it undergo forced vertical oscillations described by

$$x^2+(y-b_0\sin\sigma t)^2+z^2=a^2. \tag{19.48}$$

Then the boundary condition to be satisfied by $\varphi(x, y, z)=\varphi_1+i\varphi_2$ on the hemisphere is

$$\frac{\partial\varphi_1}{\partial r}=b_0\sigma\frac{y}{a}=b_0\sigma\cos\vartheta,\quad \frac{\partial\varphi_2}{\partial r}=0\quad\text{on}\quad x^2+y^2+z^2=a^2,\quad y\leqq 0. \tag{19.49}$$

φ must, of course, also satisfy the free-surface condition and the radiation condition, as stated in (19.7).

The method of the above-named authors is to represent φ as a series in which the first term is a source at the center, say (13.17), and the remaining terms represent only local disturbances of the sort shown in (13.21), with $m=0$ since we have radial symmetry. The source term is actually taken in the form (13.17‴). Since the source is at (0, 0, 0), $r=r_1$ in the formulas and certain terms cancel and others double. Let

$$\left.\begin{aligned}\varphi^{(0)}&=\frac{1}{r}-\frac{2\nu}{\pi}\int_0^\infty[\nu\cos ky-k\sin ky]\frac{K_0(kR)}{k^2+\nu^2}\,dk-\\&\quad-\pi\nu e^{\nu y}Y_0(\nu R)+i\pi\nu e^{\nu y}J_0(\nu R),\\\varphi^{(n)}&=\frac{-\nu}{2n}\frac{P_{2n-1}(\cos\vartheta)}{r^{2n}}+\frac{P_{2n}(\cos\vartheta)}{r^{2n+1}}.\end{aligned}\right\} \tag{19.50}$$

Then the assumed form for φ is

$$\varphi(x, y, z)=\sum_{n=0}^{\infty}a^{n+2}(A_n+iB_n)\,\varphi^{(n)}(x, y, z). \tag{19.51}$$

Substitution in the boundary condition (19.49) leads to an infinite set of linear equations for the coefficients A_n, B_n. Numerical methods may then be used to find any desired number of terms.

Having found φ approximately, one may proceed to compute the vertical hydrodynamic force on the sphere by integrating the pressure $p=-\varrho\,\partial\Phi/\partial t$

over the hemisphere. HAVELOCK carried through an approximate calculation, expressing the result in the form

$$\left.\begin{aligned} Y &= \frac{2}{3}\pi\varrho\, a^3 b_0 \sigma^2 [k \sin\sigma t - 2h\cos\sigma t] \\ &= -M\cdot k\cdot\frac{d^2 y_0}{dt^2} - M\cdot 2h\sigma\cdot\frac{dy_0}{dt}, \end{aligned}\right\} \tag{19.52}$$

where M is the mass of displaced fluid and y_0 the coordinate of the center. The parameter k is usually called the added-mass coefficient; h is called the damping parameter. Fig. 17 from HAVELOCK's paper shows k and $2h$ as functions of νa. As $\nu a \to \infty$, $2h \to 0$ and $k \to \frac{1}{2}$; $k(0) = 0.828\ldots$. The average rate at which work is being done by the sphere is $\frac{2}{3}\pi\varrho a^3 b_0^2 \sigma^3 h$ and does not involve k.

see errata

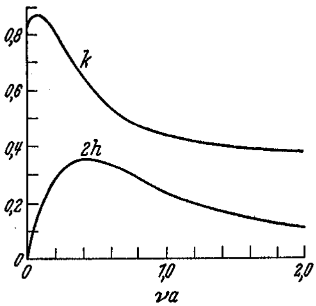

Fig. 17.

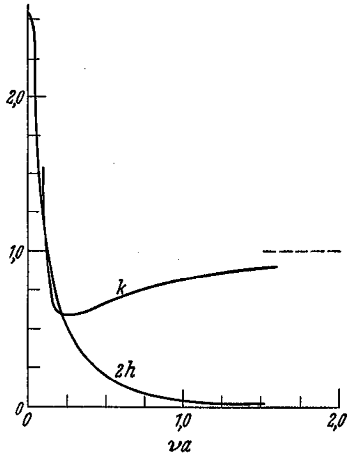

Fig. 18.

It is of interest to compare the same parameters as computed by URSELL (1949a) for a circular cylinder (per unit length). They are shown in Fig. 18. The asymptotic behavior of k is given by

$$\left.\begin{aligned} k(\nu a) &= \frac{8}{\pi^2}\left[\log\frac{1}{\nu a} + \frac{3}{2} - 2\log 2 - \gamma\right] + o(\nu a) \\ &= \frac{8}{\pi^2}\left[\log\frac{1}{\nu a} - 0.46\right] + o(\nu a) \quad \text{as} \quad \nu a \to 0, \\ k(\nu a) &= 1 - \frac{4}{3\pi\nu a} + o\left(\frac{1}{\nu a}\right) \quad \text{as} \quad \nu a \to \infty. \end{aligned}\right\} \tag{19.53}$$

β) Steady oscillations of a freely floating body in waves. Let us suppose that a rigid body is floating in an infinite ocean with prescribed plane waves approaching from a fixed direction, say from $x = +\infty$. If the motion has persisted for some time, we may suppose that the body is moving with a simple periodic motion of the same frequency as the waves. With this assumption the proper formulation of the linearized equations and boundary conditions has been derived by JOHN (1949).

Suppose the body is at rest in still water and let (x_0, y_0, z_0) be the coordinates of its center of gravity. Let $\bar{O}\bar{x}\bar{y}\bar{z}$ be a coordinate system fixed in the body with $\bar{O}$ at the center of gravity and the axes parallel to the space axes $Oxyz$. When the body is displaced, one may describe its position by giving the new position of the center of gravity $(\xi, \eta, \zeta) = (x_0 + \varepsilon x_1, y_0 + \varepsilon y_1, z_0 + \varepsilon z_1)$ and the Eulerian angles $\varepsilon\alpha$, $\varepsilon\beta$, $\varepsilon\gamma$ (we change notation from the customary φ, ϑ, ψ to avoid confusion with our other use of these letters). Thus the choice of ε implies that the amplitude of motion is small compared to some typical body length. The assumption of Sect. 10α that $\Phi = \varepsilon\,\Phi^{(1)} + \cdots$ implies that the amplitude of the prescribed incoming waves is also small compared to this length. The relationship between the two sets of coordinates may be easily found from the usual formulas concerning Eulerian angles to be of the form

$$\left.\begin{aligned}\bar{x} &= x - x_0 - \varepsilon\,[x_1 - \gamma(y - y_0) + \beta(z - z_0)] + \varepsilon^2[\cdots] + \cdots,\\ \bar{y} &= y - y_0 - \varepsilon\,[\gamma(x - x_0) + y_1 - \alpha(z - z_0)] + \cdots,\\ \bar{z} &= z - z_0 - \varepsilon\,[-\beta(x - x_0) + \alpha(z - z_0) + z_1] + \cdots.\end{aligned}\right\} \tag{19.54}$$

Let the surface of the body be described by

$$F(\bar{x}, \bar{y}, \bar{z}) = 0 \tag{19.55}$$

in body coordinates. To find the position in space coordinates one must substitute from (19.54) in (19.55). The kinematic boundary condition [see Eq. (19.1)] then becomes

$$\left.\begin{aligned}\varepsilon\{&\operatorname{grad} F(x-x_0, y-y_0, z-z_0)\cdot\operatorname{grad}\Phi^{(1)} + F_x(x-x_0, y-y_0, z-z_0)\times\\ &\times[-\dot{x}_1 - \dot{\gamma}(y-y_0) - \dot{\beta}(z-z_0)] + F_y[-\dot{y}_1 + \dot{\alpha}(z-z_0) - \dot{\gamma}(x-x_0)] +\\ &+F_z[-\dot{z}_1 + \dot{\beta}(x-x_0) - \dot{\alpha}(y-y_0)]\} + \varepsilon^2\{\cdots\} + \cdots = 0.\end{aligned}\right\} \tag{19.56}$$

Letting n_x, n_y, n_z be the components of the unit inward normal vector to the surface at rest, i.e.,

$$F(x - x_0, y - y_0, z - z_0) = 0 \tag{19.57}$$

(we shall call this surface S_0), and $\boldsymbol{q} = (\boldsymbol{r} - \boldsymbol{r}_0)\times\boldsymbol{n}$, i.e.

$$\left.\begin{aligned}&q_x = (y - y_0)\,n_z - (z - z_0)\,n_y, \quad q_y = (z - z_0)\,n_x - (x - x_0)\,n_z,\\ &q_z = (x - x_0)\,n_y - (y - y_0)\,n_x,\end{aligned}\right\} \tag{19.58}$$

we may express the first-order term in (19.56), after dropping the superscript, in the form

$$\Phi_n = \dot{x}_1 n_x + \dot{y}_1 n_y + \dot{z}_1 n_z + \dot{\alpha}\,q_x + \dot{\beta}\,q_y + \dot{\gamma}\,q_z \quad \text{for} \quad (x, y, z) \text{ on } S_0. \tag{19.59}$$

We call attention to the fact that it follows as a natural consequence of the linearization that the boundary condition is to be satisfied on the surface in its undisturbed position.

In order to state the dynamical conditions on the body we introduce the following notations. Let M be its mass and $I_x, I_y, I_z, I_{xx}, I_{xy}, \ldots$ its moments and moments and products of inertia about the body axes selected above. Let V be the volume bounded by the plane $y = 0$ and the submerged part of the surface in its rest position, and let $I^V, I_x^V, I_y^V, I_z^V, I_{xx}^V, I_{xy}^V, \ldots$ be the volume, the moments and the moments and products of inertia of this volume about the body axes in their rest position. Let A be the intersection of the body in its rest position with the surface $y = 0$, and let $I^A, I_x^A, I_z^A, I_{xx}^A, I_{xz}^A, I_{zz}^A$ denote the area, the moments and the moments and products of inertia of A with respect to axes through $(x_0, 0, z_0)$ and parallel to the body axes; e.g.,

$$I_{xz}^A = \iint_A (x - x_0)\,(z - z_0)\,dx\,dz.$$

The exact dynamical equations are

$$\left.\begin{aligned} M\ddot{\xi} &= \iint_S p\cos(n,x)\,d\sigma, \\ M\ddot{\eta} &= \iint_S p\cos(n,y)\,d\sigma - Mg, \\ M\ddot{\zeta} &= \iint_S p\cos(n,z)\,d\sigma, \end{aligned}\right\} \tag{19.60}$$

where S is the wetted surface of the body in its (to-be-determined) position at time t and

$$p = -\varrho g y - \varrho \Phi_t - \tfrac{1}{2}\varrho\,|\mathrm{grad}\,\Phi|^2;$$

and three similar equations for $\ddot{\alpha}, \ddot{\beta}, \ddot{\gamma}$. Substitution of the perturbation series gives for the zero-order terms

$$M = \varrho I^V, \quad I_x^V = I_y^V = 0, \tag{19.61}$$

i.e., ARCHIMEDES' law and the statement that the center of buoyancy and center of gravity are on the same vertical line. The first-order equations, after dropping superscripts, are

$$\left.\begin{aligned} M\ddot{x}_1 &= -\varrho\iint_{S_0}\Phi_t n_x\,d\sigma, \\ M\ddot{y}_1 &= -\varrho\iint_{S_0}\Phi_t n_y\,d\sigma + \varrho g\,(-I^A y_1 - I_x^A\gamma + I_z^A\alpha), \\ M\ddot{z}_1 &= -\varrho\iint_{S_0}\Phi_t n_z\,d\sigma, \\ -(I_{yy}+I_{zz})\ddot{\alpha} + I_{xy}\ddot{\beta} + I_{xz}\ddot{\gamma} &= \varrho\iint_{S_0}\Phi_t q_x\,d\sigma - \varrho g\,[I_z^A y_1 + I_{xz}^A\gamma - I_{zz}^A\alpha - I_y^V\alpha], \\ I_{xy}\ddot{\alpha} - (I_{xx}+I_{zz})\ddot{\beta} + I_{yz}\ddot{\gamma} &= \varrho\iint_{S_0}\Phi_t q_y\,d\sigma, \\ I_{xz}\ddot{\alpha} + I_{yz}\ddot{\beta} - (I_{xx}+I_{yy})\ddot{\gamma} &= \varrho\iint_{S_0}\Phi_t q_z\,d\sigma + \varrho g\,[I_x^A y_1 + I_{xx}^A\gamma - I_{xz}^A\alpha + I_y^V\gamma]. \end{aligned}\right\} \tag{19.62}$$

We note that the boundary conditions have been derived for general motions of the body and fluid, not just for the simply periodic ones for which they will be used below.

JOHN (1949) has used the equations to investigate the stability of a floating body. We shall not reproduce the results but remark that he shows that the usual condition for stability, namely that the metacenter lie above the center of gravity, derived from purely hydrostatic considerations, is in fact still a sufficient condition for stability when the hydrodynamic equations are considered (within the limitations of the linearized theory).

It is also shown by JOHN that the above equations have a unique solution for an initial-value problem, i.e., if at some instant the position and velocity of body and fluid are prescribed. However, for the problem with which we are concerned in this section, steady simple harmonic motion with a prescribed incoming wave, he proves uniqueness only for sufficiently large values of σ and for bodies such that a vertical line intersects the immersed surface only once (e.g., a floating sphere with its center at or above the free surface).

Knowledge of the motion of a floating body in surface waves is obviously of great importance to ship designers, and, as might be expected, there is a large amount of specialized literature. However, most of this literature may be considered irrelevant to this article for it is based upon the assumption that one may neglect the kinematic boundary condition (19.59) completely and, in the dynamic boundary condition (19.62), that one may take for Φ simply the velocity potential for the oncoming wave, thus neglecting the effect of the diffracted waves and the waves generated by the ship's own motion. This assumption is usually called the Froude-Krylov Hypothesis. W. FROUDE (1861) introduced it in connection with an investigation of ship rolling in waves and A. N. KRYLOV (1896, 1898) investigated its implications rather thoroughly for general motions. In spite of its apparent crudeness this assumption has been useful in elucidating many aspects of ship motions.

In recent years there have appeared a number of papers in which an attempt has been made to take account of the proper boundary conditions, but no attempt will be made to summarize this literature. The most systematic investigation of the matter has been made by JOHN (1949, 1950), HASKIND (1946a), and PETERS and STOKER (1957). The papers by JOHN consider the proper formulation of the linearized problem for a body with no average forward speed and the uniqueness and existence of solutions. Both HASKIND and PETERS and STOKER are primarily concerned with ships having a constant average forward speed. PETERS and STOKER treat carefully the proper formulation of the linearized problem and conclude that HASKIND'S fundamental equations are not properly formulated in that some of his terms really belong with the second-order terms and should have been discarded. The objection applies also to part of his results for a stationary ship. The other part will be summarized below.

The motion of a ship in waves when it has a nonzero translational velocity will not be considered in this article. For this theory one should refer to the cited papers, to STOKER'S *Water waves* (1957, Chap. 9), or to a recent survey by MARUO (1957). The transient oscillatory motion of a floating body in calm water will be considered later.

Let us return to the problem of steady oscillation of a floating body in oncoming waves. Since we assume steady oscillation, we shall write

$$\left.\begin{aligned} \Phi = \operatorname{Re}\{\varphi\, e^{-i\sigma t}\}, \qquad (x_1, y_1, z_1) &= \operatorname{Re}\{(a_0, b_0, c_0)\, e^{-i\sigma t}\}, \\ (\alpha, \beta, \gamma) &= \operatorname{Re}\{(\alpha_0, \beta_0, \gamma_0)\, e^{-i\sigma t}\}, \end{aligned}\right\} \tag{19.63}$$

where $\varphi = \varphi_1 + i\varphi_2$, $a_0 = a_0' + i a_0''$, etc. The unknown function φ and the constants $a_0, \ldots, \gamma_0$ are to be determined from the equations and boundary conditions.

We shall assume that Φ can be expressed as the sum of the velocity potentials of the incoming wave, say

$$\Phi^e = \frac{A\sigma}{g}\, e^{\nu y} \cos(\nu x + \sigma t) \tag{19.64}$$

if the fluid is infinitely deep, a diffracted wave $\Phi^0 = \varphi^0 e^{-i\sigma t}$ and a forced wave $\Phi_f = \varphi_f\, e^{-i\sigma t}$ resulting from the body's own motion:

$$\Phi = \Phi^e + \Phi^0 + \Phi_f. \tag{19.65}$$

We shall express Φ_f in the following form (following HASKIND):

see errata

$$\Phi_f = \operatorname{Re}\{\varphi^1 \dot{x}_1 + \varphi^2 \dot{y}_1 + \varphi^3 \dot{z}_1 + \varphi^4 \dot{\alpha} + \varphi^5 \dot{\beta} + \varphi^6 \dot{\gamma}\}. \tag{19.66}$$

Then the kinematic boundary condition (19.59) implies:

$$\left.\begin{aligned} &\varphi_n^0 = -\varphi_n^e, \\ &\varphi_n^1 = n_x, \qquad \varphi_n^2 = n_y, \qquad \varphi_n^3 = n_z, \\ &\varphi_n^4 = q_x, \qquad \varphi_n^5 = q_y, \qquad \varphi_n^6 = q_z, \end{aligned}\right\} \tag{19.67}$$

all to be satisfied on S_0, the rest position of the body. The functions φ^k, $k = 0, 1, \ldots, 6$, are to satisfy also the radiation condition and the condition at $y = -\infty$ (or at $y = -h$ for a flat bottom). The dynamical condition (19.62) will be used to determine the amplitudes and phases (i.e., the complex amplitudes), but first we introduce some notation. Let

$$\mu_{jk} + \frac{i}{\sigma} \lambda_{jk} = \varrho \iint_{S_0} \varphi^j \frac{\partial \varphi^k}{\partial n} d\sigma. \tag{19.68}$$

The constants μ_{jk} and λ_{jk} depend only upon the geometry of the body. It may be shown by an application of GREEN'S Theorem that $\mu_{kj} = \mu_{jk}$ and $\lambda_{kj} = \lambda_{jk}$.

Let us now substitute the expanded expression for Φ into, say, the first of Eqs. (19.62) (the others may be treated similarly), remembering that $n_x = \varphi_n^1$ on S_0:

$$M \ddot{x}_1 = -\varrho \iint_{S_0} (\Phi^e + \Phi^0)_t \, n_x \, d\sigma - \varrho \iint_{S_0} (\varphi^1 \ddot{x}_1 + \cdots + \varphi^6 \ddot{\gamma}) \, \varphi_n^1 \, d\sigma. \tag{19.69}$$

Consider, for example, the second term of the second integral:

$$\varrho \iint_{S_0} \varphi^2 \varphi_n^1 \ddot{y}_1 \, d\sigma = \left(\mu_{21} + \frac{i}{\sigma} \lambda_{21}\right) \ddot{y}_1 = \mu_{21} \ddot{y}_1 + \lambda_{21} \dot{y}_1, \tag{19.70}$$

where we have used the special form of $y_1 = b_0 \, e^{-i\sigma t}$. Thus, (19.69) may be written

$$(M + \mu_{11}) \ddot{x}_1 + \mu_{21} \ddot{y}_1 + \cdots + \mu_{61} \ddot{\gamma} + \lambda_{11} \dot{x}_1 + \lambda_{21} \dot{y}_1 + \cdots + \lambda_{61} \dot{\gamma} = F_{ex} + F_{0x}, \tag{19.71}$$

where $F_{ex} = f_{ex} \, e^{-i\sigma t}$ and $F_{0x} = F_{0x} \, e^{-i\sigma t}$ represent the x-components of the forces resulting from the incoming and diffracted waves and are to be computed from the first integral in (19.69). The form of (19.71) explains the names given to the μ_{ij} and λ_{ij}: the μ_{ij} are called *added masses*, the λ_{ij}, *damping coefficients*. If one now writes $x_1, \ldots, \gamma$ in their assumed forms in (19.63) and substitutes in (19.71), one obtains

$$\left.\begin{aligned} &-\sigma^2 (M + \mu_{11}) \, a_0 - \sigma^2 \mu_{21} \, b_0 - \cdots - \sigma^2 \mu_{61} \gamma_0 - \\ &\qquad - i \sigma \lambda_{11} \, a_0 - \cdots - i \sigma \lambda_{61} \gamma_0 = f_{ex} + f_{0x} \end{aligned}\right\} \tag{19.72}$$

and five similar equations. Since the amplitudes $a_0, \ldots, \gamma_0$ are complex, this gives twelve equations to determine the twelve unknown quantities. It is thus clear that, providing these equations can be uniquely solved, the problem of finding the steady oscillatory motion of a freely floating body can be reduced to the solution of several problems of the type studied in Sects. 18 and 19α. From the form of (19.72) and the similar equations, it is clear that the complex amplitudes are all proportional to the amplitude A of the incoming wave as would be expected.

HASKIND has applied the method outlined above to a body symmetric with respect to the (x, y)-plane, e.g., a ship heading into waves. The only possible motions are heaving, pitching and surging. In carrying out some numerical computations he makes a further approximation that the kinematic boundary condition on the body may be satisfied on its plane of symmetry rather than on the surface. Although this approximation is perfectly consistent with the linearized theory in certain contexts, as will be seen in Sect. 21, it is not consistent with the theory as formulated here and must be considered to be a further approximation of some sort.

Freely floating sphere. Computation of the motion of a freely floating sphere with its center at the undisturbed water level can be carried through without an unreasonable amount of numerical work. The procedure for the heaving motion has been carried up to the point of numerical computation by Barakat (1958) [in an earlier investigation by MacCamy (1954) the multipole terms in the potential for the diffracted wave were omitted]. Part of the problem has already been solved in Sect. 19α, i.e., the waves resulting from the forced motion.

Since the phase at infinity must be kept arbitrary, one must replace (19.48) by

$$x^2 + (y - b_0' \cos \sigma t - b_0'' \sin \sigma t)^2 + z^2 = a^2 \tag{19.72}$$

However, the solution of that problem may be taken over with practically no change, for the velocity potential φ^2 in the notation of (19.66) must satisfy

$$\frac{\partial}{\partial r} \varphi^2 = \frac{y}{a} = \cos\vartheta \qquad \text{for } x^2 + y^2 + z^2 = a^2, \qquad y \leqq 0. \tag{19.73}$$

Thus we need only set $b_0 \sigma = 1$ in (19.49) and later. In fact, from formula (19.52)

$$\mu_{22} = \tfrac{2}{3} \pi \varrho a^3 \cdot k, \qquad \lambda_{22} = \tfrac{2}{3} \pi \varrho a^3 \cdot 2h. \tag{19.74}$$

Finding the diffracted wave requires finding an outgoing wave satisfying

$$\left.\frac{\partial \varphi^0}{\partial r}\right|_{r=a} = -\frac{A g}{\sigma} \nu \left[\cos\vartheta - i \sin\vartheta \cos\alpha\right] e^{\nu a \cos\vartheta} e^{-i \nu a \sin\vartheta \cos\alpha}, \tag{19.75}$$

where $x = r \sin\vartheta \cos\alpha$, $y = r\cos\vartheta$, $z = r \sin\vartheta \sin\alpha$. Barakat shows that φ^0 can be found as a series in functions of the form (13.21), with $b=0$ and account taken of certain symmetries, and functions of the form (13.20) with $b=0$ and $m=n$. Let

$$\left.\begin{aligned}
G_{2k}^{2m} &= \left[\frac{P_{2k}^{2m}(\cos\vartheta)}{r^{2k+1}} - \frac{\nu}{2k-2m} \frac{P_{2k-1}^{2m}(\cos\vartheta)}{r^{2k}}\right] \cos 2m\alpha, \\
&\qquad k = 1, 2, \ldots; \qquad m = 0, \ldots, k-1; \\
G_{2k}^{2m-1} &= \left[\frac{P_{2k+1}^{2m-1}(\cos\vartheta)}{r^{2k+2}} - \frac{\nu}{2k-2m+2} \frac{P_{2k}^{2m-1}(\cos\vartheta)}{r^{2k+1}}\right] \cos(2m-1)\alpha, \\
&\qquad k = 1, 2, \ldots; \qquad m = 1, \ldots, k; \\
\Phi_n &= \left[\frac{P_n^n(\cos\vartheta)}{r^{n+1}} + (-1)^n \,\mathrm{PV}\!\int_0^\infty \frac{k+\nu}{k-\nu} k^n e^{ky} J_n(kR)\, dk + \right. \\
&\qquad \left. + 2\pi i (-1)^n \nu^{n+1} e^{\nu y} J_n(\nu R)\right] \cos n\alpha, \qquad n = 1, 2, \ldots.
\end{aligned}\right\} \tag{19.76}$$

Then φ^0 may be expressed as follows

$$\varphi^0 = \sum_{k=1}^{\infty} \sum_{m=0}^{k-1} A_{2k}^{2m} a^{2k+2} G_{2k}^{2m} + i \sum_{k=1}^{\infty} \sum_{m=1}^{k} B_{2k}^{2m-1} a^{2k+3} G_{2k}^{2m-1} + \sum_{n=0}^{\infty} C_n \Phi_n, \tag{19.77}$$

where the complex coefficients A_{2k}^{2m}, B_{2k}^{2m-1}, C_n are to be determined from (19.75). No numerical computations seem to be available.

20. Motions which may be treated as steady flows. In this section we shall consider several problems which are time-independent, either by their formulation or by introduction of moving coordinates. The flow associated with a constant discharge rate through a canal is of the first type; the waves associated with a ship which has moved with constant velocity C over a long period is typical of the second.

The boundary conditions at the free surface have been derived in Sects. 10 and 11. For three-dimensional motion the velocity potential must satisfy [see Eq. (11.3)]

$$\varphi_y(x, 0, z) + \frac{c^2}{g}\varphi_{xx}(x, 0, z) = 0; \tag{20.1}$$

the equation of the free surface is

$$y = \eta(x, z) = \frac{c}{g}\varphi_x(x, 0, z). \tag{20.2}$$

In two-dimensional motion, if the complex potential $f = \varphi + i\psi$ is used, then the boundary condition may be written

$$\mathrm{Re}\left\{f''(x + i\,0) + i\frac{g}{c^2}f'(x + i\,0)\right\} = 0. \tag{20.3}$$

If the potential has been taken in the form $F(z) = -cz + f(z)$ with $\Psi = 0$ as the free-surface streamline, then

$$\mathrm{Re}\left\{f'(x + i\,0) + i\frac{g}{c^2}f(x + i\,0)\right\} = 0; \tag{20.3'}$$

the surface is given by

$$y = \eta(x) = \frac{1}{c}\psi(x, 0). \tag{20.4}$$

On obstructions, which are now all fixed, one has as usual

$$\varphi_n = 0 \quad \text{or} \quad \psi = \text{const}. \tag{20.5}$$

Far ahead of, or far upstream of, the obstruction the motion must approach either rest, or a uniform flow, respectively.

The general theory of steady free-surface flow about a submerged obstacle in infinitely deep fluid has been considered by Kochin (1937) for both two and three dimensions. Haskind (1945a, b) has extended Kochin's treatment to fluid of constant finite depth. The methods used for waves generated by oscillating bodies carry over with only slight change, so that we shall not consider here the general aspects of the theory but consider instead several special problems.

α) Flow over an uneven bottom. Let us first derive the proper boundary condition on the bottom. We shall assume that the bottom may be represented in the form

$$y = -h + \varepsilon b^{(1)}(x) \tag{20.6}$$

and that the fluid flows from the right with discharge rate $q = ch$. As in the derivation of (10.19) we take

$$F(z) = -cz + \varepsilon f^{(1)}(z) + \varepsilon^2 f^{(2)}(z) + \cdots. \tag{20.7}$$

Then the condition that the bottom be a streamline is

$$-c(-h + \varepsilon b^{(1)} + \cdots) + \varepsilon\psi^{(1)}(x, -h + \varepsilon b^{(1)} + \cdots) + \cdots = ch. \tag{20.8}$$

Expansion in the manner of Sect. 10 and grouping of coefficients leads to the boundary condition for $\psi^{(1)}$:

$$\psi^{(1)}(x, -h) = c\,b^{(1)}. \tag{20.9}$$

We may hereafter write ψ for $\varepsilon\psi^{(1)}$ and b for $\varepsilon b^{(1)}$. We note that the choice of ε indicates that the amplitude of unevenness of the bottom must be small compared with h for the linearized theory to be applicable.

Consider now a bottom of the form [see LAMB (1932, p. 409), Wien (1900, p. 200)]

$$y = -h + b_0 \cos kx. \tag{20.10}$$

We look for a solution in the form

$$f(z) = A \cos kz + B \sin kz, \tag{20.11}$$

where A and B are complex. Substitution in (20.9), with $b^{(1)} = b_0 \cos kx$, shows that A must be pure imaginary, say iA', and B real, and further that

$$A' \cosh kh - B \sinh kh = c b_0. \tag{20.12}$$

Substitution in (20.3), i.e., $\psi_y(x, 0) - g c^{-2} \psi(x, 0) = 0$ yields

$$kB = \frac{g}{c^2} A'. \tag{20.13}$$

One then finds easily that

$$f(z) = \frac{\nu \sin kz + ik \cos kz}{k \cosh kh - \nu \sinh kh} c b_0, \qquad \nu = \frac{g}{c^2}, \tag{20.14}$$

$$\eta(x) = \frac{k b_0}{k \cosh kh - \nu \sinh kh} \cos kx.$$

An interesting consequence is that when $c^2/gh < 1$, i.e., when the flow is subcritical, the crests and troughs just oppose those of the bottom, whereas, if $c^2/gh > 1$, they occur together. If $c^2/gh = 1$, there is no steady flow. Also, when c^2/gh is close to 1, it is clear that the assumption of small perturbations is no longer satisfied.

By use of the Fourier Integral Theorem one may now construct solutions for an arbitrarily shaped bottom, within the limitations of the theorem. For from

$$b(x) = \frac{1}{\pi} \int_0^\infty dk \int_{-\infty}^\infty b(\xi) \cos k(x - \xi)\, d\xi \tag{20.15}$$

one may derive

$$\left.\begin{aligned} f(z) &= \frac{c}{\pi} \mathrm{PV} \int_0^\infty dk \int_{-\infty}^\infty b(\xi) \frac{\nu \sin k(z-\xi) + ik \cos k(z-\xi)}{k \cosh kh - \nu \sinh kh}\, d\xi, \\ \eta(x) &= \frac{1}{\pi} \mathrm{PV} \int_0^\infty \frac{k}{k \cosh kh - \nu \sinh kh}\, dk \int_{-\infty}^\infty b(\xi) \cos k(x-\xi)\, d\xi. \end{aligned}\right\} \tag{20.16}$$

An examination of the asymptotic properties of these integrals as $x \to +\infty$ shows that they do not vanish if $\nu h = gh/c^2 > 1$. Conditions for the validity of the Fourier Integral Theorem, e.g., that $b(x)$ is of bounded variation and absolutely integrable, indicate that it applies to situations in which the bottom unevenness is somewhat localized. Hence, it is reasonable to require the additional boundary condition

$$\lim_{x \to \infty} \eta(x) = 0.$$

see errata

Thus we must amend the solutions (20.10) if $gh/c^2 > 1$ by adding, respectively,

$$\left.\begin{aligned} &\frac{-\nu c}{\cosh^2 k_0 h - \nu h} \int_{-\infty}^\infty b(\xi) \cos k_0(z - \xi + ih)\, d\xi, \\ &\frac{+\nu \sinh k_0 h}{\cosh^2 k_0 h - \nu h} \int_{-\infty}^\infty b(\xi) \sin k_0(x - \xi)\, d\xi, \end{aligned}\right\} \tag{20.17}$$

where k_0 is the real solution of $k \cosh kh - \nu \sinh kh = 0$. We note that the other boundary conditions are not spoiled, for the first expression in (20.17) satisfies (20.3) and its imaginary (stream-function) part vanishes for $y = -h$ so that (20.3) is still satisfied.

Thus, if $c^2 > gh$ there is a local disturbance of the fluid in the region of unevenness which eventually smooths out. If $c^2 < gh$ there is also a local disturbance given by (20.16), but as $x \to -\infty$ there remains a disturbance given by twice the expressions in (20.17).

We remark in passing that we might have obtained this solution by distributing along the bottom dipoles of the form (13.48) with $\alpha = 0$ and with moment density $c\,b(x)$.

Various special cases of $b(x)$ may be considered. LAMB (1932, p. 410) replaces the unevenness by a single dipole. WIEN (1900, p. 202) takes $b(x) = \arctan e x$ and in the limit lets $e \to \infty$ in order to find the flow over a small step. However, KOCHIN (1938) has treated this problem by a different method and finds that WIEN has made an error by a factor of two in the downstream waves (he had not satisfied the upstream condition) [see also LAMB (1934)]. The flow about a vertical plate in the bottom may be treated by distributing vortices (13.47) along the plate with the intensity to be determined by solving an integral equation.

One will find an attractive discussion of the subject in four papers of W. THOMSON (Lord KELVIN) (1886, 1887). EKMAN (1906) has applied the same method to three-dimensional flow. First he finds the form of the free surface over a doubly periodic bottom, then applies the double Fourier integral theorem to construct flows over irregular bottoms. He analyzes the asymptotic behavior of the surface for the case of an isolated dipole on the bottom and presents graphs showing the change in wave amplitude for different radial sections. The method of analysis may also be extended to superposed fluids of different densities (see KOCHIN (1937a, b, 1938c), LONG (1953, § 4)].

β) Flow about submerged obstacles. Linearization. The procedure for linearizing may be carried through in at least two ways, leading to somewhat different boundary conditions for the body. Consider a body moving in a fluid. For the time being, in order to achieve somewhat greater generality, we shall not restrict the velocity to be constant. If the dimensions of the body are sufficiently small compared with the depth of submersion, it will not disturb the surface appreciably, and one will expect to be able to use the infinitesimal-wave approximation. However, the same end is obtained if the body approximates to a flat disc moving in its plane, a line segment moving along its line, a piece of a cylindrical surface moving along the cylinder, etc., various combinations being easily visualized. We consider the two situations separately.

Let $F(x, y, z, t) = 0$ describe the surface of a bounded body at time t, and let a be some typical dimension of the body, say its maximum diameter, and let h be the depth of submersion measured to some point $(x_0, -h, z_0)$ of the body. Now, consider the family of flows associated with the family of surfaces

$$F^{(\varepsilon)}(x, y, z, t) = F\left(\frac{x - x_0}{\varepsilon} + x_0, \frac{y + h}{\varepsilon} - h, \frac{z - z_0}{\varepsilon} + z_0, t\right) = 0 \qquad (20.18)$$

where $\varepsilon = a/h$. As $\varepsilon \to 0$ the surface $F^{(\varepsilon)} = 0$ contracts to a point and the fluid approaches a state of rest. Hence, as in Sect. 10α, it is allowable to expand Φ and η in a perturbation series

$$\Phi = \varepsilon\,\Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + \cdots, \qquad \eta = \varepsilon \eta^{(1)} + \varepsilon^2 \eta^{(2)} + \cdots. \qquad (20.19)$$

The boundary condition to be satisfied on the body, namely,

$$\operatorname{grad} F^{(\varepsilon)} \cdot \operatorname{grad} \Phi + F_t^{(\varepsilon)} = 0, \tag{20.20}$$

becomes

$$\operatorname{grad} F \cdot \operatorname{grad} \Phi^{(1)} + F_t + \varepsilon \operatorname{grad} F \cdot \operatorname{grad} \Phi^{(2)} + \cdots = 0,$$

and $\Phi^{(1)}$ must satisfy

$$\operatorname{grad} F \cdot \operatorname{grad} \Phi^{(1)} + F_t = 0 \quad \text{on} \quad F^{(\varepsilon)} = 0.$$

Thus, one finds that, in this method of linearizing, the boundary condition to be satisfied on the body is the exact one

$$\operatorname{grad} F \cdot \operatorname{grad} \Phi + F_t = 0 \quad \text{on} \quad F(x, y, z, t) = 0. \tag{20.21}$$

The boundary condition satisfied by Φ on the free surface is, of course, the linearized one. The approximation to the exact solution is better, the deeper the relative submergence.

The second method of linearization will be illustrated with the so-called *thin-ship* approximation. Let the equation of a ship hull be given in the form

$$\bar{z} = \pm F(\bar{x}, \bar{y}). \tag{20.22}$$

in coordinates fixed in the ship. Let us write this in the form

$$\bar{z} = \pm \varepsilon F^{(1)}(\bar{x}, \bar{y}) \tag{20.23}$$

where ε is, say, the beam/length. Suppose the ship moves in direction OX with velocity $c(t)$ and consider the family of flows generated by the motion of such bodies for different ε. Let the velocity potential be $\Phi(x, y, z, t; \varepsilon)$. Then, since as $\varepsilon \to 0$ the hull approaches a flat disc S_0, the ship's centerplane section, the motion of the fluid will also approach a state of rest and we may expand

$$\Phi = \varepsilon \Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + \cdots \tag{20.24}$$

and similarly for η. The assumed forms for Φ and η lead immediately as in Sect. 10α to the linearized free-surface boundary condition for $\Phi^{(1)}$. The exact condition on the hull is

$$F_x\Big(x - \int^t c(\tau)\, d\tau, y\Big)\, \Phi_x\Big(x, y, F\Big(x - \int^t c\, d\tau, y\Big), t\Big) + F_y \Phi_y - \Phi_z - c(t)\, F_x = 0. \tag{20.25}$$

After substituting (20.23) and (20.24) in (20.25), one finds that $\Phi^{(1)}$ must satisfy

$$\Phi_z^{(1)}(x, y, \pm 0, t) = \mp c(t)\, F_x^{(1)}\Big(x - \int^t c(\tau)\, d\tau, y\Big), \tag{20.26}$$

$\Phi^{(2)}$ must satisfy

$$\Phi_z^{(2)}(x, y, \pm 0, t) = \pm \left[F_x^{(1)}(x, y)\, \Phi_x^{(1)}(x, y \pm 0, t) + F_y^{(1)} \Phi_y^{(1)} - F^{(1)} \Phi_{zz}^{(1)}\right], \tag{20.27}$$

and $\Phi^{(i)}$ a relation of the form

$$\Phi_z^{(i)}(x, y, \pm 0, t) = \pm C_i\{F^{(1)}, \Phi^{(1)}, \ldots, \Phi^{(i-1)}\}, \tag{20.28}$$

where C_i is a functional of the functions in braces. We note especially that it is a consequence of the linearization that the boundary condition imposed by the presence of the body is to be satisfied on the centerplane section and not on the actual surface. One will expect this linearized theory to be more accurate the smaller ε is, i.e., the smaller the beam-to-length ratio.

It is clear that one may proceed similarly in the situations mentioned earlier. We record the results in several cases for reference.

First consider the *thin-wing approximation* for two-dimensional hydrofoils. In a coordinate system $\bar{O}\bar{x}\bar{y}$ fixed in the hydrofoil let the trailing edge of the hydrofoil be at $(-a, -h)$, and let the upper and lower surfaces be given by

$$\bar{y} = -h + u(\bar{x}) \quad \text{and} \quad \bar{y} = -h + b(\bar{x}), \qquad -a \leqq \bar{x} \leqq a, \tag{20.29}$$

respectively. Define

$$r(\bar{x}) = \tfrac{1}{2}[u(\bar{x}) + b(\bar{x})], \qquad s(\bar{x}) = \tfrac{1}{2}[u(\bar{x}) - b(\bar{x})], \tag{20.30}$$

so that now the top and bottom are given by

$$\bar{y} = -h + r(\bar{x}) + s(\bar{x}) \quad \text{and} \quad \bar{y} = -h + r(\bar{x}) - s(\bar{x}), \qquad -a \leqq \bar{x} \leqq a, \tag{20.31}$$

respectively. The class of profiles in a form analogous to (20.23) is now given by

$$\bar{y} = -h + \varepsilon[r^{(1)}(\bar{x}) \pm s^{(1)}(\bar{x})], \qquad -a \leqq \bar{x} \leqq a. \tag{20.32}$$

It is clear that as $\varepsilon \to 0$, the profiles approach the line segment $\bar{y} = -h$, $0 \leqq \bar{x} \leqq a$, so that the perturbation procedure is allowable. The analysis leads to the linearized boundary condition

$$\left.\begin{aligned} \varphi_y(x, -h \pm 0, t) = -c(t)\, r'\Big(x - \int^t c(\tau)\, d\tau\Big) \mp c(t)\, s'\Big(x - \int^t c(\tau)\, d\tau\Big), \\ -a \leqq x - \int^t c\, d\tau \leqq a. \end{aligned}\right\} \tag{20.33}$$

The *slender-body approximation* is also consistent with the linearized free-surface condition. Let the body be defined by

$$(\bar{y} + h)^2 + \bar{z}^2 - r^2(\bar{x}) = 0, \qquad |\bar{x}| < a, \qquad h > 0, \tag{20.34}$$

in a coordinate system fixed in the body. If one considers the class of bodies defined by $\varepsilon r^{(1)}(\bar{x})$, then the appropriate condition to be satisfied by $\Phi^{(1)}$ is

$$\lim_{\varepsilon \to 0} \Phi_r^{(1)}\Big(x - \int^t c(\tau)\, d\tau, -h + \varepsilon r^{(1)} \cos\vartheta, \varepsilon r^{(1)} \sin\vartheta\Big)\, r^{(1)}\Big(x - \int^t c\, d\tau\Big) = -c\, r^{(1)} r^{(1)\prime}. \tag{20.35}$$

We note that the same problem may be approached by two linearized theories. For example, in approximating the flow about a hydrofoil, one may either consider it as a relatively deeply submerged body and satisfy the exact conditions on the surface, or consider it as a thin hydrofoil and use the conditions (20.33). Each method will have its own domain of excellence, but it is not proper in the present context to say that the thin-hydrofoil approximation is less exact than the other one, even though this is true in an unbounded fluid.

The H-functions. KOCHIN's H-function, introduced in Sect. 19α, may also be used effectively for the flows considered in the present section. The definition for three dimensions is identical in appearance with (19.14). For two dimensions (19.21) isreplaced by

$$H(k) = \int_{C_1} e^{-ik\zeta} f'(\zeta)\, d\zeta. \tag{20.36}$$

However, the formulas for the force on the body are somewhat different. For three dimensions they are

$$\left.\begin{aligned} X &= -\frac{\nu^2 \varrho}{2\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} |H(\nu \sec^2\vartheta, \vartheta)|^2 \sec^3\vartheta\, d\vartheta, \\ Y &= \varrho g V - \frac{\varrho}{8\pi^2} \int_0^\infty \int_{-\pi}^{\pi} |H(k, \vartheta)|^2 k\, d\vartheta\, dk + \\ &\quad + \frac{\nu^2 \varrho}{4\pi^2} \int_{-\pi}^{\pi} \mathrm{PV} \int_{-\infty}^{1} \left| H\left(\frac{\nu(1-\lambda)}{\cos^2\vartheta}, \vartheta\right)\right|^2 \frac{1-\lambda}{\lambda} \sec^4\vartheta\, d\lambda\, d\vartheta, \\ Z &= -\frac{\nu^2 \varrho}{2\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} |H(\nu \sec^2\vartheta, \vartheta)|^2 \sec^4\vartheta \sin\vartheta\, d\vartheta, \end{aligned}\right\} \tag{20.37}$$

where V is the displaced volume of fluid and $\nu = g/c^2$. The two-dimensional formulas are

$$\left.\begin{aligned} X &= -\nu\varrho\,|H(\nu)|^2, \\ Y &= \varrho g A + \varrho c\Gamma - \frac{\varrho}{2\pi}\int_0^\infty |H(k)|^2\,dk + \frac{\varrho\nu}{\pi}\,\mathrm{PV}\int_{-\infty}^{1} |H(\nu - k\nu)|^2\,\frac{dk}{k}, \\ M &= -g\varrho A x_c - \varrho c\,\mathrm{Re}\{iH'(0)\} - \varrho\,\mathrm{Re}\left\{\frac{1}{2\pi i}\int_0^\infty H'(k)\,\overline{H(k)}\,dk + \right. \\ &\quad \left. + \nu H'(\nu)\,\overline{H(\nu)} + \frac{\nu i}{\pi}\,\mathrm{PV}\int_{-\infty}^{1} H'(\nu - k\nu)\,\overline{H(\nu - k\nu)}\,\frac{dk}{k}\right\}, \end{aligned}\right\} \tag{20.38}$$

where A is the area of the profile, (x_c, y_c) are the coordinates of its centroid, and Γ is the circulation. The remarks made in Sect. 19α concerning the use of the H-function apply also here.

S u b m e r g e d c i r c u l a r c y l i n d e r. The appropriate linearization for the circular cylinder is the one associated with deep submergence. Hence, one must try to satisfy the exact boundary condition on the cylinder.

This problem has been treated by LAMB (1913; see also 1932, § 247), HAVELOCK (1927, 1929a, 1936), SRETENSKII (1938), who considers also finite depth, KOCHIN (1937) and HASKIND (1945a), who applies KOCHIN's methods for finite depth. COOMBS (1950) considers the flow about a pair of submerged cylinders, and, as a preliminary, also about a single cylinder; numerical computations are carried though for two cases, one with the centers on a horizontal line and one with them on a vertical line. COOMB's method has wider applicability than just to circular cylinders. In all the cited papers, with the exception of HAVELOCK's and COOMBS', the problem is solved by placing at the center of the circle a dipole modified to satisfy the free-surface condition, i.e., (13.45) with $\alpha = 0$ and $M = 2\pi c a^2$, where a is the radius and c the velocity [the c of (13.45) is taken as $-ih$, h the depth of the center]. This provides, of course, only an approximate solution, for in the presence of a free surface a dipole in a stream no longer generates a circle, as is testified to by the fact that the contour actually generated is subject to a moment. HAVELOCK (1927, 1929a) gave second approximations for drag and lift and later (1936b) a complete solution.

The problem may be treated by a combination of MILNE-THOMSON's Circle Theorem (1956, p. 151) and a formula of KOCHIN's. The former states that if $f(z)$ is the complex potential for a flow with its singularities all at a distance greater than a from the origin and with no solid boundaries, then

$$f(z) + \bar{f}\left(\frac{a^2}{z}\right) \tag{20.39}$$

is the complex potential for a flow with the same singularities but now with a circle of radius a and center at the origin situated in the fluid.

KOCHIN (1937) has proved that if $f(z)$ is a single-valued complex potential for a bounded contour C under a free surface, then

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{z-\zeta}\,d\zeta + \frac{1}{2\pi i}\int_{C_1} \overline{f(\zeta)}\left[\frac{1}{z-\bar{\zeta}} - 2i\nu\,e^{-i\nu z}\int_\infty^z \frac{e^{i\nu u}}{u-\bar{\zeta}}\,du\right] d\bar{\zeta}, \tag{20.40}$$

where C_1 is any contour in the lower half-plane containing C. The formula and its proof are almost identical with that given in (17.15). The first integral in

(20.40) represents a function regular everywhere outside C, the second integral a function regular everywhere in the lower half-plane. If one starts with a function $f(z)$ whose only singularities are contained inside C, then the operation

$$f + \mathfrak{K}\{f\}, \tag{20.41}$$

where

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$$\left.\begin{aligned} \mathfrak{K}\{f\} &= \frac{1}{2\pi i}\int\limits_{C_1} \overline{f(\zeta)}\left[\frac{1}{z-\bar{\zeta}} - 2i\nu\, e^{-i\nu z}\int\limits_{\infty}^{z} \frac{e^{i\nu u}}{u-\bar{\zeta}}\,du\right] d\bar{\zeta}, \\ &= \frac{1}{2\pi i}\int\limits_{C_1} \overline{f(z)}\left[\frac{1}{z-\bar{\zeta}} - 2\pi\nu\, e^{-i\nu(z-\bar{\zeta})} + 2i\nu\, \mathrm{PV}\int\limits_{0}^{\infty} \frac{e^{-iu(z-\bar{\zeta})}}{u-\nu}\,du\right] d\bar{\zeta} \end{aligned}\right\} \tag{20.42}$$

yields a complex potential function satisfying the free-surface condition and having the same singularities as f in the lower half-plane.

On the other hand, if one starts with a complex potential $f(z)$ whose only singularities are in the upper half-plane, then

$$f + \mathfrak{M}\{f\} \tag{20.43}$$

where

$$\mathfrak{M}\{f\} = \bar{f}\left(\frac{a^2}{z+ih} + ih\right) + ich, \quad a < h; \tag{20.44}$$

will be a complex potential for a flow about a circle of radius a and center $-ih$ and with the same singularities as f in the upper half-plane, the singularities of $\mathfrak{M}\{f\}$ being all inside the circle.

We start with the flow $f_0(z) = -cz$ representing a uniform flow from the right; the free-surface condition is satisfied for $y=0$ in a trivial manner. Now form the sequence

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$$f_0,\ f_1 = \mathfrak{M}\{f_0\},\quad f_2 = \mathfrak{K}\{f\}, \ldots, f_{2n+1} = \mathfrak{M}\{f_{2n}\}, f_{2n+2} = \mathfrak{K}\{f_{2n+1}\}, \ldots. \tag{20.45}$$

Then $f_0+f_1, f_2+f_3, f_4+f_5, \ldots$ each represent flows satisfying the boundary condition on the circle; hence, also their sum if the series converges. On the other hand, $f_0, f_1+f_2, f_3+f_4, \ldots$ each represent flows satisfying the free surface condition, and, hence, also the sum if it exists.

Let us now consider the two operators $\mathfrak{M}$ and $\mathfrak{K}$. $\mathfrak{M}$ is always being applied to a function regular and bounded in the lower half-plane. Since $a^2(z+ih)^{-1}+ih$ maps the exterior of the circle on to the the interior, the maximum of $|\mathfrak{M}\{f\}|$ for z in the lower half-plane does not exceed that for $|f|$ within or on the circle. We write this as

$$|\mathfrak{M}\{f\}| \leqq \|f\| \equiv \max_C |f|. \tag{20.46}$$

In particular,

$$\|f_{2n+1}\| \leqq \|f_{2n}\|. \tag{20.47}$$

The operator $\mathfrak{K}$ is always applied here to functions regular everywhere outside and on the boundary of the circle. Hence C_1 may be contracted to C and one can establish the following estimate for z in the lower half-plane:

$$\left.\begin{aligned} |\mathfrak{K}\{f\}| &\leqq a \max_C |f| \left[\frac{1}{|z-\bar{\xi}|} + 2\pi\nu\, e^{\nu(y+\eta)} + 2\nu\, e^{\nu(y+\eta)}\left|\mathrm{Ei}(\nu|y+\eta|)\right|\right] \\ &\leqq a\left[\frac{1}{h-a} + 2\pi\nu\, e^{-\nu(h-a)} + 2\nu\, e^{-\nu(h-a)}\,\mathrm{Ei}\left(\nu(h-a)\right)\right]\max_C |f| \\ &\leqq K\,\|f\|, \end{aligned}\right\} \tag{20.48}$$

where in the second inquality h must be large enough that $\nu(h-a)>0.4$. For fixed values of νa one may select h/a large enough so that K is as small as one wishes, in any case, less than 1. Thus, in particular,

$$\begin{aligned} \|f_{2n}\| &\leqq K\,\|f_{2n-1}\| < \|f_{2n-1}\| \\ &\leqq K^n\,\|f_0\|, \\ \|f_{2n+1}\| &\leqq \|f_{2n}\| \leqq K^n\,\|f_0\|. \end{aligned} \tag{20.50}$$

From this it follows easily that the series

$$f_0+f_1+f_2+\cdots+f_{2n}+f_{2n+1}+\cdots \tag{20.51}$$

with terms defined by (20.45) converges uniformly in the part of the lower half-plane exterior to $|z+ih|<a$.

One may extend the method to flows about more general cylinders by combining the operator $\mathfrak{M}$ with another defined in terms of conformal mapping of the given profile into a circle. The procedure carried through above is a natural generalization of the procedure used by HAVELOCK in his first two papers (1926, 1929a) to find the second approximation. However, in his later paper (1939b) he used a different procedure, one which has also been used by URSELL in analogous problems. This consists in expressing the potential as a sum of multipoles situated at the center and, of course, already modified so as to satisfy the condition on the free surface and as $x\to\infty$. This leads to an infinite set of equations for the coefficients. The method is quite suitable for approximate computation.

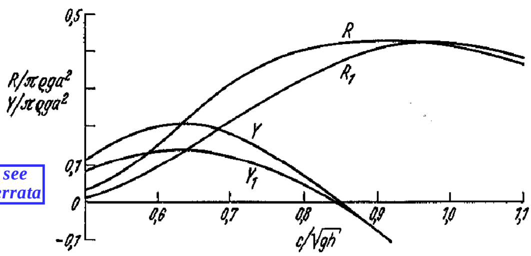

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Fig. 19.

After computation of $H(k)$, the formulas (20.38) can be used to find the force and moment. In the computation of H only the terms with odd indices contribute. This leads to a considerable saving in effort. For example, if one had approximated the flow by the first three terms of (20.51) and computed the force by integrating the pressure over the cylinder, the result would be the same as that obtained by using the H-function evaluated for f_1 alone, and without the need of finding f_2. HAVELOCK has frequently made use of this device without specifically introducing the H-function. Fig. 19 from HAVELOCK (1936b) shows $R=-X$ and Y plotted in units of $\pi g\varrho a^2$ with abscissa $1/\sqrt{\nu h}$ for $a/h=\frac{1}{2}$. The curves labelled R_1 and Y_1 give the result when only the first approximation is used, i.e.,

$$\left.\begin{aligned} H(k) &= 2\pi c\,a^2 k\,e^{-kh} \\ R_1 &= \pi\varrho g a^2\cdot\pi\left(\frac{a}{h}\right)^2\left(\frac{2gh}{c^2}\right)^2 e^{-2gh/c^2}, \\ Y_1 &= -\pi\varrho g a^2\cdot\left(\frac{a}{h}\right)^2\frac{c^2}{2gh}\left[1+\frac{2gh}{c^2}-\left(\frac{2gh}{c^2}\right)^2-\left(\frac{2gh}{c^2}\right)^3 e^{-2gh/c^2}\,\mathrm{Ei}\left(\frac{2gh}{c^2}\right)\right]. \end{aligned}\right\} \tag{20.52}$$

Computation of M gives, on this approximation, the anomalous result

$$M_1 = h\left[1-\frac{c^2}{g\,h}\right]\cdot R.$$

In the formula for Y the terms resulting from buoyancy and circulation are omitted. HAVELOCK (1928b) has investigated the form of the surface over a moving dipole, i.e., over a sphere to the same degree of approximation.

Some submerged three-dimensional bodies. The flow about submerged ellipsoids and bodies of revolution in general has obvious interest in connection with the wave resistance of submarines. As a result there is a considerable amount of both theoretical and experimental work available, and even some tables for computation of wave resistance. Most of the theoretical work does not go beyond the approximation in which one represents the body by the singularity distribution appropriate to an unbounded fluid, but with the potential function for the singularity modified to satisfy the conditions on the free surface and at $x = +\infty$. Thus, to find the flow about a submerged sphere one will in this approximation use a modified dipole with axis in the direction Ox and moment $\frac{1}{2}ca^3$. One should realize, however, that the boundary condition on the body appropriate to deep submergence has not been satisfied. The necessary refinements could be carried through for the sphere in a manner similar to that used for the circular cylinder. Since the sphere (and even more, the circular cylinder) is a poor shape to which to apply perfect-fluid theory, such a computation is of only moderate interest. Both POND (1951, 1952) and HAVELOCK (1952) have considered methods for improving the accuracy with which the boundary condition on bodies of revolution is satisfied. This is particularly important in estimating the moment about the transverse horizontal axis, but, as POND shows, of less importance for the wave resistance.

HAVELOCK (1931a) treated by the approximate method the wave resistance of prolate and oblate spheroids moving both along and at right angles to their axes. Later (1931b) he extended the results to general ellipsoids moving in the direction of the longest axis. WEINBLUM (1936) has considered bodies of revolution using the slender-body approximation, but satisfying it only in the approximate sense described above; his aim was to find forms of minimum wave resistance. WEINBLUM (1951) returned to the problem, taking up in particular numerical computation of the wave resistance for a given shape. Tables and graphs are given to facilitate the computation for certain classes of bodies. Experiments were made by WEINBLUM, AMTSBERG and BOCK (1936) on three forms at several depths. Presumably, more recent experiments exist whose results are not publicly available. A general survey of the theory may be found in BESSHO (1957).

If one has once computed the function $H(k,\vartheta)$ for a source and a dipole, it is usually straightforward to compute it for bodies generated by distributions of sources and dipoles, and hence to compute the force. Let S_1 be a surface containing a single submerged source of strength m at the point (a, b, c), $b<0$ [i.e., (13.36) multiplied by $-m$]; one finds

$$H(k,\vartheta) = 4\pi m\, e^{kb}\, e^{ik(a\cos\vartheta + c\sin\vartheta)}. \tag{20.53}$$

For a dipole of moment M in the direction Ox one finds

$$H(k,\vartheta) = 4\pi i M k\, e^{kb}\, e^{ik(a\cos\vartheta + c\sin\vartheta)} \cos\vartheta. \tag{20.54}$$

These may now be superposed as necessary for either discrete or continuous distributions. Thus, if we write $G(x, y, z; \xi, \eta, \zeta)$ for the function (13.36) with (a, b, c) replaced by (ξ, η, ζ), and if we can express φ for some problem by

$$\varphi = \iint_S \gamma(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta)\, d\sigma, \tag{20.55}$$

then [cf. Eq. (19.20)]

$$H(k,\vartheta) = -4\pi \iint_S e^{k\eta}\, e^{ik(\xi\cos\vartheta + \zeta\sin\vartheta)}\, \gamma(\xi,\eta,\zeta)\, d\sigma. \tag{20.56}$$

Prolate spheroid. We give an example of the preceding remarks. A prolate spheroid of major semi-axis a and minor semi-axes b moving with velocity c in the direction of its major axis can be represented in an unbounded fluid by a distribution of dipoles of moment density

$$\mu(\xi) = A\,c\,(a^2 e^2 - \xi^2), \qquad |\xi| < a\,e, \tag{20.57}$$

where

$$A^{-1} = \frac{4e}{1-e^2} - 2\log\frac{1+e}{1-e}, \qquad e^2 = 1 - \frac{b^2}{a^2},$$

placed along the major axis between the two foci. Hence with the center at depth h one has in this approximation

$$\left.\begin{aligned} H(k\ \vartheta) &= 4\pi\,i\,A\,c\,k\,\mathrm{e}^{-kh}\cos\vartheta \int_{-ae}^{ae} (a^2e^2-\xi^2)\,\mathrm{e}^{ik\xi\cos\vartheta}\,d\xi \\ &= 8\sqrt{2\pi}\,i\,A\,c\,(a\,e)^{\frac{3}{2}}\,\frac{\mathrm{e}^{-kh}}{k^{\frac{1}{2}}\cos^{\frac{1}{2}}\vartheta}\,J_{\frac{3}{2}}(a\,e\,k\cos\vartheta). \end{aligned}\right\} \tag{20.58}$$

Substituting in the first formula of (20.37) one finds

$$R = -X = +128\pi\,\varrho\,\nu\,c^2\,a^3\,e^3\,A^2 \int_0^{\frac{1}{2}\pi} \mathrm{e}^{-2\nu h\sec^2\vartheta}\,[J_{\frac{3}{2}}(a\,e\,\nu\sec\vartheta)]^2\sec^2\vartheta\,d\vartheta. \tag{20.59}$$

Fig. 20 from HAVELOCK (1931a) shows a graphical presentation of $R/\pi\varrho a b^3$ for spheroids with various ratios of a/b and for $h=2b$. In comparing the different curves one should keep in mind the selected vertical scale; one based on displaced fluid, i.e. $R/\frac{4}{3}\pi\varrho g a b^2$, would give the comparison a different aspect.

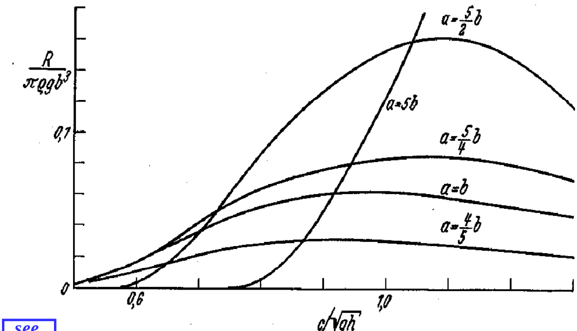

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Fig. 20.

As mentioned earlier, it has been shown by both POND and HAVELOCK that this approximate treatment of the boundary condition on the body is inadequate for computation of the moment. Fig. 21 is from POND (1951, 1958) and shows the computed moment about the center for a Rankine ovoid, i.e., for the body generated in an unbounded fluid by a source and sink of equal strengths moving in the direction of their axis. The dashed curves show the result with the approximate computation; the solid curves were computed by a method in which the boundary condition on the body is more closely satisfied. The length l of the body is 10.5 times the maximum diameter $d=2b$. A positive moment is nose-up.

Slender bodies. It is known that, for bodies of revolution given in the form (20.34) the slender-body boundary condition (20.35) can be satisfied in an infinite fluid by a distribution of sources along the axis of strength density

$$\gamma(x) = \frac{1}{4}\,c\,\frac{d}{dx}\,r^2(x). \tag{20.60}$$

If one assumes that this same distribution of the modified sources (13.36) will satisfy approximately (20.35) then

$$\varphi(x, y, z) = \int_{-a}^{a} \gamma(\xi)\,G(x, y, z; \xi, -h, 0)\,d\xi \tag{20.61}$$

and

$$H(k, \vartheta) = -4\pi\, \mathrm{e}^{-kh} \int_{-a}^{a} \mathrm{e}^{ik\xi\cos\vartheta} \gamma(\xi)\, d\xi. \tag{20.62}$$

From this one finds easily from (20.37)

$$\left.\begin{aligned} R &= -X \\ &= 16\pi\varrho\nu^2 \int_0^{\pi/2} d\vartheta \sec^3\vartheta\, \mathrm{e}^{-2\nu h\sec^2\vartheta} \int_{-a}^{a}\int_{-a}^{a} \gamma(\xi)\gamma(\xi')\cos[k(\xi-\xi')\sec\vartheta]\, d\xi\, d\xi' \\ &= 16\pi\varrho\nu^2 \int_0^{\pi/2} \sec^3\vartheta\, \mathrm{e}^{-2\nu h\sec^2\vartheta}[P^2+Q^2]\, d\vartheta, \end{aligned}\right\} \tag{20.63}$$

where

$$P(\vartheta) = \int_{-a}^{a} \gamma(\xi)\cos(\nu\xi\sec\vartheta)\, d\xi, \qquad Q(\vartheta) = \int_{-a}^{a} \gamma(\xi)\sin(\nu\xi\sec\vartheta)\, d\xi.$$

As mentioned earlier, WEINBLUM (1951) has published tables allowing one to compute R for γ-s representable as certain polynomials. An earlier paper (1936) considers the minimization of R among certain classes of polynomial γ-s. POND (1952) treats the necessary refinements to (20.61) in order to compute the moment. CUMMINS (1954) finds the additional effect of a train of waves on the surface.

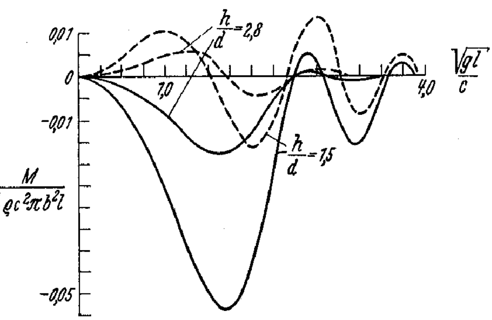

Fig. 21.

Thin ships. Let the equation of the hull be given as in (20.22) by $z = \pm F(x, y)$ in a coordinate system moving with the ship which we take to move with constant velocity c in the direction Ox. If we assume that a steady state has been reached, the boundary condition for the hull appropriate to the thin-ship approximation is, from (20.26) with $\Phi(x, y, z, t) = \varphi(x - ct, y, z)$ and a change to amoving coordinate system,

$$\varphi_z(x, y, \pm 0) = \mp c F_x(x, y) \tag{20.64}$$

for $(x, y, 0)$ in S_0, the centerplane section of the ship at rest. For $(x, y, 0)$ not in S_0 one has $\varphi_z(x, y, \pm 0) = 0$ from symmetry considerations. φ must, of course, also satisfy (20.1) and the condition of vanishing motion as $x \to \infty$.

The boundary conditions may be satisfied immediately by distributing sources (13.26) over S_0. If we again denote the potential function in (13.36) by $G(x, y, z; \xi, \eta, \zeta)$, then, for infinitely deep fluid, the solution is

$$\varphi(x, y, z) = \frac{c}{2\pi} \iint_{S_0} G(x, y, z; \xi, \eta, 0)\, F_x(\xi, \eta)\, d\sigma. \tag{20.65}$$

This follows easily from known theorems in potential theory[1]. (The part of G regular in $y \leqq 0$ does not interfere with the satisfying of (20.64) since the z-derivative of these terms vanishes for $z = 0$.)

The quantity of chief interest is the resistance resulting from the waves. This may be computed by using again (20.56) and (20.37) (and remembering to

[1] See, e.g., O. D. KELLOGG: Foundations of potential theory, pp. 160—166. Berlin: Springer 1929.

take account of both halves of the hull), or by direct integration of the pressure over the hull after taking account of linearization, i.e.

$$R = 2\varrho c \iint_{S_0} \varphi_x(x, y, 0)\, F_x(x, y)\, dx\, dy. \tag{20.66}$$

If the latter form is used, only the single-integral term in G gives a non-vanishing contribution. In either case one finds immediately, again for infinitely deep fluid,

$$\left.\begin{aligned} R &= \frac{4g^2\varrho}{\pi c^2} \int_0^{\pi/2} \sec^3\vartheta\, [P^2(\vartheta) + Q^2(\vartheta)]\, d\vartheta, \\ P &= \iint_{S_0} F_x(x, y)\, e^{\nu y \sec^2\vartheta} \cos(\nu x \sec\vartheta)\, dx\, dy, \\ Q &= \iint_{S_0} F_x(x, y)\, e^{\nu y \sec^2\vartheta} \sin(\nu x \sec\vartheta)\, dx\, dy. \end{aligned}\right\} \tag{20.67}$$

The result may be, and has been, put into a variety of different forms by change of variable and order of integration. We give one of them. Let $\lambda = \sec\vartheta$. Then one may verify that

$$\left.\begin{aligned} R &= \frac{4g^2\varrho}{\pi c^2} \iint_{S_0} dx\, dy \iint_{S_0} d\xi\, d\eta\, F_x(x, y)\, F_\xi(\xi, \eta)\, M(\nu(x-\xi),\ \nu(y+\eta)), \\ M(x, y) &= \int_1^\infty \frac{\lambda^2}{\sqrt{\lambda^2 - 1}}\, e^{\lambda^2 y} \cos\lambda x\, d\lambda. \end{aligned}\right\} \tag{20.68}$$

This expression for R in terms of the hull form and velocity was first given by MICHELL (1898), but derivations by different methods have since been given by many other, e.g. HAVELOCK (1932, 1951), SRETENSKII (1937), KOCHIN (1937), LUNDE (1951), and TIMMAN and VOSSERS (1955). It is usually called "MICHELL's integral".

Because MICHELL's integral gives R directly in terms of the hull geometry it has been intensively investigated by several persons in order to throw light upon the influence of variations of hull form upon wave resistance. Foremost among these investigators has been HAVELOCK, who in a series of notable papers (1923, 1925a, b, 1926a, 1932a, b, 1935) studied the effects of various systematic variations described by the titles of the papers. Much of this work is summarized in HAVELOCK (1926). In addition, there are numerous papers by G.P. WEINBLUM and W.C.S. WIGLEY devoted to comparison of experiment and theory. One can find surveys of much of this and related work, as well as further bibliography, in WIGLEY (1930, 1935, 1949), WEINBLUM (1950), HAVELOCK (1951), LUNDE (1957), and WEHAUSEN (1957), LUNDE's 1951 paper contains derivations of practically all the general theoretical results, including the effect of finite depth, walls, and acceleration. TAKAO INUI (1954) has given an extensive survey of Japanese investigations on wave resistance and related topics, and in a later paper (1957) a complete survey.

In order to allow better exploitation of MICHELL's integral much attention has been given in recent years to its numerical computation. One can find a general discussion in BIRKHOFF and KOTIK (1954), and various special proposals in KABACHINSKII (1947), REINOV (1951), GUILLOTON (1951) and WEINBLUM (1955). The last two papers both contain sets of tables to be used in evaluating MICHELL's integral.

In making a comparison of the theoretically predicted wave resistance with measured wave resistance one must examine critically the experimental method for estimating the wave resistance. The standard method consists in measuring the total resistance, estimating the part of the total resulting from the effects of viscosity, and attributing the difference to wave making. Thus the accuracy of the experimentally estimated wave resistance depends upon the accuracy of the estimated viscous resistance and upon the validity of the assumption that the two may be added. In the case of a very thin body this estimate can be made accurately and, in addition, the physical assumption in the thin-ship linearization is well realized. Fig. 22 from a report by WEINBLUM, KENDRICK and TODD (1952) shows a comparison between estimated and computed values of $R_w/\frac{1}{2}\varrho c^2 S$ for a towed "friction plane" 21 feet long with parabolic ends and 3 foot draft. These experimental data present MICHELL'S integral in a most favorable light. For more ship-like forms the separation of viscous from wave-making resistance is more difficult and the compared values seldom show such striking quantitative agreement, although it is still fair in many cases. We call attention to the fact that MICHELL'S integral predicts the same wave-making resistance no matter in which direction the ship moves.

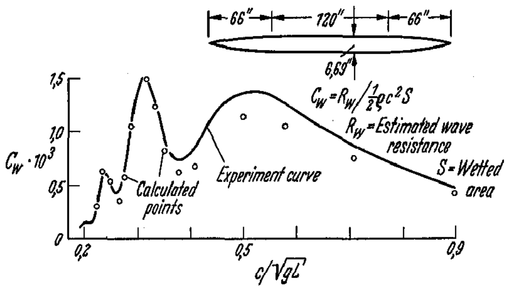

Fig. 22.

So far we have discussed vessels moving in an infinitely deep fluid. However, if for our function G we had taken (13.37) instead of (13.36), the same analysis would have led us to the following expression, first given by SRETENSKII (1937):

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$$\left.\begin{aligned} R &= \frac{2\varrho g c}{\pi} \int_{\mu_h}^{\infty} [P^2(\mu) + Q^2(\mu)] \sqrt{\frac{\mu}{\mu - \nu \tanh \mu h}}\, d\mu, \\ P(\mu) &= \iint_{S_0} F_x(x,y)\, \frac{\cosh \mu(y+h)}{\cosh \mu h} \cos\left(x\sqrt{\nu\mu \tanh \mu h}\right) dx\, dy, \\ Q(\mu) &= \iint_{S_0} F_x(x,y)\, \frac{\cosh \mu(y+h)}{\cosh \mu h} \sin\left(x\sqrt{\nu\mu \tanh \mu h}\right) dx\, dy. \end{aligned}\right\} \quad (20.69)$$

Here μ_h is the nonzero solution of $\mu = \nu \tanh \mu h$ if such exists, i.e. if $c^2/gh > 1$; otherwise $\mu_h = 0$. As $h \to \infty$, $\mu_h \to \nu$ and one obtains one of the forms of MICHELL'S integral.

An expression for the wave resistance of a thin ship moving down the center of a rectangular canal was derived independently by SRETENSKII (1936, 1937) and KELDYSH and SEDOV (1937). The result may be found in LUNDE (1951).

One may naturally ask how the wave pattern illustrated in Fig. 1 for a moving source is related to that for a ship. In the thin-ship approximation, the ship is replaced by a distribution of sources on the centerplane, so that each infinitesimal area of the centerplane contributes to such a pattern according to its strength. However, in many large vessels the middle part of the ship is cylindrical, so that $F_x = 0$ in this region and only the bow and stern regions contribute a nonvanishing source density. Thus, if one replaces the ship by a single source in the bow region and a single sink in the stern region, the resulting wave pattern will approximate to that of a ship, the approximation being better at higher values of the Froude number $c/\sqrt{gL}$. Depending upon the value of $c/\sqrt{gL}$, the transverse wave systems from the two singularities may either reinforce or partially cancel one another. When they are in phase, a larger amount of energy is being left behind in the wave system and the resistance curve shows a maximum, when they are out of phase a minimum, the so-called "humps and hollows" of the resistance curve; these show clearly in Fig. 22. Replacing the ship by a source and sink is, of course, a gross simplification. However, it serves to explain qualitatively certain aspects of a ship's wave pattern and resistance curve, and, in fact, can be given a certain

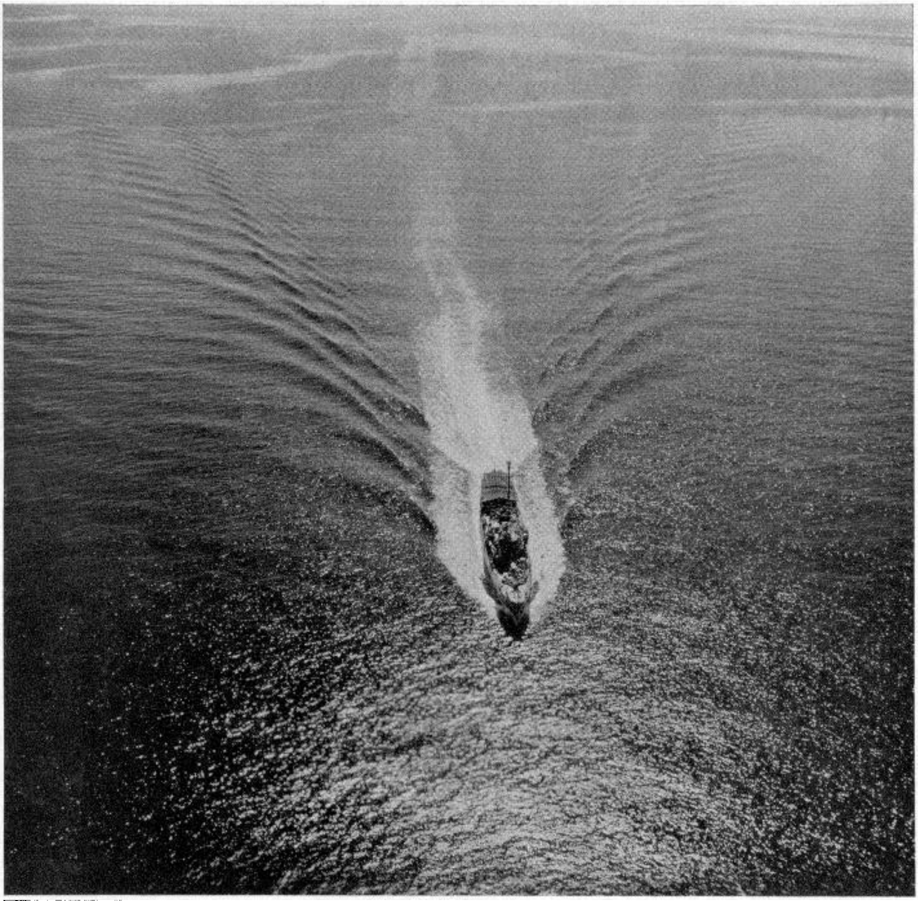

Fig. 23. Ship waves.

amount of validity as an approximate computation of MICHELL's integral for sufficiently large $c/\sqrt{gL}$. For very large values of $c/\sqrt{gL}$, the wave length of the transverse waves along the path, $2\pi c^2/g$, becomes much larger than L and one may approximate the ship by a dipole. Many photographs of the wave pattern made by a fast motor boat fall into this class. The photograph reproduced in Fig. 23 shows clearly the diverging waves from the bow and stern, a third set possibly originating at the forward shoulder, and also the transverse waves, which presumably are here nearly in phase.

The angular opening of the wedge containing the wave pattern should be, according to (13.42) and Fig. 1, 38° 56′ in deep water. This is confirmed only approximately in photographs; difficulty in determining boundaries makes precise confirmation difficult. For ships moving in water of finite depth h the angular opening changes as shown in Fig. 2, and for supercritical velocities, i.e. $c^2 > gh$, there are no transverse waves.

JINNAKA (1957) has recently published a brief survey of the theory of ship waves.

Thin hydrofoils. We take the hydrofoil as described in (20.29) and treat the problem two-dimensionally. Assuming constant velocity c and steady motion and taking our coordinate system moving with the hydrofoil, the boundary condition (20.33) becomes

$$\varphi_y(x, -h \pm 0) = -c\,r'(x) \mp c\,s'(x), \qquad -a < x < a. \tag{20.70}$$

This problem has been treated very thoroughly by KELDYSH and LAVRENT'EV (1936). They follow a procedure quite analogous to that used in Sect. 19α to find the waves generated by a vertical oscillator not in a wall. Distribute vortices of intensity $\gamma(x)$ and sources of strength $\sigma(x)$ along the line $-a < x < a$, $y = -h$, but taking them, of course, modified as in (13.43) in order to satisfy the free-surface condition and the conditions at infinity. To start with, we take $\sigma(x) = -2cs'(x)$. It then follows from the theorem of PLEMELJ-SOKHOTSKII [cf. Eq. (17.18)] that

$$\varphi_y(x, -h+0) - \varphi_y(x, -h-0) = -2cs'(x), \tag{20.71}$$

a step toward satisfying (20.70). We now look for a complex potential in the form

$$f(z) = \int_{-a}^{a} [-2c\,s'(\xi)\,f_s(z;\xi - i\,h) + \gamma(\xi)\,f_v(z;\xi - i\,h)]\,d\xi, \tag{20.72}$$

where we have separated the source and vortex potentials in (13.43). The boundary condition (20.70) now yields an integral equation for $\gamma(x)$ in much the same manner that (17.18) was derived:

$$\left.\begin{aligned}\operatorname{Im}\int_{-a}^{a}[-2cs'(\xi)\,f_s'(x-i\,h;\,\xi-i\,h)+\gamma(\xi)\,f_v'(x-i\,h;\,\xi-i\,h)]\,d\xi = c\,r'(x),\\ -a<x<a.\end{aligned}\right\} \tag{20.73}$$

Noting from the third expression in (13.43) that f_s and f_v are functions of the difference $x-\xi$, we define

$$\begin{aligned}2\pi i\,f_s'(x-i\,h, -i\,h) &= i\left[\frac{1}{x} - \frac{1}{x-2i\,h}\right] - 2\nu\,e^{-i\nu x}\int_{\infty}^{x}\frac{e^{i\nu t}}{t-2i\,h}\,dt\\ &= H(x) + i\,J(x),\end{aligned}$$

$$\begin{aligned}2\pi i\,f_v'(x-i\,h, -i\,h) &= \frac{1}{x} + \frac{1}{x-2i\,h} - 2i\,\nu\,e^{-i\nu x}\int_{\infty}^{x}\frac{e^{i\nu t}}{t-2i\,h}\,dt\\ &= K(x) + i\,I(x).\end{aligned}$$

Here, for example,

$$K(x) = \frac{1}{x} + \frac{x}{x^2+4h^2} + 2\nu\int_{\infty}^{x}\frac{2h\cos\nu(t-x) + t\sin\nu(t-x)}{t^2+4h^2}\,dt.$$

The integral equation (20.73) can now be written in the following form:

$$\int_{-a}^{a}\gamma(\xi)\,K(x-\xi)\,d\xi = -2\pi\,c\,r'(x) + \int_{-a}^{a}2c\,s'(\xi)\,H(x-\xi)\,d\xi, \tag{20.74}$$

where the right-hand side is a known function.

This integral equation is the hydrofoil analogue of the thin-wing integral equation of airfoil theory:

$$\int_{-a}^{a} \gamma(\xi) \frac{1}{x-\xi} \, d\xi = -2\pi c r'(x). \tag{20.75}$$

In the latter equation the kernel is simpler and in addition only the function r' describing the camber and the angle of attack occurs on the right side. Since the wing thickness does not enter into the determination of γ in (20.75) it may be neglected, for only γ is needed to find the lift. The situation is clearly different for hydrofoils. Even a symmetric wing with zero angle of attack may have a circulation, and hence lift. This is a consequence, of course, of the presence of the free surface and the associated wave motion.

KOCHIN (1936) has also considered hydrofoils, but from a somewhat different viewpoint. He has essentially used the "deep-submersion" linearization described first in this section. Thus he must satisfy the exact boundary conditions on the wing as well as the Kutta-Joukowski condition. However, one cannot say here, as one could for an infinite fluid, that his method is more exact than that of KELDYSH and LAVRENT'EV. Their approximation is more accurate the thinner the wing, for a given submersion. KOCHIN'S is more accurate the deeper the submersion, for a given wing.

Eq. (20.74) is not sufficient to determine $\gamma(x)$ uniquely. One must still add some further condition. We shall assume a finite velocity at the trailing edge, i.e.

$$\varphi_y(-a-0, -h) = \int_{-a}^{a} [\gamma(\xi) K(-a-\xi) - 2c s'(\xi) H(-a-\xi)] \, d\xi \quad \text{finite}. \tag{20.76}$$

KELDYSH and LAVRENT'EV propose solving the integral equation (without actually proving that a solution exists) by expanding K, H and γ in a power series in $\tau = a/\nu h$ and then determining recursively the coefficients. Let

$$\left.\begin{aligned}
K(x) &= \frac{1}{x} + \frac{\tau}{a} \sum_{n=0}^{\infty} K_n(2\nu h)\, \tau^n \left(\frac{x}{a}\right)^n, \\
H(x) &= \frac{\tau}{a} \sum_{n=0}^{\infty} H_n(2\nu h)\, \tau^n \left(\frac{x}{a}\right)^n, \\
\gamma(x) &= \sum_{n=0}^{\infty} \gamma_n(x)\, \tau^n.
\end{aligned}\right\} \tag{20.77}$$

Then (20.74) gives the following sequence of integral equations.

$$\left.\begin{aligned}
&\int_{-a}^{a} \gamma_0(\xi) \frac{d\xi}{x-\xi} = -2\pi c r'(x), \\
&\int_{-a}^{a} \gamma_1(\xi) \frac{d\xi}{x-\xi} = \frac{1}{a} H_0 \int_{-a}^{a} 2c s'(\xi)\, d\xi = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\int_{-a}^{a} \gamma_{n+1}(\xi) \frac{d\xi}{x-\xi} = \frac{1}{a^{n+1}} H_n \int_{-a}^{a} 2a s'(\xi)\, (x-\xi)^n d\xi - \\
&\qquad - \sum_{k+l=n} \frac{1}{a^{l+1}} K_l \int_{-a}^{a} \gamma(\xi)\, (x-\xi)^l d\xi, \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{20.78}$$

This procedure has the obvious advantage of reducing the solution to the airfoil integral equation for which an explicit solution satisfying the trailing-edge condition is known. If we denote temporarily the right-hand sides of the Eqs. (20.78) by $F_n(x)$, respectively, then the general solution is[1]

$$\gamma_n(x) = \frac{1}{\pi^2 \sqrt{a^2 - x^2}} \left[\int_{-a}^{a} \frac{F_n(\xi)\sqrt{a^2-\xi^2}}{\xi - x} d\xi + \pi \int_{-a}^{a} \gamma_n(\xi)\, d\xi \right], \tag{20.79}$$

where the value of $\int_{-a}^{a} \gamma_n \, d\xi$ is undetermined. In terms of the series expansion, condition (20.76) states that

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} \tau^n \int_{-a}^{a} \gamma_n(\xi) \frac{d\xi}{x-\xi} + \sum_{n=1}^{\infty} \tau^n \sum_{k+l=n-1} a^{-l-1} K_l \int_{-a}^{a} \gamma_k(\xi)\,(x-\xi)^l\, d\xi - \\ &\qquad - \sum_{n=1}^{\infty} a^{-n-1} H_{n-1} \int_{-a}^{a} 2c\, s'(\xi)\,(x-\xi)^{n-1}\, d\xi \end{aligned}\right\} \tag{20.80}$$

must remain finite for $x \to -a$. We assume $s'(-a)$ finite. γ_k may possibly have a singularity of the form $1/\sqrt{a+x}$ near $x = -a$. However, the integral

$$\int_{-a}^{a} \frac{(x-\xi)^l}{\sqrt{a^2-\xi^2}} d\xi = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (x - a \sin\alpha)^l \, d\alpha,$$

is a polynomial in x for $l \geqq 0$. Thus the last two summations of (20.80) remain finite at $x = -a$. However, the first summations potentially contributes terms like

see errata

$$\int_{-a}^{a} \frac{d\xi}{(x-\xi)\sqrt{a^2-\xi^2}} = \frac{\pi}{\sqrt{x^2-a^2}},$$

In order to avoid this singularity at $x = -a$, we select the total circulation $\int_{-a}^{a} \gamma_n \, d\xi$ so that

$$\int_{-a}^{a} \gamma_n(\xi)\, d\xi = -\frac{1}{\pi} \int_{-a}^{a} \frac{F_n(\xi)\sqrt{a^2-\xi^2}}{\xi + a} d\xi. \tag{20.81}$$

Substituting into (20.79), one finds finally

$$\gamma_n(x) = \frac{1}{\pi^2} \sqrt{\frac{a+x}{a-x}} \int_{-a}^{a} F_n(\xi) \sqrt{\frac{a-\xi}{a+\xi}} \frac{d\xi}{\xi - x}. \tag{20.82}$$

$\gamma(x)$ itself is given by the sum displayed in (20.77). Although the singularity at the trailing edge has been removed, there is still one at the leading edge; this occurs also in thin-airfoil theory and corresponds roughly to the fact that the conditions of linearization (i.e. of small disturbance) are not satisfied near the leading-edge stagnation point.

Keldysh and Lavrent'ev compute the integrals which will be necessary if $r(x)$ and $s(x)$ are given as polynomials and apply their computational method to a flat-plate and circular-arc airfoil at a small angle of attack.

[1] See, e.g., W. Schmeidler: Integralgleichungen ..., pp. 55—56. Leipzig: Akademische Verlagsgesellschaft 1950.

In order to find the force and moment on the wing it is convenient to fall back on the H-function. One finds easily

$$\left.\begin{aligned} H(k) &= e^{-kh}\int_{-a}^{a}[\gamma(\xi) - 2ics'(\xi)]\,e^{-i\lambda s}\,ds, \\ |H(k)|^2 &= e^{-2hk}\int_{-a}^{a}\int_{-a}^{a}\{[\gamma(\xi)\gamma(x) + 4c^2 s'(\xi)\,s'(x)]\cos k(\xi - x) + \\ &\quad + 2c[\gamma(\xi)\,\sigma'(x) - \gamma(x)\,\sigma'(\xi)]\sin k(\xi - x)\}\,d\xi\,dx. \end{aligned}\right\} \tag{20.83}$$

Formulas (20.38) allow one to complete the calculation for special cases.

The theory analogous to that described above for fluid of finite depth h_0 has been carried through by TIKHONOV (1940). He has applied the method to calculate the lift and drag coefficients for a flat plate at a small angle of attack.

Rather than reproduce the graphical presentations of KELDYSH and LAVRENT'EV and TIKHONOV for the flat plate, we shall give instead the lift and drag coefficients for a submerged vortex. Here one may give relatively simple analytic expressions, and the qualitative behavior of the curves is similar to that of a flat plate. The formulas for lift L and drag D are as follows:

$$\left.\begin{aligned} D_\infty &= \varrho\nu\Gamma^2 e^{-2\nu h}, \\ L_\infty &= \varrho\Gamma c - \frac{\varrho\Gamma^2}{4\pi h} + \frac{\varrho\Gamma^2}{\pi}\nu e^{-2\nu h}\,\mathrm{Ei}(2\nu h), \\ D_{h_0} &= \varrho\nu\Gamma^2\frac{\sinh^2 m_0(h_0 - h)}{\nu h_0 - \cosh^2 m_0 h_0} \quad \text{if} \quad \nu h_0 > 1, \quad = 0 \quad \text{if} \quad \nu h_0 < 1, \\ L_{h_0} &= \varrho\Gamma c - \frac{\varrho\Gamma^2}{4\pi}\frac{1}{h_0 - h} + \frac{\varrho\Gamma^2}{2\pi}\int_0^\infty(\nu + k)\,e^{-kh_0}\frac{\sinh 2k(h_0 - h)}{\nu\sinh k h_0 - k\cosh k h_0}\,dk, \end{aligned}\right\} \tag{20.84}$$

where m_0 is the real root of $m = \nu\tanh m h_0$. For finite depth the expression for D stems from the last term in (13.47). The dimensionless coefficients $C_D = Dh/\varrho\Gamma^2$ and $C_L = (L - \varrho c\Gamma)\,h/\varrho\Gamma^2$ are shown in Fig. 24a for infinite depth as functions of c^2/gh and in Figs. 24b and c as functions of c^2/gh_0 for various values of $\beta = h/h_0$. For infinite depth C_L starts with a value $1/4\pi$ and tends asymptotically to $-1/4\pi$, crossing the axis at $c^2/gh = 2.47$. For finite depth the coefficients have a discontinuity at $c^2/gh_0 = 1$. As $c^2/gh_0 \to 0$, $C_D \to 0$, and as $c^2/h_0 \to 1$, $C_D \to \frac{3}{2}\beta(1 - \beta)^2$. For $c^2/gh_0 > 1$, C_L is always negative and increasing with a vertical asymptote at $c^2/gh_0 = 1$ and a horizontal one as $c^2/gh_0 \to \infty$ at $-\beta/4\sin\beta\pi$; these curves start at $\frac{1}{4}\beta\cot\beta\pi$.

Further development of hydrofoil theory has taken place in several directions. HASKIND (1945a) has extended KOCHIN'S "deep-submersion" theory to water of finite depth. However, he does not discuss the steps necessary for fulfilment of the Kutta-Joukowski condition, as does KOCHIN. The lifting-line theory for airfoils of finite span has been extended to hydrofoils by WU (1954), BRESLIN (1957), and HASKIND (1956). PARKIN, PERRY and WU (1956) and LAITONE (1954, 1955) have investigated both theoretically and experimentally the effect of bringing a given hydrofoil so close to the surface that the infinitesimal-wave approximation breaks down completely. There exists also a considerable amount of work on flow about cavitating hydrofoils. However, since the effect of gravity is neglected, this work is not considered in the present article. Experimental data relevant to the theoretical development outlined above are scanty. Reports by BENSON and LAND (1942) and by LAND (1943) give results of an experimental investigation of the effect of depth of submersion. However, the investigations were not designed to test the validity of the theory and do not, for example, include the region of maximum C_D. AUSMAN (1953), in connection with an experimental investigation of the pressure distribution on the upper surface of a hydrofoil, measured the lift coefficient and compared it with that predicted by the thin-hydrofoil theory. The theory failed when gh/c^2 became too small because the associated free surface over the hydrofoil no longer approximated infinitesimal waves, or, in other words, the thin hydrofoil was not thin enough

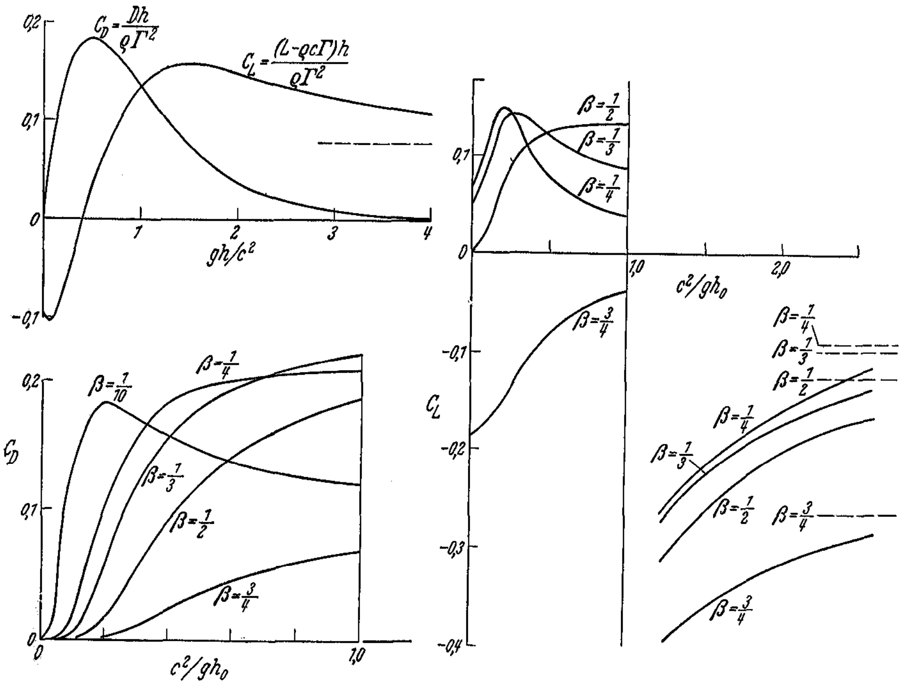

Fig. 24 a—c.

for these values of gh/c^2 for the theory to be applicable. It should also be emphasized that for small values of gh/c^2 the occurrence of cavitation on the upper surface must be taken into account for a complete theory. Recent measurements by NISHIYAMA (1959) show good agreement even for small values of gh/c^2. A comprehensive survey of hydrofoil theory is given in a recent paper by NISHIYAMA (1957).

γ) Planing surfaces. The following discussion is limited to two-dimensional motion, for the theory of three-dimensional planing surfaces for flows with gravity does not appear to have been developed.

For the linearized problem it is natural to consider the planing surface or glider as an approximation to a flat plate moving along the surface of the undisturbed fluid, i.e. the curvature, angle of attack and vertical displacement are all assumed small. In order to formalize the perturbation procedure, let the planing surface be represented by

$$y = h + F(x), \quad |x| \leqq a, \quad F(-a) = 0, \tag{20.85}$$

in coordinates fixed in space, and let the fluid have velocity $-c$ at $x = +\infty$. Thus, we are going to consider the flow to be a perturbation of a uniform flow.

First let us consider briefly in a qualitative fashion the exact solution. There will be a stagnation point A somewhere behind the leading edge and a jet will be thrown out ahead of the glider. We take it to be of thickness b and to make an angle β with OX. If $\Phi = -cx + \varphi(x, y)$ and $\Psi = -cy + \psi(x, y)$ are potential and stream function, respectively, we shall take the free surface ahead of the glider to be given by $\Psi = -bc$ and behind the glider by $\Psi = 0$ (see Fig. 25). Then b/a, and AL/a will all be functions of ga/c^2 and gh/c^2. It will be assumed as one of the boundary conditions of the problem, in analogy with the Kutta-Joukowski condition, that the velocity is continuous at the trialing edge. It is obvious that the flow near the leading edge cannot conform to the requirement that it be a small perturbation of a uniform flow. However, we shall give arguments below to indicate that, except in the neighborhood of the leading edge, this effect is of the second order.

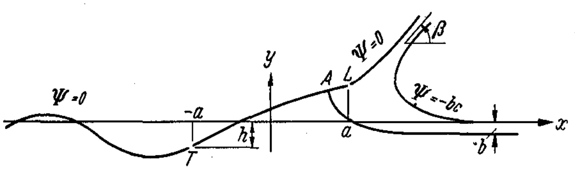

Fig. 25.

In order to get some idea of the relative size of the jet, consider the simpler problem of a flat plate of length l and angle of attack α gliding on a weightless fluid. This problem can be solved exactly [see, e.g., MILNE-THOMSON (1956, § 12.26); A. E. GREEN (1935, 1936)]. The asymptotic expression, for small α, of both the ratios b/l and AL/l is $\frac{\pi}{2}\frac{\alpha^2}{1+\cos\beta}$, i.e. they are both of the second order. We shall suppose that this relation continues to hold when gravity is acting.

We now carry through the perturbation procedure of Sect. 10α [see especially Eq. (10.16)], writing

$$\left.\begin{aligned} \Phi &= -cx + \varepsilon\varphi^{(1)} + \cdots, \quad \Psi = -cy + \varepsilon\psi^{(1)} + \cdots, \quad \eta = \varepsilon\eta^{(1)} + \cdots, \\ h + F(x) &= \varepsilon F^{(1)}(x) + \varepsilon h^{(1)} + \varepsilon^2 h^{(2)} + \cdots, \\ b &= \varepsilon^2 b^{(2)} + \cdots. \end{aligned}\right\} \tag{20.86}$$

Substitution in the exact boundary conditions then yields the following linearized conditions:

$$\left.\begin{aligned} \psi^{(1)}(x,0) - \frac{c^2}{g}\psi_y^{(1)}(x,0) &= 0, \qquad |x| > a, \\ \psi^{(1)}(x,0) &= c\,(h^{(1)} + F^{(1)}(x)), \qquad |x| < a. \end{aligned}\right\} \tag{20.87}$$

The free surface is given by

$$\eta^{(1)}(x) = \frac{1}{c}\psi^{(1)}(x,0) = \frac{c}{g}\varphi_x^{(1)}(x,0), \qquad |x| > a. \tag{20.88}$$

We require as usual that the disturbance vanish as $x \to \infty$. One will expect that the behavior near the leading edge will reflect in some manner the inconsistency of the exact solution with the notion of a small perturbation. It will turn out that it will be necessary to allow a singularity at the leading edge. (Almost the same situation exists in the thin-hydrofoil or thin-wing theory since the stagnation point near the leading edge also prevents the flow in that region from being a small perturbation of a uniform flow.) A singularity at the trailing edge, although mathematically possible, has been specifically proscribed. The strength of the singularity at the leading edge and the elevation h of the trailing edge will be determined as functions of ga/c^2 in the course of solving the problem.

The problem as formulated above has been considered, for infinite depth, by SRETENSKII (1933, 1940), SEDOV (1937), KOCHIN (1938), and MARUO (1951). HASKIND has extended SEDOV's analysis to finite depth (1943a), and later (1955) has treated a glider moving on a wavy surface. YU. S. CHAPLYGIN (1940) has apparently carried through a fairly comprehensive numerical analysis for a flat plate making use of SEDOV's method of analysis [see SRETENSKII (1951, p. 83)]. SRETENSKII's papers are expounded in terms of a flat plate, but the method clearly has wider applicability, as he remarks in his first paper. SRETENSKII's 1940 paper gives the results of rather extensive calculations for flat plates. MARUO's paper is conceptually very similar to those of SRETENSKII, but his method is not quite as efficient for computation. However, he also gives computational results and includes a correction to take account of the failure of the linearized theory near the leading edge. More recently the problem has been considered again by authors unaware of the earlier work. SQUIRE (1957) has analyzed a gliding flat plate by a method similar to that used by SRETENSKII and MARUO. Certain integrals involved in this method have been tabulated by MILLER (1957). CUMBERBATCH (1958) has used a method similar to SEDOV's. Both authors add new results to the earlier work.

Both SEDOV and KOCHIN introduce the complex potential $f(z)=\varphi+i\psi$ and thereafter the function $f'+i\nu f$, $\nu=g/c^2$. Although the two methods are not by any means the same, they have much in common with the treatment of hydrofoils given above. Consequently, we shall outline below the method followed by SRETENSKII.

As a preliminary we need a result from Sect. 21 below. Suppose that a pressure distribution $p(x)$,which we take to be absolutely integrable, is given on the free surface. Then the complex velocity potential must satisfy

$$\operatorname{Re}\{f'(x+i0)+i\nu f(x+i0)\}=\frac{1}{\varrho c}\,p(x), \tag{20.89}$$

and the free surface is given by

$$y=\eta(x)=\frac{1}{c}\,\psi(x,0)=\frac{c}{g}\,\varphi_x(x,0)-\frac{1}{\varrho g}\,p(x). \tag{20.90}$$

The function $f(z)$ which satisfies (20.89) and which vanishes as $x\to\infty$ can be written in several forms, of which we select the following [see Eq. (21.38)]:

$$f(z)=\frac{1}{\pi i\varrho c}\int_{-\infty}^{\infty}d\xi\,p(\xi)\,\mathrm{PV}\int_{0}^{\infty}\frac{-\,\mathrm{e}^{-i\lambda(z-\xi)}}{\lambda-\nu}\,d\lambda-\frac{1}{\varrho c}\int_{-\infty}^{\infty}p(\xi)\,\mathrm{e}^{-i\nu(z-\xi)}\,d\xi. \tag{20.91}$$

The free surface is given by

$$\left.\begin{aligned}\eta(x)=\lim_{y\to-0}\frac{1}{\pi\varrho c^2}\int_{-\infty}^{\infty}d\xi\,p(\xi)\,\mathrm{PV}\int_{0}^{\infty}\frac{\cos\lambda(x-\xi)}{\lambda-\nu}\,\mathrm{e}^{\lambda y}\,d\lambda+\\ +\frac{\mathrm{e}^{\nu y}}{\varrho c}\int_{-\infty}^{\infty}p(\xi)\sin\nu(x-\xi)\,d\xi;\end{aligned}\right\} \tag{20.92}$$

the reason for leaving y explicitly in the formulas will appear below.

When a glider is moving on a free surface, the streamline $y=c^{-1}\psi(x,0)$ will consist partly of free surface, where $p(x)=0$, and partly of the wetted surface of the glider, where $p(x)$ is some unknown function.

Eq. (20.92) may then be written as the following integral equation for this unknown function $p(x)$:

$$h + F(x) = \lim_{y \to -0} \frac{1}{\pi \varrho c^2} \int_{-a}^{a} d\xi\, p(\xi)\, \mathrm{PV} \int_0^\infty \frac{\cos\lambda(x-\xi)}{\lambda - \nu} e^{\lambda y}\, d\lambda + \frac{e^{\nu y}}{\varrho c} \int_{-\infty}^{\infty} p(\xi) \sin\nu(x-\xi)\, d\xi, \quad |x| < a. \tag{20.93}$$

Once $p(x)$ has been determined, one may substitute back into (20.92) in order to find the form of the free surface for $|x| > a$.

It is possible to work directly with (20.93), and this is the procedure followed by MARUO. However, SRETENSKII differentiates twice with respect to x and adds ν^2 times (20.93). This yields

$$\begin{aligned} F''(x) + \nu^2 F(x) + \nu^2 h &= \lim_{y \to -0} \frac{-1}{\pi \varrho c^2} \int_{-a}^{a} d\xi\, p(\xi) \int_0^\infty (\lambda + \nu) \cos\lambda(x-\xi)\, e^{\lambda y}\, d\lambda \\ &= \lim_{y \to 0} \frac{1}{\pi \varrho c^2} \int_{-a}^{a} p(\xi) \left(\nu + \frac{\partial}{\partial y}\right) \frac{y}{(x-\xi)^2 + y^2}\, d\xi \\ &= -\frac{\nu}{\varrho c^2} p(x) - \lim_{y \to 0} \frac{1}{\pi \varrho c^2} \frac{\partial}{\partial x} \int_{-a}^{a} p(\xi) \frac{x-\xi}{(x-\xi)^2 + y^2}\, d\xi \\ &= \frac{-\nu}{\varrho c^2} p(x) - \frac{1}{\pi \varrho c^2} \frac{\partial}{\partial x} \mathrm{PV} \int_{-a}^{a} \frac{p(\xi)}{x-\xi}\, d\xi. \end{aligned} \tag{20.94}$$

Although this last equation is a necessary condition for $p(x)$, it obviously cannot determine it uniquely, for the last term of (20.93), assuring vanishing of the disturbance far ahead of the glider, was lost in the formation of (20.94). Thus one still has need for (20.93). Eq. (20.94) is essentially the equation derived by SRETENSKII.

Let us now integrate (20.94) with respect to x from $x = -a$ to x, and denote

$$P(x) = \frac{1}{\varrho c^2} \int_{-a}^{x} p(\xi)\, d\xi.$$

Then Eq. (20.94) becomes

$$F'(x) - F'(-a) + \nu^2 \int_{-a}^{x} F(\xi)\, d\xi + \nu^2 h(x+a) = -\nu P(x) - \frac{1}{\pi} \int_{-a}^{a} \frac{P'(\xi)}{x-\xi}\, d\xi, \tag{20.95}$$

where an additive constant has been discarded since h itself is an undetermined constant. Eq. (20.95) is just PRANDTL's integro-differential equation for the circulation about an airfoil of finite span[1]. Thus known methods for solving the airfoil equation can be carried over to the study of this equation. However, the solutions themselves cannot be taken over directly, for different boundary conditions are impcsed: in the airfoil equation the unknown function is the circulation $\Gamma(x)$ and it is usually assumed that $\Gamma(-x) = \Gamma(x)$ and $\Gamma(-a) = \Gamma(a) = 0$;

[1] See, e.g., N. I. MUSKHELISHVILI: Singular integral equations, Chap. 17. Groningen: Noordhoff 1953.

in the present problem $P(x)$ is not necessarily symmetric and $P(-a) = P'(-a) = 0$, but $P(a)$ is not restricted except to be finite. The theory of the Prandtl integro-differential equation without the customary additional requirements associated with airplane wings has been developed by L. G. MAGNARADZE [Soobshch. Akad. Nauk Gruzin. SSR **3**, 503—508 (1942)].

The equation can be solved by an extension of GLAUERT's method[1]. This is the method which has been used by both MARUO and SRETENSKII. However, each expands $P' = p$ rather than P in a Fourier series in order to obtain the correct behavior at the two end points. Introduce the new variables ϑ and γ by the equations

$$x = -a\cos\vartheta, \qquad \xi = -a\cos\gamma$$

and assume the following expansion for $p(x)$:

$$\left.\begin{aligned} \frac{1}{\varrho c^2} p(x) &= \frac{1}{\varrho c^2} p(-a\cos\vartheta) = a_0 \tan\frac{1}{2}\vartheta + a_1\sin\vartheta + \cdots + a_n\sin n\vartheta + \cdots \\ &= a_0\sqrt{\frac{a-x}{a+x}} + a_1\sqrt{a^2-x^2} + \cdots. \end{aligned}\right\} \tag{20.96}$$

MARUO substitutes (20.96) into (20.93), SRETENSKII into (20.94). The latter, which seems less laborious, leads to

$$\left.\begin{aligned} &a\left[F''(-a\cos\vartheta) + \nu^2 F(-a\cos\vartheta) + \nu^2 h\right]\sin\vartheta \\ &\quad = -\nu a a_0(1-\cos\vartheta) - \nu a\sum_{n=1}^{\infty} a_n \sin\vartheta\sin n\vartheta - \sum_{n=1}^{\infty} n a_n \sin n\vartheta. \end{aligned}\right\} \tag{20.97}$$

We shall not discuss SRETENSKII 's further steps to determine the coefficients a_n. However, they lead to expressions of the following kinds for the coefficients:

$$\left.\begin{aligned} a_{2n-1} &= A_{2n-1} a\nu^2 h + B_{2n-1} a\nu a_0 + C_{2n-1}, \\ a_{2n} &= B_{2n} a\nu a_0 + C_{2n}, \quad n = 1, 2, \ldots, \end{aligned}\right\} \tag{20.98}$$

where A_n, B_n, C_n are functions of νa. Substitution of the coefficients into (20.93) and into (20.93) differentiated once with respect to x and evaluated at $x = -a$ results in equations of the form

$$\left.\begin{aligned} \nu h &= Q_1 a_0 + R_1 \nu h + S_1, \\ F'(-a) &= Q_2 a_0 + R_2 \nu h + S_2, \end{aligned}\right\} \tag{20.99}$$

where Q_i, R_i, S_i are functions of νa; these equations may be used to determine νh and a_0 as functions of νa. As long as $a_0 \neq 0$ there will be a singularity at the leading edge.

Once $p(x)$ has been determined approximately, one can compute the lift, drag and moment about, say, the center. To the order of approximation appropriate to the linear theory they are

$$\left.\begin{aligned} L &= \int_{-a}^{a} p(x)\,dx, \\ R &= \int_{-a}^{a} p(x)\,f'(x)\,dx, \\ M &= \int_{-a}^{a} p(x)\,x\,dx. \end{aligned}\right\} \tag{20.100}$$

[1] H. GLAUERT: The elements of airfoil and airscrew theory, Chap. XI. Cambridge 1943.

For the flat-plate glider it is possible to give the following asymptotic expressions for these quantities when $\nu a \to 0$.

$$\left.\begin{aligned} L &= \pi a \varrho c^2 \alpha \left[1 - \nu a \left(\pi + \frac{4}{\pi}\right)\right] + O(\nu^2 a^2 \log \nu a), \qquad R = \alpha L, \\ M &= \frac{1}{2} \pi a^2 \varrho c^2 \alpha \left[1 - \nu a \left(\pi + \frac{8}{3\pi}\right)\right] + O(\nu^4 a^4 \log \nu a). \end{aligned}\right\} \tag{20.101}$$

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There were first given by SEDOV, but are also derived in the papers by KOCHIN and SRETENSKII.

Fig. 26 reproduces several of SRETENSKII's computed pressure distributions for a flat plate. The predictions of the linearized theory cannot, of course, be expected to be

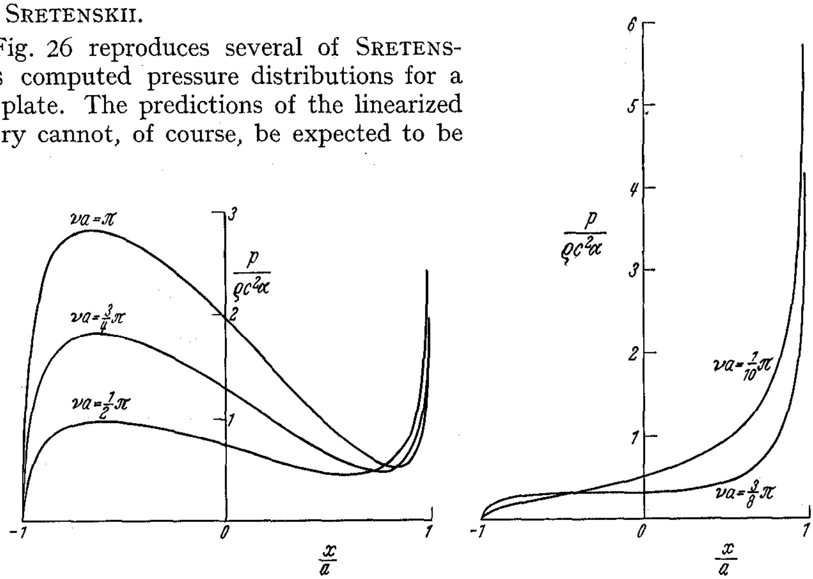

Fig. 26.

accurate near the leading edge. MARUO has corrected his computed points in this region by using the exact theory for a weightless fluid. Both MARUO and SRETENSKII give further computational results which we do not reproduce. MARUO (1959) has also provided experimental confirmation of the predicted pressure distributions.

21. Waves resulting from variable pressure distributions. In the situations considered up to now in this chapter the pressure at the free surface has been taken as constant. We now consider the result of allowing the pressure over the free surface to be a given function of both position and time. Otherwise the fluid is taken to be infinite in horizontal extent and to be either infinitely deep or of uniform depth h. The time variation in pressure will be limited to two cases. In Sect. 21α a periodically varying pressure is considered; in Sect. 21β the pressure is taken to move with uniform velocity; Sect. 21γ gives some references to a combination of these two. In Sect. 22 waves from pressure distributions will be considered again in connection with initial-value problems. Since the methods for finding the velocity potential are similar in most respects to those used in finding the velocity potential for a source, we shall, with one exception, give the results without proof.

Just as in the cases of the stationary source of periodic strength and the moving source of constant strength treated in Sect. 13γ, we must in the present situation impose boundary conditions at infinity in order to ensure a unique solution. The imposed conditions, namely the radiation condition and the vanishing of the fluid motion far ahead, respectively, are selected as being physically reasonable. However, one may proceed differently, derive formulas analogous to (13.50) and (13.51) and then find the limit as $t \to \infty$. The resulting velocity potentials automatically satisfy the correct boundary conditions at infinity. This method has been used, for example, by G. GREEN (1948) in the two-dimensional problems considered in the following two sections, and also by STOKER (1953, 1954).

The theory of wave generation by pressure distributions has an obvious application in oceanographic problems. However, the theory was apparently first developed in an attempt to explain the wave pattern produced by a ship. We shall not attempt to disentangle the history of the subject. For the material covered in Sect. 21β we call attention to a survey by J. K. LUNDE (1951b) which also contains a useful bibliography.

α) Pressure distributions periodic in time. Three dimensions. The boundary conditions have already been given in Sect. 11. If Φ and p are represented by

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$$\left.\begin{aligned} &\Phi(x, y, z, t) = \mathrm{Re}\,\varphi(x, y, z)\, e^{-i\sigma t}, \qquad p(x, y, z\, t) = p(x, z)\, e^{-i\sigma t}, \\ &\varphi = \varphi_1 + i\varphi_2, \qquad p = p_1 + i p_2, \end{aligned}\right\} \tag{21.1}$$

then the condition on the free surface may be written

$$\varphi_y(x, 0, z) - \frac{\sigma^2}{g}\varphi(x, 0, z) = \frac{i\sigma}{\varrho g} p(x, z), \tag{21.2}$$

and the form of the surface is given by

$$\eta(x, z, t) = \mathrm{Re}\left\{\frac{i\sigma}{g}\varphi(x, 0, z) - \frac{1}{\varrho g} p(x, z)\right\} e^{-i\sigma t}. \tag{21.3}$$

In addition, a radiation condition is assumed at infinity [see fifth Eq. (13.9)] and a condition appropriate to the depth of fluid. We shall also assume $p(x, z)$ to be absolutely integrable.

The velocity potential can be expressed as follows:

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$$\left.\begin{aligned} \varphi(x, y, z) = &\frac{-i\sigma}{2\pi\varrho g}\int_{-\infty}^{\infty}\int_{\infty}^{\infty} p(\xi, \zeta)\, d\xi\, d\zeta\; \mathrm{PV}\!\int_{\infty}^{\infty}\frac{k\, e^{ky}}{k-\nu} J_0\!\left(k\sqrt{(x-\xi)^2 + (z-\zeta)^2}\right) dk + \\ &+ \frac{\sigma\nu\, e^{\nu y}}{2\varrho g}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\xi, \zeta)\, J_0\!\left(\nu\sqrt{(x-\xi)^2 + (z-\zeta)^2}\right) d\xi\, d\zeta, \qquad \nu = \frac{\sigma^2}{g}; \end{aligned}\right\} \tag{21.3}$$

and in cylindrical coordinates $x = R\cos\alpha$, $z = R\sin\alpha$ in the form

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$$\left.\begin{aligned} \varphi(R, \alpha, y) = &\frac{-i\sigma}{2\pi\varrho g}\int_0^{2\pi}\int_0^{\infty} p(R', \alpha')\, R'\, dR'\, d\alpha'\; \mathrm{PV}\!\int_{-\infty}^{\infty}\frac{k\, e^{ky}}{k-\nu}\times \\ &\times J_0\!\left(k\sqrt{R^2 + R'^2 - 2RR'\cos(\alpha'-\alpha)}\right) dk + \frac{\sigma\nu\, e^{\nu y}}{2\varrho g}\int_0^{2\pi}\int_0^{\infty} p(R', \alpha')\times \\ &\times J_0\!\left(\nu\sqrt{R^2 + R'^2 - 2RR'\cos(\alpha'-\alpha)}\right) R'\, dR'\, d\alpha'. \end{aligned}\right\} \tag{21.4}$$

The addition theorem for J_0 allows one to write[1]

$$\left.\begin{aligned} J_0\left(k\sqrt{R^2+R'^2-2RR'\cos(\alpha'-\alpha)}\right) &= \sum_{n=0}^{\infty} \varepsilon_n J_n(kR)\, J_n(kR') \cos n(\alpha'-\alpha)\,, \\ \varepsilon_0 = 1, \qquad \varepsilon_n &= 2, \qquad n \geqq 1\,. \end{aligned}\right\} \tag{21.5}$$

If p is independent of α, one may derive easily

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$$\left.\begin{aligned} \varphi(R,y) = \frac{-i\sigma}{\varrho g} \int_0^{\infty} p(R')\, R'\, dR'\, \mathrm{PV}\!\int \frac{k\, e^{ky}}{k-\nu} J_0(kR)\, J_0(kR')\, dk + \\ + \frac{\pi\sigma\nu\, e^{\nu y}}{\varrho g} J_0(\nu R) \int_0^{\infty} p(R')\, J_0(\nu R')\, R'\, dR'. \end{aligned}\right\} \tag{21.6}$$

see errata The asymptotic form for large R of (21.6) is a relatively simpel expression:

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$$\varphi(R,y) \sim \frac{\pi\sigma\nu\, e^{\nu y}}{\varrho g} \sqrt{\frac{2}{\pi\nu R}}\, e^{i\left(\nu R-\frac{\pi}{4}\right)} \int_0^{\infty} p(R')\, J_0(\nu R')\, R'\, dR'. \tag{21.7}$$

We note in passing that the potential function (21.3) or (21.4) can also be obtained see errata as a distribution of sources on the surface [see HUDIMAC (1953, p. 78)]. This may be easily verified as follows. In (13.17″) let $b=0$. Then, using (13.12), one obtains (substituting ξ, ζ for a, c)

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$$2\,\mathrm{PV}\!\int_0^{\infty} \frac{k}{k-\nu} e^{ky} J_0\left(k\sqrt{(x-\xi)^2+(z-\zeta)^2}\right) dk + i\, 2\pi\nu\, e^{\nu y} J_0\left(\nu\sqrt{(x-\xi)^2+(z-\zeta)^2}\right).$$

see errata A distribution over the plane $y=0$ of these sources of strength $+i\sigma p(\xi,\zeta)/4\pi\varrho g$ yields (21.3) (we recall that a source of strength m behaves like $-m/r$ near the singularity).

The rate at which the pressure distribution does work upon the fluid can be calculated directly or by using Eq. (8.2). Consider the volume of fluid contained in a large cylinder of radius R_0. Then, from (8.2), after appropriate linearization, the rate of increase of energy of the fluid is given by

$$\begin{aligned} \frac{dE}{dt} = \mathrm{Re}\Bigg\{ &\int_0^{2\pi}\int_0^{R_0} p(R,\alpha,t)\, \Phi(R,\alpha,0,t)\, R\, dR\, d\alpha + \\ &+ \varrho \int_0^{2\pi}\int_{-\infty}^{0} \Phi_t(R_0,\alpha,y,t)\, \Phi_R(R_0\,\alpha,y,t)\, R_0\, dy\, d\alpha \Bigg\}. \end{aligned}$$

Now substitute (21.1) and take the average over a period, which will clearly be zero. The result may be written:

$$\left.\begin{aligned} 0 = \left[\frac{dE}{dt}\right]_{\mathrm{av}} = \mathrm{Re}\Big\{ -\tfrac{1}{2} \iint p(R,\alpha)\, \overline{\varphi}_y(R,\alpha,0)\, R\, dR\, d\alpha + \\ + \tfrac{1}{2}\varrho\sigma \iint i\, \varphi_R(R_0,\alpha,y)\, \overline{\varphi}(R_0,\alpha,y)\, R_0\, dy\, d\alpha \Big\}. \end{aligned}\right\} \tag{21.8}$$

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[1] See G. N. WATSON: A treatise on the theory of Bessel functions, p. 353. Cambridge 1944.

The first integral gives the average rate W_{av} at which the pressure distribution is working upon the fluid. It must equal minus the second integral. If $p(R,\alpha)=p(R)$, then we may apply (21.7) to obtain a relatively simple expression for the average rate over the whole fluid:

$$W_{\mathrm{av}}=\frac{\pi^2\sigma\nu^2}{\varrho g}\left|\int_0^\infty p(R')\,J_0(\nu R')\,R'\,dR'\right|^2. \tag{21.9}$$

To carry through the computation when p is not circularly symmetric is more complicated arithmetically, but can be carried through by use of (21.5).

One can find an investigation of the waves resulting from a doubly modulated pressure distribution over a rectangular domain,

$$p=A\,e^{-i\sigma t}\cos m x\cos n z,\qquad |x|\leqq a,\qquad |z|\leqq b,$$

in a paper of SRETENSKII (1956).

If the fluid is of uniform depth h, the expression for the velocity potential is

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$$\left.\begin{aligned}\varphi(x,y,z)=\frac{-i\sigma}{2\pi\varrho g}\iint_{-\infty}^{\infty}p(\xi,\zeta)\,d\xi\,d\zeta\,\mathrm{PV}\int_0^\infty\frac{k\cosh k(y+h)}{k\sinh kh-\nu\cosh kh}\,\times\\ \times J_0\!\left(k\sqrt{(x-\xi)^2+(z-\zeta)^2}\right)+\frac{\sigma}{2\varrho g}\,\frac{m_0\cosh m_0(y+h)\sinh m_0 h}{\nu h+\sinh^2 m_0 h}\,\times\\ \times\iint_{-\infty}^{\infty}p(\xi,\zeta)\,J_0\!\left(m_0\sqrt{(x-\xi)^2+(z-\zeta^2)}\right)d\xi\,d\zeta,\end{aligned}\right\} \tag{21.10}$$

where, as usual, m_0 is the real solution of

$$m_0\tanh m_0 h-\nu=0.$$

Other forms of this expression similar to (21.4), (21.6) and (21.7) can be found with no difficulty. We give only the analogue of (21.9):

$$W_{\mathrm{av}}=\frac{1}{2}\pi^2\frac{\sigma\nu m_0}{\varrho g}\,\frac{\sinh 2m_0 h}{\nu h+\sinh^2 m_0 h}\left|\int_0^\infty p(R')\,J_0(\nu R')\,R'\,dR'\right|^2. \tag{21.11}$$

The identities following (13.18) may be used to put both (21.10) and (21.11) into other forms.

Two dimensions. The derivation of the velocity potential will be carried through, at least in part, since it illustrates a nice application of the Plemelj-Sokhotskii formulas. Two complex units will be introduced, as described at the end of Sect. 11. That is, we shall write

$$\left.\begin{aligned}\Phi(x,y,t)&=\varphi_1(x,y)\cos\sigma t+\varphi_2(x,y)\sin\sigma t=\mathrm{Re}_j\,\varphi\,e^{-j\sigma t},\\ p(x,t)&=p_1(x)\cos\sigma t+p_2(x)\sin\sigma t=\mathrm{Re}_j\,p\,e^{-j\sigma t},\\ \varphi&=\varphi_1+j\varphi_2,\qquad p=p_1+j p_2,\end{aligned}\right\} \tag{21.12}$$

and also introduce a stream function $\psi=\psi_1+j\psi_2$ and a complex potential

$$f(z)=f_1(z)+j f_2(z),\qquad f_k=\varphi_k+i\psi_k,\qquad k=1,2. \tag{21.13}$$

Then the boundary condition on the free surface may be formulated as follows

$$\mathrm{Im}_i\{f'(x-i0)+i\nu f(x-i0)\}=-\frac{\sigma}{\varrho g}\,j\,p(x). \tag{21.14}$$

The definition of $g \equiv f' + i\nu f$ may be extended to the whole complex z-plane by reflection, i.e.

$$g(x+iy) = \overline{g(x-iy)}, \quad y > 0.$$

Then the condition (21.14) may be written in the form

$$\mathrm{Im}_i \{f'(x \pm i\,0) + i\nu f(x \pm i\,0)\} = \pm \frac{\sigma}{\varrho g} j\, p(x). \tag{21.15}$$

We shall suppose $p(x)$ to be absolutely integrable on the infinite interval. In addition, we shall suppose that either $p(x)$ satisfies a Hölder condition (and is hence continuous) on the whole infinite interval, or else that there are a finite number of segments $(-\infty, b_1)$, $(a_2, b_2), \ldots, (a_r, \infty)$, $b_n < a_{n+1}$, such that $p(x)$ satisfies a Hölder condition on any closed segment interior to one of the above segments, and at an end-point may be expressed in the form

$$p(x) = \frac{q(x)}{(x-c)^\alpha}, \quad 0 \leqq \alpha < 1, \quad c = a_i \text{ or } b_i,$$

where $q(x)$ satisfies a Hölder condition at the end. Here a Hölder condition means that for any pair of points x_1, x_2, $p(x)$ satisfies

$$|p(x_1) - p(x_2)| < A\,|x_1 - x_2|^\mu, \quad \mu > 0.$$

In the first case f' will be assumed to have no singularities in the whole lower half-plane. In the second case the behavior of $f'(z)$ near an end-point c will be restricted so that it must satisfy

$$|f'(z)| < \frac{C}{|z-c|^\alpha}, \quad 0 \leqq \alpha < 1.$$

As usual it will be assumed that $|f'|$ is bounded as $z \to \infty$ and that only outgoing waves are generated.

The solution of this boundary-value problem for the function $g(z) = f' + i\nu f$ is determined, up to an additive real constant which may be discarded here, by[1]

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$$f'(z) + i\nu f(z) = j \frac{\sigma}{\pi \varrho g} \int_{-\infty}^{\infty} \frac{p(s)}{z-s}\, ds, \quad y < 0. \tag{21.16}$$

After integrating the differential equation and selecting the solution so as to represent outgoing waves at $x = \pm\infty$, one obtains finally [the derivation is similar to that of (13.28)]

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$$f(z) = j \frac{\sigma}{\pi \varrho g} e^{-i\nu z} \int_{-\infty}^{\infty} p(s)\, ds \int_{\infty}^{z} \frac{e^{i\nu\zeta}}{\zeta - s}\, d\zeta + \frac{\sigma}{\varrho g}(1 + ij)\, e^{-i\nu z} \int_{-\infty}^{\infty} p(s)\, e^{i\nu s}\, ds, \tag{21.17}$$

where the path of integration for ζ is taken in the lower half-plane. The asymptotic form of the time-dependent velocity potential is given by

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$$\mathrm{Re}_j f(z)\, e^{-j\sigma t} \sim \frac{\sigma}{\varrho g} e^{-i(\nu z \mp \sigma t)} \int_{-\infty}^{\infty} e^{i\nu s} [p_1(s) \mp i\, p_2(s)]\, ds \quad \text{as} \quad x \to \pm\infty, \tag{21.18}$$

[1] See, e.g., N. I. MUSKHELISHVILI: Singular integral equations, §§ 43, 78. Groningen: Noordhoff 1953.

and the asymptotic form of the free surface by

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$$\eta(x,t) \sim \mathrm{Im}_i \left\{ \frac{\sigma^2}{\varrho g^2} \mathrm{e}^{-i(\nu x \mp \sigma t)} \int_{-\infty}^{\infty} \mathrm{e}^{i\nu s} [p_1 \mp i p_2]\, ds \right\} \quad \text{as} \quad x \to \pm\infty. \tag{21.19}$$

From this last expression one can easily derive the average rate at which the pressure system is transferring energy to the fluid:

$$\left.\begin{aligned} W_{\mathrm{av}} &= \frac{1}{4}\,\frac{\sigma\nu}{\varrho g}\left\{\left|\int_{-\infty}^{\infty}(p_1 - i p_2)\,\mathrm{e}^{i\nu s}\,ds\right|^2 + \left|\int_{-\infty}^{\infty}(p_1 + i p_2)\,\mathrm{e}^{i\nu s}\,ds\right|^2\right\} \\ &= \frac{1}{2}\,\frac{\sigma\nu}{\varrho g}\left\{\left[\int_{-\infty}^{\infty} p_1 \cos\nu s\,ds\right]^2 + \left[\int_{-\infty}^{\infty} p_1 \sin\nu s\,ds\right]^2 + \right. \\ &\quad \left. + \left[\int_{-\infty}^{\infty} p_2 \cos\nu s\,ds\right]^2 + \left[\int_{-\infty}^{\infty} p_2 \sin\nu s\,ds\right]^2\right\}. \end{aligned}\right\} \tag{21.20}$$

The expression for $f(z)$ can be put into several different forms by changing variables and deforming the path of integration appropriately. Thus, if one introduces a new variable λ by $\nu(\zeta - z) = -\lambda(z - s)$ and deforms the resulting path to the x-axis, one obtains

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$$f(z) = -j\,\frac{\sigma}{\pi\varrho g}\int_{-\infty}^{\infty} p(s)\,ds\;\mathrm{PV}\!\int_0^{\infty}\frac{\mathrm{e}^{-ik(z-s)}}{k-\nu}\,dk + \frac{\sigma}{\varrho g}\,\mathrm{e}^{-i\nu z}\int_{-\infty}^{\infty} p(s)\,\mathrm{e}^{i\nu s}\,ds. \tag{21.21}$$

A different deformation of the path leads to

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$$\left.\begin{aligned} f(z) &= -j\,\frac{\sigma}{\pi\varrho g}\int_{-\infty}^{x} p(s)\,ds\int_0^{\infty}\frac{\mathrm{e}^{-\mu(z-s)}}{\mu - i\nu}\,d\mu - j\,\frac{\sigma}{\pi\varrho g}\int_{x}^{\infty} p(s)\,ds\int_0^{\infty}\frac{\mathrm{e}^{\mu(z-s)}}{\mu + i\nu}\,d\mu + \\ &\quad + \frac{\sigma}{\varrho g}(1 + ij)\int_{-\infty}^{x} p(s)\,\mathrm{e}^{-i\nu(z-s)}\,ds + \frac{\sigma}{\varrho g}(1 - ij)\int_{x}^{\infty} p(s)\,\mathrm{e}^{-i\nu(z-s)}\,ds. \end{aligned}\right\} \tag{21.22}$$

For fluid of depth h an expression for the complex velocity potential analogous to (21.10) and (21.21) is

$$\left.\begin{aligned} f(z) &= -j\,\frac{\sigma}{\pi\varrho g}\int_{-\infty}^{\infty} ds\,p(s)\;\mathrm{PV}\!\int_0^{\infty}\frac{\cos k(z - s + ih)}{k\sinh kh - \nu\cosh kh}\,dk + \\ &\quad + \frac{\sigma}{\varrho g}\,\frac{\sinh m_0 h}{\nu h + \sinh^2 m_0 h}\int_{-\infty}^{\infty} p(s)\cos m_0(z - s + ih)\,ds. \end{aligned}\right\} \tag{21.23}$$

One will find both the two- and the three-dimensional case of a periodic pressure distribution over infinitely deep water discussed in LAMB (1904, pp. 387—393). STOKER (1957, Chap. 4) discusses in considerable detail the two-dimensional problem of waves generated by a periodic uniform pressure applied over a finite interval.

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β) Moving pressure distributions. In this section we shall suppose that a fixed pressure distribution is moving with a constant velocity c. Thus the motion may be treated as time-independent in a coordinate system moving with the pressure distribution. The boundary condition at the free surface is given by [see (11.3)]

$$\varphi_y(x,0,z) + \frac{1}{\nu}\varphi_{xx}(x,0,z) = \frac{c}{\varrho g} p_x(x,z), \quad \nu = \frac{g}{c^2}, \tag{21.24}$$

and the form of the free surface by

$$\eta(x,z) = \frac{c}{g}\varphi_x(x,0,z) - \frac{1}{\varrho g} p(x,z). \tag{21.25}$$

In addition, we shall assume vanishing of the fluid motion far ahead, i.e. as $x \to +\infty$, and the usual conditions appropriate to infinite or finite depth. p will be assumed to be absolutely integrable and to vanish for sufficiently large values of x^2+z^2; however, the latter condition can be weakened.

Results will be given without proof since their derivations are similar to those in Sects. 13γ and 21α. The results for two and three dimensions will be separated.

Three dimensions. The expression for the velocity potential for infinite depth of fluid can be given as follows:

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$$\left.\begin{aligned}\varphi(x,y,z) = {} & \frac{1}{\pi^2\varrho c}\iint\limits_{-\infty}^{\infty} d\xi\, d\zeta\, p(\xi,\zeta)\int\limits_0^{\frac{1}{2}\pi} d\vartheta \sec\vartheta\, \mathrm{PV}\int\limits_0^{\infty} dk \frac{k\, e^{ky}}{k-\nu\sec^2\vartheta}\times \\ & \times\sin[k(x-\xi)\cos\vartheta]\cos[k(z-\zeta)\sin\vartheta] - \\ & -\frac{\nu}{\pi\varrho c}\iint\limits_{-\infty}^{\infty} d\xi\, d\zeta\, p(\xi,\zeta)\int\limits_0^{\frac{1}{2}\pi} d\vartheta \sec^3\vartheta\, e^{-\nu\sec^2\vartheta}\times \\ & \times\cos[\nu(x-\xi)\sec\vartheta]\cos[\nu(z-\zeta)\sec^2\vartheta\sin\vartheta].\end{aligned}\right\} \tag{21.26}$$

The rate at which the pressure distribution is transferring energy to the fluid is given by

$$W = -\iint\limits_{-\infty}^{\infty} p(x,z)\,\varphi_y(x,-0,z)\,dx\,dz. \tag{21.27}$$

This may be computed directly from (21.26). The first term gives no contribution since it is an odd function of $x-\xi$ [cf. the evaluation of (20.66)]. The final result may be expressed as follows:

$$\left.\begin{aligned} W = {} & \frac{\nu^2}{\pi\varrho c}\int\limits_0^{\frac{1}{2}\pi} d\vartheta \sec^5\vartheta \iint\limits_{-\infty}^{\infty} dx\,dz \iint\limits_{-\infty}^{\infty} d\xi\,dz\, p(x,y)\,p(\xi,\zeta)\times \\ & \times\cos\left(\nu\sec^2\vartheta\,[(x-\xi)\cos\vartheta + (z-\zeta)\sin\vartheta]\right). \\ = {} & \frac{\nu^2}{\pi\varrho c}\int\limits_0^{\frac{1}{2}\pi} d\vartheta\sec^5\vartheta\,[P^2(\vartheta)+Q^2(\vartheta)], \\ P(\vartheta) = {} & \iint\limits_{-\infty}^{\infty} dx\,dz\,p(x,z)\cos[\nu\sec^2\vartheta\,(x\cos\vartheta + z\sin\vartheta)], \\ Q(\vartheta) = {} & \iint\limits_{-\infty}^{\infty} dx\,dz\,p(x,z)\sin[\nu\sec^2\vartheta\,(x\cos\vartheta + z\sin\vartheta)]. \end{aligned}\right\} \tag{21.28}$$

If the pressure distribution is given in cylindrical coordinates, $p = p(R,\alpha)$, $x = R\cos\alpha$, $z = R\sin\alpha$, then one may express, say, $P(\vartheta)$ in the form

$$P(\vartheta) = \int\limits_0^{\infty} dR \int\limits_0^{2\pi} d\alpha\, R\, p(R,\alpha)\cos[\nu R\sec^2\vartheta\cos(\alpha-\vartheta)] \tag{21.29}$$

and a similar formula for $Q(\vartheta)$. If p depends only upon R, then $Q(\vartheta)\equiv 0$ and

$$P(\vartheta) = \int_0^\infty 2\pi R\, p(R)\, J_0(\nu R \sec^2\vartheta)\, dR. \tag{21.30}$$

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If the fluid is of depth h, the velocity potential is given by

$$\left.\begin{aligned}
\varphi(x, y, z) = {} & \frac{1}{\pi^2 \varrho c} \iint_{-\infty}^{\infty} d\xi\, d\zeta\, p(\xi,\zeta) \int_0^{\frac{1}{2}\pi} d\vartheta \sec\vartheta \times \\
& \times \mathrm{PV}\int_0^\infty dk \frac{k \cosh k(y+h)\operatorname{sech} k h}{k-\nu\sec^2\vartheta\tanh k h} \sin[k(x-\xi)\cos\vartheta]\cos[k(z-\zeta)\sin\vartheta] - \\
& - \frac{1}{\pi\varrho c}\iint_{-\infty}^{\infty} d\xi\, d\zeta\, p(\xi,\zeta)\int_{\vartheta_0}^{\frac{1}{2}\pi} d\vartheta \sec\vartheta \frac{k_0\cosh k_0(y+h)\operatorname{sech} k_0 h}{1-\nu h\sec^2\vartheta\operatorname{sech}^2 k_0 h} \times \\
& \qquad \times \cos[k_0(x-\xi)\cos\vartheta]\cos[k_0(z-\zeta)\sin\vartheta],
\end{aligned}\right\} \tag{21.31}$$

where $k_0 = k_0(\vartheta)$ is the positive real root of

$$k - \nu\sec^2\vartheta\tanh k h = 0, \qquad \vartheta_0 < \vartheta < \tfrac{1}{2}\pi,$$

and

$$\vartheta_0 = \begin{cases} \operatorname{Arc}\cos\sqrt{\nu h} & \text{if } \nu h \equiv \dfrac{g h}{c^2} < 1, \\ 0 & \text{if } \nu h > 1. \end{cases}$$

The rate of transfer of energy may again be computed from (21.27) and again only the second integral gives a nonvanishing contribution. The result may be expressed in several forms analogous to (21.28) to (21.30):

$$\left.\begin{aligned}
W = {} & \frac{c}{\pi\varrho g}\int_{\vartheta_0}^{\frac{1}{2}\pi} d\vartheta \frac{k_0^3\cos\vartheta}{1-\nu h\sec^2\vartheta\operatorname{sech}^2 k_0 h}\iint_{-\infty}^{\infty} dx\, dz \iint_{-\infty}^{\infty} d\xi\, d\zeta\, p(x,z)\, p(\xi,\zeta) \times \\
& \qquad \times \cos(k_0[(x-\xi)\cos\vartheta + (z-\zeta)\sin\vartheta]), \\
= {} & \frac{c}{\pi\varrho g}\int_{\vartheta_0}^{\frac{1}{2}\pi} d\vartheta \frac{k_0^3\cos\vartheta}{1-\nu h\sec^2\vartheta\operatorname{sech}^2 k_0 h}\,[P^2(\vartheta) + Q^2(\vartheta)], \\
P(\vartheta) = {} & \iint_{-\infty}^{\infty} p(x,z)\cos[k_0(x\cos\vartheta + z\sin\vartheta)]\, dx\, dy, \\
Q(\vartheta) = {} & \iint_{-\infty}^{\infty} p(x,z)\sin[k_0(x\cos\vartheta + z\sin\vartheta)]\, dx\, dy.
\end{aligned}\right\} \tag{21.32}$$

In cylindrical coordinates formulas (21.29) and (21.30) carry over to the present situation with ν replaced by k_0.

The asymptotic form of the free surface for either infinite or finite depth is much more complicated to analyze than for the stationary periodically oscillating pressure distribution of the preceding section. Although it is not strictly necessary to do so, it has been customary in this type of analysis to consider the special case of a "concentrated pressure point". To derive the velocity potential for the pressure point consider the pressure distribution defined by

$$p(R) = \left\{\begin{matrix} \dfrac{p_0}{\pi R_0^2}, & R \leqq R_0, \\ 0, & R > R_0. \end{matrix}\right\} \tag{21.33}$$

Substitute in (21.26) or (21.31) and take the limit as $R_0 \to 0$. Then (21.26) becomes

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$$\left.\begin{aligned}\varphi(x, y, z) = &\frac{p_0}{\pi^2 \rho c} \int_0^{\frac{1}{2}\pi} d\vartheta \sec\vartheta \, \mathrm{PV} \int_0^{\infty} dk \frac{k\, e^{ky}}{k - \nu \sec^2} \sin(k x \cos\vartheta) \cos(k z \sin\vartheta) - \\ &- \frac{\nu p_0}{\pi \rho c} \int_0^{\frac{1}{2}\pi} d\vartheta \sec^3\vartheta \, e^{y \nu \sec^2\vartheta} \cos(\nu x \sec\vartheta) \cos(\nu z \sec^2\vartheta \sin\vartheta).\end{aligned}\right\} \tag{21.34}$$

The velocity potential for the pressure point in fluid of finite depth is derived similarly. The equation representing the free surface may now be obtained immediately from (21.25):

$$\eta(x, z) = \frac{c}{g} \varphi_x(x, 0, z), \qquad x^2 + z^2 > 0. \tag{21.35}$$

The velocity potential (21.34) is very similar to that of a submerged source in steady motion [see Eq. (13.36)] and the method sketched in Sect. 13 for the derivation of the asymptotic expression (13.42) can be carried over directly to the moving pressure point.

The result, expressed in cylindrical coordinates, is as follows:

$$\left.\begin{aligned}&\text{for } 0 \leqq \alpha < \pi - \operatorname{Arc\,sin} \tfrac{1}{3} = \alpha_c: \\ &\quad \eta(R, \alpha) = O(\nu R)^{-2}; \\ &\text{for } \alpha = \alpha_c: \\ &\quad \eta(R, \alpha_c) = -\frac{p_0 \nu^2}{\pi \rho g} 2^{-\frac{2}{3}} 3^{\frac{4}{3}} \Gamma\left(\frac{1}{3}\right) (\nu R)^{-\frac{1}{3}} \sin\left(\frac{\sqrt{3}}{2} \nu R\right) + O\left((\nu R)^{-\frac{2}{3}}\right); \\ &\text{for } \alpha_c < \alpha < \pi: \\ &\quad \eta(R, \alpha) = \frac{p_0 \nu^2}{\pi \rho g} \sqrt{\frac{2\pi}{\nu R}} \frac{1}{[1 - 9\sin^2\alpha]^{\frac{1}{4}}} \left\{\sec^{\frac{5}{2}}\vartheta_1 \sin\left(\nu R \mu_1 - \frac{\pi}{4}\right) + \right. \\ &\qquad\qquad \left. + \sec^{\frac{5}{2}}\vartheta_2 \sin\left(\nu R \mu_2 + \frac{\pi}{4}\right)\right\} + O\left((\nu R)^{-1}\right); \\ &\text{for } \alpha = \pi: \\ &\quad \eta(R, \pi) = -\frac{p_0 \nu^2}{\pi \rho g} \sqrt{\frac{2\pi}{\nu R}} \sin\left(\nu R + \frac{\pi}{4}\right).\end{aligned}\right\} \tag{21.36}$$

The variables $\vartheta_1, \vartheta_2, \mu_1, \mu_2$ are the same ones defined following (13.42), where certain properties are also given. For the pressure point Fig. 1a is not quite accurate as a description of the wave crests in the region $|\pi - \alpha| < \alpha_c$ since the phase in (21.36) has been shifted by $\frac{1}{2}\pi$; the wave crests in Fig. 1a should be moved back a distance $\frac{1}{2}\pi$. Also, in the neighborhood of $\alpha = \alpha_c$ the expressions in (21.36) are inaccurate; in this region η may be expressed in terms of Airy functions [see URSELL (1960)].

The wave pattern resulting from a moving pressure distribution has been the subject of many investigations, starting apparently with KELVIN (1906). His aim was to explain the typical wave pattern found behind a ship. The procedure is quite reasonable as a method for obtaining a qualitative prediction of a ship's wave pattern, since a moving ship has associated with it a pressure distribution around the wetted hull. The obvious disadvantage of the method is that it gives no connection between the geometry of the hull and the wave pattern. For this the "thin ship" approximation of Sect. 20β is better within its range of applicability. The single pressure point can be taken to represent approximately a ship moving at high speed (more accurately, at high Froude number $c/\sqrt{Lg}$, where L is the length), say a fast motor boat.

For detailed investigation of the asymptotic expression one should refer to HOGNER (1923), PETERS (1949), BARTELS and DOWNING (1955) (who do not restrict themselves to a pressure point) and STOKER (1957, Chap. 8). The necessary modifications for finite depth have been made by HAVELOCK (1908) and TETURÔ INUI (1936) and are described qualitatively in the discussion following (13.42). One can find an exposition of the theory of waves generated by moving pressure distributions in a report by LUNDE (1951 b). Several papers by HAVELOCK (1909, 1914b, 1919, 1922) take up the wave resistance ($=W/c$) of a pressure distribution. HOGNER (1928) has also considered the wave resistance and gives essentially (21.28).

In the preceding considerations we have assumed that, as $x^2+z^2\to\infty$, $p(x,z)$ approached zero sufficiently quickly so that it might be represented as a Fourier integral. It is also possible to proceed somewhat differently, assume $p(x,y)$ periodic in one or both variables and use a Fourier series representation. This has been done, for example, by VOIT (1957a), who has considered for both infinite and finite depth a moving pressure distribution of the following form:

$$p(x,z)=\begin{cases}P(z)\sum\limits_{n=1}^{\infty}a_n\cos n\,k\,x, & |z|<h\\ 0, \quad |z|>h. & \end{cases}$$

The waves resulting from a pressure point moving parallel to beaches forming angles of 30 and 45° with the surface have been treated by HANSON (1926); in the same paper he also treats the waves formed by a pressure point moving over a two-layered fluid. A detailed investigation of this last topic is given in a paper of SRETENSKII'S (1934).

Two dimensions. By introducing a stream function $\psi(x,y)$ and a complex potential $f=\varphi+i\psi$, the free surface boundary condition can be put into a form analogous to (21.14), namely,

$$\mathrm{Re}\{f'(x-i\,0)+i\,\nu\,f(x-i\,0)\}=\frac{1}{\varrho\,c}\,p(x),\qquad \nu=\frac{g}{c^2}. \tag{21.37}$$

In addition, we assume $|f'|$ bounded as $z\to\infty$ and also $\lim\limits_{x\to\infty}|f'|=0$. We shall assume $p(x)$ subject to the same limitations as in Sect. 21α.

One may apply the same method of analysis to derive the following forms for the complex velocity potential:

$$\left.\begin{aligned}f(z)&=\frac{\mathrm{e}^{-i\nu z}}{\pi\,i\,\varrho\,c}\int\limits_{-\infty}^{\infty}ds\,p(s)\int\limits_{\infty}^{z}\frac{\mathrm{e}^{i\nu\zeta}}{\zeta-s}\,d\zeta\\&=-\frac{1}{\pi\,i\,\varrho\,c}\int\limits_{-\infty}^{\infty}ds\,p(s)\,\mathrm{PV}\int\limits_{0}^{\infty}\frac{\mathrm{e}^{-i\lambda(z-s)}}{\lambda-\nu}\,d\lambda-\frac{\mathrm{e}^{-i\nu z}}{\varrho\,c}\int\limits_{-\infty}^{\infty}p(s)\,\mathrm{e}^{i\nu s}\,ds\\&=-\frac{1}{\pi\,i\,\varrho\,c}\int\limits_{-\infty}^{x}ds\,p(s)\int\limits_{0}^{\infty}\frac{\mathrm{e}^{-\mu(z-s)}}{\mu-i\nu}\,d\mu-\frac{1}{\pi\,i\,\varrho\,c}\int\limits_{x}^{\infty}ds\,p(s)\int\limits_{0}^{\infty}\frac{\mathrm{e}^{\mu(z-s)}}{\mu+i\nu}\,d\mu-\\&\quad-\frac{2}{\varrho\,c}\,\mathrm{e}^{-i\nu z}\int\limits_{x}^{\infty}p(s)\,\mathrm{e}^{i\nu s}\,ds,\end{aligned}\right\}\tag{21.38}$$

where the path of integration for ζ in the first expression is taken in the lower half-plane. The rate at which the pressure distribution transfers energy to the fluid is easily found from formula (21.27) and the second expression for $f(z)$ to be

$$W = \frac{\nu}{\varrho c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x)\, p(\xi) \cos \nu (x-\xi)\, dx\, d\xi. \tag{21.39}$$

If the fluid is of depth h, the complex velocity potential may be given in a form analogous to the second formula of (21.37):

$$\left.\begin{aligned} f(z) = {} & \frac{\nu c}{\pi \varrho g} \int_{-\infty}^{\infty} ds\, p(s)\, \mathrm{PV} \int_{-\infty}^{\infty} dk \frac{\sin k(z - s + i h) \operatorname{sech} k h}{k - \nu \tanh k h} - \\ & - \frac{\nu c}{\varrho g} \int_{-\infty}^{\infty} p(s) \frac{\cos k_0 (z - s + i h) \operatorname{sech} k_0 h}{1 - \nu h \operatorname{sech}^2 k_0 h}\, ds, \end{aligned}\right\} \tag{21.40}$$

where k_0 is the real positive root of

$$k_0 - \nu \tanh k_0 h = 0$$

and exists only if $\nu h = g h/c^2 > 1$; if $\nu h \leqq 1$, the last term in (21.40) must be deleted. The rate at which the pressure is doing work upon the fluid is given by

$$W = \begin{cases} \dfrac{k_0^2 c}{\varrho g} \displaystyle\iint_{-\infty}^{\infty} p(x)\, p(\xi) \frac{\cos k_0 (x-\xi)}{1 - \nu h \operatorname{sech}^2 k_0 h}\, dx\, d\xi & \text{if } \nu h > 1, \\ 0 \quad \text{if } \nu h < 1. \end{cases} \tag{21.41}$$

The absence of the second term in (21.40) and the vanishing of W when $\nu h < 1$ correspond to the absence of an infinite train of trailing waves. A similar phenomenon occurs behind a moving singularity in two dimensions [cf. (13.46) to (13.48) and the following remarks].

For either (21.37) or (21.39) the form of the free surface can be written down immediately from the formula

$$\eta(x) = \frac{1}{c} \psi(x, 0). \tag{21.41}$$

We shall not carry out the details. The asymptotic form of the surface behind a two-dimensional "pressure point", or also a distributed pressure, is much easier in two than in three dimensions and we again omit a detailed statement. However, the problem has been treated by Kelvin (1905) and is discussed in Lamb (1932, § 242 to 245), both for infinite and finite depth. It is also discussed in the paper of Peters (1949) already cited in connection with the three-dimensional problem.

Derivations of the complex velocity potential may be found in the papers of Sretenskii (1934, 1940), Sedov (1936), Kochin (1939) and Haskind (1943a) already cited in connection with planing surfaces. We refer also to papers of Dean (1947) and Timman and Vossers (1955).

γ) Moving periodic pressure distributions. It is clearly possible to combine the cases considered in Sects. 21 α and β and consider the waves resulting from a pressure distribution expressible in the form

$$p(x, z, t) = p_1(x - c t, z) \cos \sigma t + p_2(x - c t, z) \sin \sigma t,$$

where the coordinates are fixed in space. The resulting velocity potential will be analogous to (13.52) for the three-dimensional case, if one is dealing with a "pressure point".

We shall not reproduce the formulas here. However, the analogues of (13.49) and (13.53) for pressure distributions may be found in Eqs. (22.48) and (22.49) or in the cited report of LUNDE (1951b), and from these the required velocity potential may be found. For two-dimensional motion the details, carried out by this procedure, may be found in papers by KAPLAN (1957) and WU (1957).

22. Initial-value problems. In the special problems considered in Sects. 17 to 21, the dependence upon the time has been precipitated out, either by assuming the motion steady in a moving coordinate system or by assuming a harmonic dependence upon the time. In this section we shall consider motions in which the displacement and velocity of the surface are specified at some instant of time, say $t=0$, the motions of any solid boundaries are given for each instant $t \geqq 0$ (except at the very end where freely floating bodies are considered) and the pressure distribution over the free surface is a given function for $t \geqq 0$.

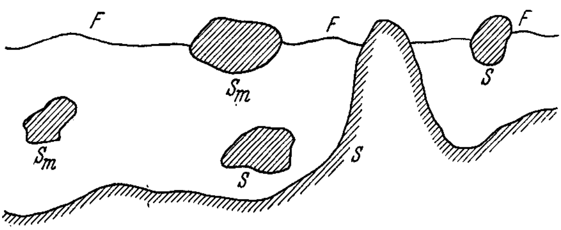

Fig. 27.

It is not usually possible in the most general situations to give explicit solutions for such problems. However, VOLTERRA (1934) has proved a uniqueness theorem and has shown how to reduce the problem to finding an appropriate GREEN's function. His results were later rediscovered and extended to a wider class of problems by FINKELSTEIN (1957) [see also STOKER (1957, Chap. 6)]. However, the use of GREEN's functions for initial-value problems extends back even earlier, at least to the papers of HADAMARD (1910, 1916) and BOULIGAND (1913). These theorems are discussed in Sect. 22α.

One of the classical problems in this category is associated with the names CAUCHY (1827) and POISSON (1815). In this problem the fluid is infinite in horizontal extent, without obstructions, and either infinitely deep or of uniform depth h. At the initial instant $t=0$, the form of the surface and its vertical velocity are given and one seeks the subsequent motion. Such problems have already been discussed at some length in Sect. 15. However, in that section interest centered upon investigation of certain aspects of the subsequent motion rather than upon obtaining the solution. In addition, the treatment of that section was limited to two-dimensional motion, although the methods could have been extended to three-dimensional motion.

The history of this problem, including an exposition of the methods used by various authors, is included in a paper by RISSER (1924, pp. 113—144). Another expository account can be found in VERGNE (1928, Chap. I). The problem is discussed here in Sect. 22β.

In Sect. 22γ several special initial-value problems are discussed.

α) Some general theorems. Let the fluid be bounded by the free surface F, fixed surfaces S and the surfaces of a finite number of bodies of bounded extent undergoing specified motions of small amplitude about equilibrium positions S_m (see Fig. 27). Let the pressure distribution on the free surface F, also be a given function $p(x, z, t)$. Furthermore, at time $t=0$ let the initial displacement and vertical velocity of the free surface be given functions

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$$\eta(x, z, 0), \qquad \eta_t = (x, z, 0). \tag{22.1}$$

The boundary conditions to be satisfied by the velocity potential $\Phi(x, y, z, t)$ are [see Eq. (11.1)]

$$\left.\begin{aligned}
&\Phi_{tt}(x, 0, z, t) + g\,\Phi_y(x, 0, z, t) = -\frac{1}{\varrho}\,p_t(x, z, t) \quad \text{on} \quad F,\\
&\Phi_n = 0 \quad \text{on} \quad S,\\
&\Phi_n = V_n(t) \quad \text{on} \quad S_m,\\
&\Phi_t(x, 0, z, 0) = -g\,\eta(x, z, 0) - \frac{1}{\varrho}\,p(x, z, 0) \quad \text{on} \quad F,\\
&\Phi_y(x, 0, z, 0) = \eta_t(x, z, 0) \quad \text{on} \quad F.
\end{aligned}\right\} \tag{22.2}$$

Here F means that part of the plane $y=0$ occupied by fluid when everything is at rest. In addition, it will be assumed that, for each t, there is a bound B and a distance r_0 such that $|\Phi|$, $|\Phi_t|$, $|\text{grad}\,\Phi|$ and $|\text{grad}\,\Phi_t|$ are each less than B for $x^2+y^2+z^2>r_0^2$.

Let us now suppose that it is possible to find a source function G of the following nature:

$$G(x, y, z; \xi, \eta, \zeta; t, \tau) = \frac{1}{r} + H(x, y, z; \xi, \eta, \zeta; t, \tau), \tag{22.3}$$

where as usual $r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$ and H is harmonic for $y\leqq 0$, and in addition G satisfies

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$$\left.\begin{aligned}
&G(x, y, z; \xi, \eta, \zeta; t, t) = G_t(x, y, z; \xi, \eta, \zeta; t, t) = 0,\\
&G_n(x, y, z; \xi, \eta, \zeta; t, \tau) = 0 \quad \text{for} \quad (x, y, z) \text{ on } S \text{ and for all } t.
\end{aligned}\right\} \tag{22.4}$$

This function has already been constructed in two cases. If there are no fixed boundaries and the fluid is infinitely deep, the function defined in (13.49) satisfies the conditions after slight modifications: replace (a, b, c) by (ξ, η, ζ), set $m(t)=1$, and extend the definition of Φ [of (13.49)] to negative t by $\Phi(x, y, z, -t) = \Phi(x, y, z, t)$. Then we take

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$$G(x, y, z; \xi, \zeta; t, \tau) = \Phi(x, y, z; t-\tau) = G(x, y, z; \xi, \eta, \zeta; \tau, t).$$

Similarly, the function defined in (13.53) allows one to construct G when the fixed boundary consists of a horizontal bottom at $y=-h$. For the first G, FINKELSTEIN (1957, Appendix) has shown that G is $O(R^{-2})$ and G_R and G_y are $O(R^{-3})$ as $R\to\infty$, where $R^2=(x-\xi)^2+(z-\zeta)^2$; for the second G, FINKELSTEIN (1957, § 3) has shown that G, G_R and G_y are $o\left(\exp\left(\frac{-\pi}{2h}+\varepsilon\right)R\right)$ as $R\to\infty$ for arbitrary $\varepsilon>0$.

Now apply GREEN's theorem to the functions Φ_t and G and the region of fluid bounded by the surfaces S_m, the fixed boundaries S and a large sphere Ω of radius ϱ and center at the origin, where ϱ is chosen large enough to include all the surfaces S_m. Only parts of F, S and Ω will serve as bounding surfaces, and we shall call these parts F', S' and Ω', respectively. Then

$$\Phi_t(x, y, z, t) = \frac{1}{4\pi}\iint\limits_{F'+S_m+S'+\Omega'} [G(\xi, \eta, \zeta; x, y, z; t, \tau)\,\Phi_{t\nu}(\xi, \eta, \zeta, t) - \Phi_t G_\nu]\,d\sigma, \tag{22.5}$$

where ν is the exterior normal. The right-hand side is actually independent of τ since τ enters only through the function H which is harmonic. The integral over

S' vanishes since both Φ_n and G_n are zero on S. We shall assume that the behavior of G and Φ as $R \to \infty$ is such that the integral over Ω' vanishes as $\varrho \to \infty$. If the fluid is of bounded extent, the situation considered by VOLTERRA (1934), this presents, of course, no difficulty. In the two cases for which G has been given above, it has been shown by FINKELSTEIN that this is true. For finite depth the proof presents no difficulty once the estimates for G are obtained; for infinite depth the analysis is more troublesome and we refer to his paper or to STOKER (1957, pp. 193/194) for proof. After letting $\varrho \to \infty$, one then has

$$\Phi_t(x, y, z, t) = \frac{1}{4\pi} \iint_F [G\,\Phi_{t\eta} - G_\eta \Phi_t]\, d\sigma + \frac{1}{4\pi} \iint_{S_m} [G\,\Phi_{t\nu} - G_\nu \Phi_t]\, d\sigma. \tag{22.6}$$

In the integral over F we may replace G_η by $-g^{-1} G_{tt}$ because of the boundary condition at F. Now interchange t and τ and integrate with respect to τ between limits 0 and t. This gives, following an integration by parts,

$$\left.\begin{aligned} \Phi(x, y, z, t) - \Phi(x, y, z, 0) &= \frac{1}{4\pi} \iint_F \left\{ \left[G\,\Phi_\eta + \frac{1}{g}\,\Phi_t G_t \right] \Big|_0^t - \int_0^t \left[G_t \Phi_\eta + \frac{1}{g}\,\Phi_{tt} G_t \right] d\tau \right\} + \\ &\quad + \frac{1}{4\pi} \int_0^t d\tau \iint_{S_m} [G\,\Phi_{tn} - G_n \Phi_t]\, d\sigma \\ &= \frac{1}{4\pi} \iint_F \Big\{ G(\xi, 0, \zeta, x, y, z; t, t)\, \Phi_y(\xi, 0, \zeta, t) + \frac{1}{g}\, \Phi_t(\ldots, t)\, G_t(\ldots; t, t) - \\ &\quad - G(\ldots; 0, t)\, \Phi_y(\ldots, 0) - \frac{1}{g}\, \Phi_t(\ldots, 0)\, G_t(\ldots; 0, t) + \\ &\quad + \frac{1}{\varrho g} \int_0^t G_t(\ldots; \tau, t)\, p_t(\xi, \zeta, \tau)\, d\tau \Big\}\, d\xi\, d\zeta + I \end{aligned}\right\} \tag{22.7}$$

where I stands for the last integral. (G_t always represents the derivative with respect to the seventh variable.) Recalling the properties of G in (22.4), one finds

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$$\left.\begin{aligned} \Phi(x, y, z, t) &= \Phi(x, y, z, 0) + \frac{1}{4\pi} \iint_F \Big\{ -G(\xi, 0, \zeta; x, y, z; 0, t)\, \eta_t(\xi, \zeta, 0) + \\ &\quad + G_t(\xi, 0, \zeta, x, y, z; 0, t) \left[\eta(\xi, \zeta, 0) + \frac{1}{\varrho g}\, p(\xi, \zeta, 0) \right] + \\ &\quad + \frac{1}{\varrho g} \int_0 G(\xi, 0, \zeta; x, y, z, \tau, t)\, p_t(\xi, \zeta, \tau)\, d\tau \Big\}\, d\xi\, d\zeta + I \\ &= \Phi(x, y, z, 0) + \frac{1}{4\pi} \iint_F \Big\{ -G_t(\xi, 0, \zeta, x, y, z; 0, t)\, \eta_t(\xi, \zeta, 0) + \\ &\quad + G_t(\xi, 0, \zeta, x, y, z; 0, t)\, \eta(\xi, \zeta, 0) - \\ &\quad - \frac{1}{\varrho g} \int_0 G_{tt}(\xi, 0, \zeta, x, y, z; \tau, t)\, p(\xi, \zeta, \tau)\, d\tau \Big\}\, d\xi\, d\zeta + I, \end{aligned}\right\} \tag{22.8}$$

where $\Phi(x, y, z, 0)$ is determined up to an additive constant as the solution to a Neumann problem, since $\Phi_n(x, y, z, 0)$ is given on all boundaries and bounded at infinity. In the integral I we note that $\Phi_{tn} = V_n'(t)$ is known on S_m, but Φ_t is not.

If there are no moving bodies in the fluid, then the integral I is not present and Φ is determined by the initial displacement and velocity of the free surface and the given pressure distribution over it. This is VOLTERRA's result as extended to unbounded fluids by FINKELSTEIN. If surfaces S_m are present, one may still use (22.8) to derive an integral equation in the same way that (16.13) was derived. For as (x, y, z) is made to approach a point (x_0, y_0, z_0) of S_m,

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$$\frac{1}{4\pi}\iint_{S_m} G_\nu(\xi,\eta,\zeta,x,y,z;\tau,t)\,\Phi_t(\xi,\eta,\zeta,\tau)\,d\sigma \to \frac{1}{2}\Phi_t(x_0,y_0,z_0,t) + \frac{1}{4\pi}\iint_{S_m} G_\nu(\xi,\eta,\zeta,x_0,y_0,z_0;\tau,t)\,\Phi_t\,d\sigma\,[1].$$

Thus, after carrying out the integration with respect to τ, one has an integral equation for $\Phi(x, y, z, t)$ for each value of $t>0$. This may be used to find the value of Φ, and hence Φ_t, on the surface S_m, providing that the integral equation can be solved. One may then use (22.8) to determine $\Phi(x, y, z, t)$ for all values of (x, y, z) in the fluid. The integral equation has the same appearance as (22.8) except that the first two terms have coefficients $\frac{1}{2}$ and (x, y, z) is understood to be a point of S_m. This further extension of VOLTERRA's analysis is also due to FINKELSTEIN.

Uniqueness of $\Phi(x, y, z, t)$, at least up to an additive constant, may be proved a follows. Let Φ_1 and Φ_2 be two solutions satisfying the boundary conditions. Then $\Phi = \Phi_1 - \Phi_2$ satisfies (22.8) with f, F, p and V_n all identically zero, i.e.

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$$\Phi(x, y, z, t) = \text{const} - \frac{1}{4\pi}\int_0^t d\tau \iint_{S_m} G_n\,\Phi_t\,d\sigma.$$

If we assume that G_n is $O(R^{-1-\varepsilon})$ as $R\to\infty$, then Φ_t and grad Φ will have the same behavior and the integrals we shall write below may be shown to exist. As has been mentioned above, G_n vanishes much quicker than is required in the cases when the fluid is infinitely deep and when the fixed surface consists of a horizontal bottom; if the fluid is bounded in extent, no such condition is necessary to make the integrals converge.

Consider then, following VOLTERRA,

$$\begin{aligned}\Omega &= \frac{1}{2}\frac{\partial}{\partial t}\iint_F \frac{1}{g}\Phi_t^2\,d\sigma \\ &= \iint_F \frac{1}{g}\Phi_t\Phi_{tt}\,d\sigma = -\iint_F \Phi_t\Phi_y\,d\sigma \\ &= -\iint_{F+S+S_m} \Phi_t\Phi_n\,d\sigma\end{aligned}$$

since Φ_n vanishes on S and S_m. Now apply GREEN's theorem and denote the volume occupied by fluid by T:

$$\Omega = -\iiint_T \operatorname{grad}\Phi_t\cdot\operatorname{grad}\Phi\,d\tau = -\frac{1}{2}\frac{\partial}{\partial t}\iiint_T (\operatorname{grad}\Phi^2)\,d\tau.$$

Hence

$$\frac{\partial}{\partial t}\left\{\iint_F \frac{1}{g}\Phi_t^2\,d\sigma + \iiint_T (\operatorname{grad}\Phi)^2\,d\tau\right\} = 0$$

and

$$\iint_F \frac{1}{g}\Phi_t^2\,d\sigma + \iiint_T (\operatorname{grad}\Phi)^2\,d\tau = \text{const}. \qquad (22.9)$$

[1] Cf. O.D. KELLOGG: Foundations of potential theory, p. 167. Berlin: Springer 1929.

Since $\Phi_n = 0$ on F, S and S_m for $t=0$, $\Phi(x, y, z, 0) = C$, a constant; hence grad $\Phi = 0$ for $t=0$. Also $\Phi_t(x, y, z, 0) = 0$. Hence the constant in (22.9) is zero and Φ_t and grad Φ vanish for all t. Thus $\Phi(x, y, z, t) =$ const and the solution of the initial-value problem is determined up to a constant.

β) The Cauchy-Poisson problem. In this classical problem of water-wave theory, the pressure over the free surface is constant, say zero, the fluid is infinitely deep or bounded below by a horizontal bottom, no obstructions are present and the initial displacement and velocity of the free surface are given. The two- and three-dimensional cases will be separated in order to illustrate different methods of approach.

Three dimensions. The velocity potential may be obtained directly from (22.8) after setting $p(x, z, t)$ and I equal to zero. However, the explicit expressions for G and G_t are needed. As was noted in Sect. 22α, these can be written down immediately from (13.49) for infinite depth and (13.53) for depth h. The resulting expressions, after setting $\eta = 0$, are as follows:

infinite depth:

$$\left.\begin{aligned} G(x, y, z; \xi, 0, \zeta; 0, t) &= 2\int_0^\infty \left[1 - \cos\left(\sqrt{g k}\, t\right)\right] e^{k y} J_0(kR)\, dk, \\ G_t(x, y, z; \xi, 0, \zeta; 0, t) &= -2\int_0^\infty \sin\left(\sqrt{g k}\, t\right) e^{k y} J_0(kR) \sqrt{g k}\, dk; \end{aligned}\right\} \tag{22.10}$$

depth h:

$$\left.\begin{aligned} G(x, y, z; \xi, 0, \zeta; 0, t) &= 2\int_0^\infty \left[1 - \cos\left(\sqrt{g k \tanh k h}\, t\right)\right] \frac{\cosh k(y+h)}{\sinh k h} J_0(kR), \\ G_t(x, y, z; \xi, 0, \zeta; 0, t) &= -2\int_0^\infty \sqrt{g k \tanh k h}\, \sin\left(\sqrt{g k \tanh k h}\, t\right) \frac{\cosh k(y+h)}{\sinh k h} J_0(kR), \end{aligned}\right\} \tag{22.11}$$

where $R^2 = (x-\xi)^2 + (z-\zeta)^2$.

There still remains to find $\Phi(x, y, z, 0)$ where

$$\Phi_y(x, 0, z, 0) = \eta_t(x, z, 0)$$

and

$$\lim_{y\to-\infty} \Phi_y(x, y, z. 0) = 0 \quad \text{or} \quad \Phi_y(x_1 - h, z, 0) = 0.$$

The solution of these two problems is well known:

infinite depth:

$$\left.\begin{aligned} \Phi(x, y, z, 0) &= \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{\eta_t(\xi, \zeta, 0)}{[(x-\xi)^2 + y^2 + (z-\zeta)^2]^{\frac{1}{2}}}\, d\xi\, d\zeta \\ &= \frac{1}{2\pi} \iint_{-\infty}^{\infty} \eta(\xi, \zeta, 0)\, d\xi\, d\zeta \int_0^\infty e^{k y} J_0(kR)\, dk; \end{aligned}\right\} \tag{22.12}$$

depth h:

$$\Phi(x, y, z, 0) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \eta_t(\xi, \zeta, 0)\, d\xi\, d\zeta \int_0^\infty \frac{\cosh k(y+h)}{\sinh k h} J_0(kR)\, dk, \tag{22.13}$$

where R is defined as above.

Substituting the several expressions in (22.8), one obtains the expressions for the velocity potential:

infinite depth:

$$\left.\begin{aligned}\Phi(x,y,z,t) &= \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}e^{ky}\cos\sigma t\,J_0(kR)\,dk - \\ &\quad -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma\,e^{ky}\sin\sigma t\,J_0(kR)\,dk, \\ \sigma^2 &= gk;\end{aligned}\right\}\quad(22.14)$$

depth h:

$$\left.\begin{aligned}\Phi(x,y,z,t) &= \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\frac{\cosh k(y+h)}{\sinh kh}\cos\sigma t\,J_0(kR)\,dk - \\ &\quad -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma\,\frac{\cosh k(y+h)}{\sinh kh}\sin\sigma t\,J_0(kR)\,dk, \\ \sigma^2 &= gk\tanh kh.\end{aligned}\right\}\quad(22.15)$$

The equations describing the free surface are as follows:

infinite-depth:

$$\left.\begin{aligned}\eta(x,z,t) &= \frac{1}{2\pi g}\iint\limits_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma\sin\sigma t\,J_0(kR)\,dk + \\ &\quad +\frac{1}{2\pi g}\iint\limits_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma^2\cos\sigma t\,J_0(kR)\,dk, \\ \sigma^2 &= gk;\end{aligned}\right\}\quad(22.16)$$

depth h:

$$\left.\begin{aligned}\eta(x,z,t) &= \frac{1}{2\pi g}\iint\limits_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma\sin\sigma t\coth kh\,J_0(kR)\,dk + \\ &\quad +\frac{1}{2\pi g}\iint\limits_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi\,d\zeta\int\limits_0^{\infty}\sigma^2\cos\sigma t\coth kh\,J_0(kR)\,dk, \\ \sigma^2 &= gk\tanh kh.\end{aligned}\right\}\quad(22.17)$$

It has been shown by Kochin (1935) that the integrals with respect to k in (22.16) can be evaluated. Consider the integral

$$K = \int\limits_0^{\infty}\sigma^{-1}\sin\sigma t\,J_0(kR)\,dk, \qquad \sigma^2 = gk. \tag{22.18}$$

Then the first integral with respect to k in (22.16) is $-K_{tt}$ and the second one is $-K_{ttt}$ To evaluate K make first the following change of variables:

$$\varkappa^2 = kR, \qquad \omega^2 = gt^2/4R. \tag{22.19}$$

Then

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$$K = \frac{2}{\sqrt{gR}} \int_0^\infty \sin 2\omega\varkappa\, J_0(\varkappa^2)\, d\varkappa$$
$$= \frac{2}{\sqrt{gR}} \int_0^\infty d\varkappa \int_0^1 \frac{\sin 2\omega\varkappa \cos v\varkappa^2}{\sqrt{1-v^2}}\, dv$$
$$= \frac{2}{\sqrt{gR}} \int_0^\infty d\varkappa \int_0^1 [\sin(2\omega\varkappa + v\varkappa^2) + \sin(2\omega\varkappa - v\varkappa^2] \frac{dv}{\sqrt{1-v^2}}.$$

In the first integral let $u = v\varkappa + \omega$, in the second let $u = v\varkappa - \omega$. Then

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$$K = \frac{2}{\sqrt{gR}} \int_{+\omega}^\infty du \int_0^1 \sin\left(\frac{u^2-\omega^2}{v}\right) \frac{dv}{v\sqrt{1-v^2}} - \frac{2}{\sqrt{gR}} \int_{-\omega}^\infty du \int_0^1 \sin\left(\frac{u^2-\omega^2}{v}\right) \frac{dv}{v\sqrt{1-v^2}}$$
$$= + \frac{2}{\sqrt{gR}} \int_{-\omega}^{\omega} du \int_1^\infty \frac{\sin(\omega^2-u^2)\,v'}{\sqrt{v'^2-1}}\, dv' \qquad (v' = 1/v),$$
$$= \frac{4}{\sqrt{gR}} \frac{\pi}{2} \int_0^\omega J_0(\omega^2 - u^2)\, du,$$

and, after setting $u = \sqrt{2}\omega \sin \frac{1}{2}\vartheta$,

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$$K = \frac{\sqrt{2}\pi}{\sqrt{gR}}\, \omega \int_0^{\frac{1}{2}\pi} J_0\left(2\frac{\omega^2}{2}\cos\vartheta\right) \cos\frac{1}{2}\vartheta\, d\vartheta.$$

Finally, from an identity in WATSON's *Bessel functions* [§ 5.43, Eq. (1)] one finds

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$$K(\omega) = \frac{\pi^2}{\sqrt{2gR}}\, \omega\, J_{\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) J_{-\frac{1}{4}}\left(\frac{1}{2}\omega^2\right). \tag{22.20}$$

In order to use the results in (22.16) one needs the first three derivatives with respect to t. Since

$$\frac{\partial}{\partial t} = \frac{1}{2}\sqrt{\frac{g}{R}}\frac{\partial}{\partial\omega},$$

the derivatives can be computed by taking derivatives with respect to ω and multiplying by an appropriate factor. After some rather tedious computation one finds

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$$\left.\begin{aligned}
\frac{\partial}{\partial t}K(\omega) &= -\frac{1}{2\sqrt{2}}\pi^2\frac{\omega^2}{R}\left[J_{\frac{1}{4}}\left(\frac{1}{2}\omega^2\right)J_{\frac{3}{4}}\left(\frac{1}{2}\omega^2\right) - J_{-\frac{1}{4}}\left(\frac{1}{2}\omega^2\right)J_{-\frac{3}{4}}\left(\frac{1}{2}\omega^2\right)\right],\\
\frac{\partial^2}{\partial t^2}K(\omega) &= -\frac{1}{2}\pi^2\omega^3\sqrt{\frac{g}{2R^3}}\left[J_{\frac{1}{4}}\left(\frac{1}{2}\omega^2\right)J_{-\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) + J_{\frac{3}{4}}\left(\frac{1}{2}\omega^2\right)J_{-\frac{3}{4}}\left(\frac{1}{2}\omega^2\right)\right],\\
\frac{\partial^3}{\partial t^3}K(\omega) &= -\frac{\pi^2 g}{2R^2}\omega^2\left[J_{\frac{1}{4}}\left(\frac{1}{2}\omega^2\right)J_{-\frac{1}{4}}\left(\frac{1}{2}\omega^2\right) - \right.\\
&\quad \left. - \omega^2\left\{J_{\frac{1}{4}}\left(\frac{1}{2}\omega^2\right)J_{\frac{3}{4}}\left(\frac{1}{2}\omega^2\right) - J_{-\frac{1}{4}}\left(\frac{1}{2}\omega^2\right)J_{-\frac{3}{4}}\left(\frac{1}{2}\omega^2\right)\right\}\right].
\end{aligned}\right\} \tag{22.21}$$

These are KOCHIN's formulas, but derived somewhat differently from his original paper; still another derivation may be found in KOCHIN, KIBEL and ROZE (1948, Chap. 8, § 21). Similar formulas for (22.17) do not seem to have been discovered.

It should be noted that in the final form of (22.16) the dependence upon t is through the dimensionless variable $\omega^2 = gt^2/4R$. Hence, if one examines the contribution to the surface profile from a given locality, say the neighborhood of (ξ, ζ), then a given phase of this contribution, say a maximum, will be described by $gt^2/4R = \text{const}$; i.e., the phase is moving away from (ξ, ζ) with constant acceleration proportional to g. The amplitude of the contribution is modulated by either $R^{-\frac{3}{2}}$ or R^{-2} according as one is considering the first or second summand in (22.16). KOCHIN'S 1935 paper is of some methodological interest inasmuch as he started his analysis with dimensional considerations. This method will be introduced for the two-dimensional case.

One may obtain without great difficulty series expansions for the k-integrals in (22.14) and (22.16), as was first done by CAUCHY and POISSON. We refer to LAMB'S *Hydrodynamics* (1932, § 255) for the derivation and exact expressions. They can also be derived from the the known expansions for $J_{\frac{1}{4}}$, etc., as can asymptotic expressions for large ω. One may also carry out an analysis of the changing shape of the surface profile following the methods of Sect. 15.

It is evident that one can solve explicitly other similar initial-value problems for which the GREEN'S function can be given. For example, the method of images allows one to give an explicit solution for various cases when vertical walls are present as boundaries. Such cases have been considered by RISSER (1925). The Cauchy-Poisson problem in the presence of a vertical half-plane, $z = 0$, $x > 0$, has been treated by BOIKO (1938), but by more complex methods.

Two dimensions. Rather than repeat the methods used for three-dimensional motion, we shall introduce a method making use of the complex potential and thus special to two-dimensional motion. It is analogous to the method used in deriving (13.28).

Let $f(z, t) = \Phi(x, y, t) + i\Psi(x, y, t)$ be the complex velocity potential. The initial conditions will be taken in the form

$$-\frac{1}{g}\operatorname{Re} f_t(x - i0, 0+) = \eta(x, 0), \quad -\operatorname{Im} f'(x - i0, 0+) = \eta_t(x, 0). \tag{22.22}$$

Let us consider infinite depth first. For $t > 0$ we assume that $f(z, t)$ is regular and $|f'| < M(t)$, $|f_{tt}| < M(t)$ for $y < 0$ and that both f' and f_{tt} approach zero as $y \to -\infty$. Consider now the function

$$G(z, t) = f_{tt}(z, t) + ig f'(z, t). \tag{22.23}$$

From the assumptions about f it follows that, for $t > 0$, $G(z, t)$ is regular for $y < 0$, that $|G| < B(t)$ for $y < 0$ and that $G \to 0$ as $y \to -\infty$. Moreover, it follows from the condition at the free surface, (11.5), that $\operatorname{Re} G(x - i0, t) = 0$. Hence, the definition of G may be extended into the upper half-plane by defining

$$G(x + iy) = -\overline{G(x - iy)}. \tag{22.24}$$

But then since G is regular and bounded in the whole finite z-plane, it follows from LIOUVILLE'S theorem that $G = \text{const}$; the constant must equal zero from the assumed behavior as $y \to -\infty$. Hence the fundamental differential equation for the Cauchy-Poisson problem in two dimensions is

$$f_{tt}(z, t) + ig f'(z, t) = 0, \quad t > 0, \tag{22.25}$$

an observation usually credited to LEVI-CIVITA (cf. TONOLO, 1913).

Let us now find the analogous equation for finite depth. The function f will be assumed regular in the strip $0>y>-h$. The boundary condition on the bottom is

$$\operatorname{Im} f'(x-ih,t)=0. \tag{22.26}$$

Hence f may be extended analytically into the strip $-2h<y<-h$ by defining

$$f(x-i(y+2h))=\overline{f(x+iy)}, \quad 0>y\geqq -h. \tag{22.27}$$

We may also, as before, extend the function $f_{tt}+igf'$ into the strip $h\geqq y\geqq 0$. The condition $\operatorname{Re}\{f_{tt}+igf'\}=0$ for $y=0$ implies $\operatorname{Re}\{f_{tt}-igf'\}=0$ for $y=-2h$. Hence the function $f_{tt}-igf'$ can be extended by reflection into the strip $-3h\leqq y\leqq -2h$. Now consider the function

$$\left.\begin{aligned} H(z,t) &= [f_{tt}(z+ih,t)+f_{tt}(z-ih,t)]+ig[f'(z+ih,t)-f'(z-ih,t)] \\ &= \{f_{tt}(z+ih,t)+igf'(z+ih,t)\}+\{f_{tt}(z-ih,t)-igf'(z-ih,t)\}. \end{aligned}\right\} \tag{22.28}$$

As a result of the various extended regions of definition, one may verify easily that H is defined for all z in the strip $-2h<y<0$ and is regular there. Moreover, it follows that

$$H(x-ih,t)=0; \tag{22.29}$$

for from (22.27) it follows that the two pairs of summands in the first form of (22.28) are real for $z=x-ih$, whereas from the boundary conditions at $y=0$ and $y=-2h$ it follows that the terms in curly brackets in the second form of (22.28) have zero real parts. Since $H(z,t)$ is regular in the strip $0>y>-2h$ and vanishes on $y=-h$, it must vanish identically in the strip. Hence we have the following differential-difference equation of CISOTTI (1918):

$$f_{tt}(z+ih,t)+f_{tt}(z-ih,t)+ig[f'(z+ih,t)-f'(z-ih,t)]=0. \tag{22.30}$$

Let us now turn to the solution of (22.25) with initial conditions (22.22). We shall follow closely an exposition of SEDOV's (1948, 1957). However, the idea of the derivation is KOCHIN's (1935) and, in fact, really goes back to TONOLO (1913). The use of dimensional analysis can be extended to the three-dimensional problem; this was also done by KOCHIN.

We first remark that the initial-value problem can be solved by solving it for two special cases of (22.22), namely, first with $\eta(x,0)\equiv 0$ and then with $\eta_t(x,0)\equiv 0$. The sum of these two solutions will satisfy (22.22). Next we note that $\eta(x,0)$ has the dimension "length" and $\eta_t(x,0)$ the dimension "velocity", and that the solution f in each of the two initial-value problems will be proportional to some typical parameters associated with $\eta(x,0)$ or $\eta_t(x,0)$, respectively. Let us suppose that a is such a parameter with dimension $L^p T^q$ and that f is proportional to a. Since f has dimension L^2T^{-1} and g has dimension LT^{-2}, the Π theorem of dimensional analysis then states that f can be expressed as follows:

$$f(z,t)=az^{\alpha}g^{\beta}\chi\left(\frac{igt^2}{4z}\right), \tag{22.31}$$

where

$$\alpha=\tfrac{3}{2}-p-\tfrac{1}{2}q, \quad \beta=\tfrac{1}{2}(q+1). \tag{22.32}$$

(The factor $i/4$ in the argument of χ is chosen for later convenience.) Now substitute (22.31) into (22.25). One finds after some computation that

$$f_{tt}+igf'=iaz^{\alpha-1}g^{\beta+1}[\zeta\chi''(\zeta)+(\tfrac{1}{2}-\zeta)\chi'(\zeta)+\alpha\zeta]=0, \tag{22.33}$$

where $\zeta = igt^2/4z$. The differential equation obtained by setting the expression in square brackets equal to zero determines χ in terms of confluent hypergeometric functions:

$$\chi(\zeta) = A\,{}_1F_1(-\alpha, \tfrac{1}{2}; \zeta) + B\zeta^{\frac{1}{2}}\,{}_1F_1(\tfrac{1}{2}-\alpha, \tfrac{3}{2}; \zeta). \tag{22.34}$$

From this it follows that

$$\left.\begin{aligned} f(z,t;\alpha) &= a z^{\alpha} g^{\beta}\left[A\,{}_1F_1\left(-\alpha, \frac{1}{2}; \frac{igt^2}{4z}\right) + B\left(\frac{igt^2}{4z}\right)^{\frac{1}{2}} {}_1F_1\left(\frac{1}{2}, -\alpha, \frac{3}{2}; \frac{igt^2}{4z}\right)\right] \\ &= A f_1(z,t;\alpha) + B f_2(z,t;\alpha). \end{aligned}\right\} \tag{22.35}$$

Remembering that

$${}_1F_1(\gamma, \delta; 0) = 1, \qquad {}_1F_1'(\gamma, \delta; 0) = \gamma/\delta,$$

one may easily derive the following:

$$\left.\begin{aligned} f(z,0) &= A f_1(z,0) = A\,a g^{\beta} z^{\alpha}, \\ f'(z,0) &= A f_1'(z,0) = A\,a\alpha g^{\beta} z^{\alpha-1}, \\ f_t(z,0) &= B f_{2t}(z,0) = \tfrac{1}{2} B a i^{\frac{1}{2}} g^{\beta+\frac{1}{2}} z^{\alpha-\frac{1}{2}}. \end{aligned}\right\} \tag{22.36}$$

The solution (22.35) may be further generalized by replacing t by $t-t_0$ and z by $z-x_0$ (i.e., by a different choice of the dimensionless variable ζ). One may then further superimpose these solutions. For the purpose at hand it will be sufficient to retain $t_0=0$. Then we may form the solution

$$f(z,t) = \int_{-\infty}^{\infty} A(x_0)\, f_1(z-x_0, t; \alpha_1)\, dx_0 + \int_{-\infty}^{\infty} B(x_0)\, f_2(z-x_0, t; \alpha_2)\, dx_0. \tag{22.37}$$

One finds from (22.36) that

$$\left.\begin{aligned} f(z,0) &= a_1 g^{\beta_1} \int_{-\infty}^{\infty} A(x_0)\,(z-x_0)^{\alpha_1}\, dx_0, \\ f'(z,0) &= a_1 \alpha_1 g^{\beta_1} \int_{-\infty}^{\infty} A(x_0)\,(z-x_0)^{\alpha_1-1}\, dx_0, \\ f_t(z,0) &= \tfrac{1}{2} a_2 i^{\frac{1}{2}} g^{\beta_2+\frac{1}{2}} \int_{-\infty}^{\infty} B(x_0)\,(z-x_0)^{\alpha_2-\frac{1}{2}}\, dx_0. \end{aligned}\right\} \tag{22.38}$$

Let us now make some special choices of a, and hence of α and β. As a parameter describing the initial profile of the surface take

$$a_2 = \int_{-\infty}^{\infty} \eta(x,0)\, dx; \tag{22.39}$$

as a parameter describing the initial velocity distribution take

$$a_1 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} \eta_t(\xi, 0)\, d\xi. \tag{22.40}$$

Then a_1 has the dimension L^3T^{-1}, corresponding to $\alpha_1 = -1$, $\beta_1 = 0$, and a_2 the dimension L^2, corresponding to $\alpha_2 = -\frac{1}{2}$, $\beta_2 = \frac{1}{2}$. With these choices of α_1 and α_2 in (22.37) we take

$$\left.\begin{aligned} A(x_0) &= \frac{-1}{a_1 \pi} \int_{-\infty}^{x_0} \eta_t(\xi, 0)\, d\xi, \\ B(x_0) &= \frac{-2}{a_2 \pi i^{\frac{3}{2}}}\, \eta(x_0, 0). \end{aligned}\right\} \tag{22.41}$$

Then the last two equations of (22.38) become (after an integration by parts in the first one)

$$f'(z, 0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta_t(x_0, 0)}{x_0 - z}\, dx_0,$$

$$f_t(z, 0) = \frac{g}{\pi i} \int_{-\infty}^{\infty} \frac{\eta(x_0, 0)}{x_0 - z}\, dx_0.$$

From the Plemelj-Sokhotskii theorem we have

$$\left.\begin{aligned} f'(x - i\,0, 0) &= -i\,\eta_t(x, 0) + \frac{1}{\pi}\,\mathrm{PV} \int_{-\infty}^{\infty} \frac{\eta_t(x_0, 0)}{x_0 - x}\, dx_0, \\ f_t(x - i\,0, 0) &= -g\,\eta(x, 0) + \frac{g}{\pi i}\,\mathrm{PV} \int_{-\infty}^{\infty} \frac{\eta(x_0, 0)}{x_0 - x}\, dx_0. \end{aligned}\right\} \tag{22.42}$$

Thus the initial conditions (22.22) are satisfied.

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There remains to point out that for the special choices of $\alpha_1 = -1$ and $\alpha_2 = -\frac{1}{2}$ the corresponding confluent hypergeometric functions in (22.35) may be expressed in terms of Fresnel integrals or integrals of these. In fact, if we write (22.37) the form
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$$f(z, t) = \int_{-\infty}^{\infty} \Omega_1(z - x_0, t) \int_{-\infty}^{x_0} \eta_t(\xi, 0)\, d\xi\, dx_0 + \int_{-\infty}^{\infty} \Omega_2(z - x_0, t)\, \eta(x_0, 0)\, dx_0, \tag{22.43}$$

then

$$\left.\begin{aligned} \Omega_1(z, t) &= -\frac{2i}{z}\, e^{-i\frac{\pi}{2}\omega^2}\, \omega - \frac{2i}{z} \int_0^{\omega} du \int_0^{u} e^{-i\frac{\pi}{2}(v^2 - \omega^2)}\, dv, \\ \Omega_2(z, t) &= 2i \sqrt{\frac{2g}{\pi z}} \int_0^{\omega} e^{-i\frac{\pi}{2}(u^2 - \omega^2)}\, du, \end{aligned}\right\} \tag{22.44}$$

where

$$\omega^2 = \frac{g\, t^2}{2\pi z}.$$

One should also consult the discussion in LAMB's Hydrodynamics (1932, § 238, 239), where graphs are given which display the behavior of the surface profile corresponding to an initial elevation concentrated in the neighborhood of one point, i.e., essentially $-g^{-1}\,\Omega_{2t}(x - i\,0, t)$, and to a concentrated impulse, i.e., essentially $-g^{-1}\Omega_{1t}(x - i\,0, t)$. However, general aspects of the development of the surface profile have already been discussed in Sect. 15α.

It should be noted that the velocity potential (22.37) represents a much wider class of time-dependent gravity-wave motions than does (22.43). The initial-value problems corresponding to other values of α have been determined by SEDOV (1948) but the discussion will not be repeated here.

A class of solutions of (22.30) analogous to that found by SEDOV for (22.25) does not seem to have been given in the published literature. CISOTTI (1920) expands $f(z, t)$ in a power series in t, thus replacing (22.30) by a recursive set of difference equations. We refer to the original paper for his discussion of this set of equations.

In (15.22) we have already given the velocity potential and surface profile corresponding to a given initial profile; the derivation was based upon a Fourier analysis of the initial profile and the result was valid for either finite or infinite depth. The same procedure may be used for an initial velocity distribution. The combined result for the complex velocity potential and surface profile is given below in such a way as to include the possible presence of surface tension:

infinite depth:

$$\left.\begin{aligned} f(z,t) &= \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty d\xi\, \frac{1}{k}\left[-\sigma(k)\,\eta(\xi,0)\sin\sigma t + \eta_t(\xi,0)\cos\sigma t\right] e^{ik(z-\xi)},\\ \eta(x,t) &= \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty d\xi \left[\eta(\xi,0)\cos\sigma t + \frac{\sigma}{g k}\,\eta_t(\xi,0)\sin\sigma t\right]\cos k(x-\xi), \end{aligned}\right\} \quad (22.45)$$

where

$$\sigma^2 = g k + T k^3/\varrho;$$

depth h:

$$\left.\begin{aligned} f(z,t) &= \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty d\xi\, \frac{1}{k\sinh kh}\times\\ &\quad\times\left[-\sigma(k)\,\eta(\xi,0)\sin\sigma t + \eta_t(\xi,0)\cos\sigma t\right]\cos k(z-\xi+ih),\\ \eta(x,t) &= \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty d\xi\,\times\\ &\quad\times\left[\eta(\xi,0)\cos\sigma t + \frac{\sigma(k)}{g k\tanh kh}\,\eta_t(\xi,0)\sin\sigma t\right]\cos k(x-\xi), \end{aligned}\right\} \quad (22.46)$$

where

$$\sigma^2 = (g k + T k^3/\varrho)\tanh kh.$$

If $T=0$, the coefficients of $\eta_t(\xi,0)$ in the formulas for $\eta(x,t)$ reduce to σ^{-1}. When $T=0$, (22.45) is, of course, another form of (22.43).

It has already been indicated in Sect. 15 that the Cauchy-Poisson problem can also be solved for superposed fluids. SRETENSKII (1955) has investigated a further generalization in which the two fluids are each flowing with constant velocities for $t<0$ and then when $t=0$ a disturbance is suddenly created at the interface.

γ) Some other time-dependent problems. It is possible to solve a number of initial-value problems either by using Eq. (22.8) or by using the time-dependent Green functions (13.49) or (13.53) directly. The special situations treated, below fall roughly into the following four categories: wave motions resulting from a pressure distribution suddenly brought into existence at time $t=0$; waves resulting from a body set into motion at time $t=0$; waves resulting from an underwater explosion or a sudden movement of the bottom (tsunamis); and waves resulting from an initially displaced freely floating body.

Time-dependent pressure distributions. Suppose that the fluid is undisturbed for $t<0$ and that starting with $t=0$ the pressure over the free surface is a given function $p(x,z,t)$. The consequent motion of the fluid may be easily obtained, for this is just the problem formulated in (22.2) if we put $\eta(x,z,0)=\eta_t(x,z,0)=0$. Formula (22.8) then gives the velocity potential in the form

$$\Phi(x,y,z,t) = \frac{-1}{4\pi\varrho g}\iint_F d\xi\, d\zeta \int_0^t G_{tt}(\xi,0,\zeta,x,y,z;\tau,t)\,p(\xi,\zeta,\tau)\,d\tau + I. \quad (22.47)$$

see errata

In the two situations for which explicit GREEN's functions have been given, Eqs. (22.10) and (22.11), we may give explicit solutions for Φ:

infinite depth:

$$\Phi(x, y, z, t) = \frac{-1}{2\pi\varrho} \iint\limits_{-\infty}^{\infty} d\xi\, d\zeta \int\limits_{0}^{t} p(\xi, \zeta, \tau)\, d\tau \int\limits_{0}^{\infty} \cos\left(\sqrt{g k}\,(\tau - t)\right) e^{k y} J_0(k R)\, k\, dk; \tag{22.48}$$

depth h:

see errata

$$\left.\begin{aligned} \Phi(x, y, z, t) = \frac{-1}{2\pi\varrho} \iint\limits_{-\infty}^{\infty} d\xi\, d\zeta \int\limits_{0}^{t} p(\xi, \zeta, \tau)\, d\tau \int\limits_{0}^{\infty} \left(\cos \sqrt{g k \tanh k h}\,(\tau - t)\right) \times \\ \times \frac{\cosh k (y + h)}{\cosh k h} J_0(k R)\, k\, dk, \end{aligned}\right\} \tag{22.49}$$

where, as usual, $R^2 = (x - \xi)^2 + (z - \zeta)^2$.

The velocity potential for a moving pressure distribution is obtained from these expressions simply by letting

$$p(\xi, \zeta, \tau) = p_0(\xi - c\tau, \zeta, \tau).$$

If $p_0(\xi - c\tau, \zeta, \tau) = p_0(\xi - c\tau, \zeta) \cos\sigma\tau$ the resulting Φ is the velocity potential for a steadily moving pressure distribution of oscillating strength. LUNDE (1951b) has investigated the special case when $p(\xi, \zeta, \tau) = p\left(\sqrt{(\xi - c\tau)^2 + \zeta^2}\right)$ and has shown that as $t \to \infty$ the expressions (22.48) and (22.49), after a change to moving coordinates, approach asymptotically to the expressions (21.26) or (21.31) properly modified for circular symmetry (the assumed symmetry is not essential). The computation is interesting but will not be carried out here. This procedure for obtaining (21.26) or (21.31) yields the velocity potential without necessitating the extra boundary condition requiring the motion to vanish as $x \to +\infty$.

see errata

As was mentioned in connection with the solution of the Cauchy-Poisson problem, the GREEN's function for some other simple configurations can be found by the method of reflection.

The complex velocity potentials for two-dimensional motion which correspond to (22.48) and (22.49) are as follows:

infinite depth:

$$f(z, t) = \frac{-1}{\pi\varrho} \int\limits_{-\infty}^{\infty} d\xi \int\limits_{0}^{t} p(\xi, \tau)\, d\tau \int\limits_{0}^{\infty} \cos\left(\sqrt{g k}\,(\tau - t)\right) e^{-i k (z - \xi)}\, dk; \tag{22.50}$$

depth h:

$$f(z, t) = \frac{-1}{\pi\varrho} \int\limits_{-\infty}^{\infty} d\xi \int\limits_{0}^{t} p(\xi, \tau)\, d\tau \int\limits_{0}^{\infty} \cos\left(\sqrt{g k \tanh k h}\,(\tau - t)\right) \frac{\cos k (z - \xi + i h)}{\cosh k h}\, dk. \tag{22.51}$$

Certain special cases have been investigated in more detail. STOKER (1953) [see also WURTELE (1955)] has treated the motion resulting when a pressure distribution constant in time for $t > 0$ is suddenly applied to a uniformly moving stream of depth h. The velocity potential may be obtained from (22.51) by taking $p(\xi, \tau) = p_0(\xi - c\tau)$ and transferring to moving coordinates. His aim, as was that of LUNDE in the computations described above, was to show that the potential (21.40) can be derived without a special assumption about its behavior as $x \to +\infty$. The same can be carried through with (22.50) to derive (21.38). If one assumes $p(\xi, \tau) = p_1(\xi) \cos\sigma\tau + p_2(\xi) \sin\sigma\tau$, then one may also derive (21.21) or (21.23) from (22.49) or (22.50), respectively, as asymptotic expressions for large t without having to impose a radiation condition. Voit (1957b) has investigated the surface profile for large t when $p(\xi, \tau) = p(\tau)$ for $\xi < c\tau < cT$, $p(\xi, \tau) = 0$ for $\xi \geqq c\tau$ or for $\tau > T$.

Waves resulting when a body is set into motion. Many of the problems solved in Sects. 17 to 20 by means of source distributions can be formulated as initial-value problems and solved by the same procedure if one uses the appropriate time-dependent Green's function. We shall consider briefly several examples, omitting details.

see errata

In (19.28) the velocity potential was given for the motion resulting from an oscillator in a wall, described by (19.26). It was assumed there that a steady situation had been reached in which the motion was purely harmonic in the time. Suppose instead that the motion of the oscillator described by (19.26) is to start at $t=0$ and that for $t<0$ the oscillator and fluid are at rest. It is easy to verify that the time-dependent velocity $\Phi(x, y, z, t)$ potential is still given by (19.28) if for the Green's function G one uses (13.50) with $m=1$. The last term in (13.50) will give the transient aspects of the motion. For two-dimensional motion the time-dependent wave-maker has been considered by Kennard (1949), who also gives an estimate of time necessary for the transient terms to die out.

In (20.65) the velocity potential has been given for a "thin" ship moving with constant velocity c; it is assumed there that a steady state has been reached. Let us now suppose the same ship to move with velocity $c(t)$, $t>0$, but that it and the fluid have been at rest for $t<0$. As in (20.64) we take a coordinate system moving with the ship. Then from (20.26) it follows that the velocity potential $\Phi(x, y, z, t)$ must satisfy the boundary condition

$$\Phi_z(x, y, \pm 0, t) = \mp c(t)\, F_x(x, y).$$

A Green's function enabling us to construct Φ can be easily obtained from either (13.49) or (13.53). However, let us take $c(t) = C$, a constant, for $t>0$, i.e. we suppose the ship to attain instantaneously its final velocity. The Green's function for this situation has already been written out explicitly in (13.51). Setting there $u_0 = c$, $a_0 = \xi$, $b_0 = \eta$, $c_0 = \zeta$ and calling the resulting function $G(x, y, z, \xi, \eta, \zeta, t)$, the velocity potential for the problem at hand is

$$\Phi(x, y, z, t) = \frac{c}{2\pi} \iint\limits_{S_0} G(x, y, z, \xi, \eta, 0, t)\, F_x(\xi, \eta)\, d\xi\, d\eta. \tag{22.52}$$

see errata

Having found Φ, one may then compute the force upon the ship and obtain formulas analogous to (20.67) or (20.69). The computations for infinite depth was originally made by Sretenskii (1937); Lunde (1951a) gives an exposition of this result and extends it to include thin ships moving in an infinite expanse of fluid of depth h and down the center of a canal of width b and depth h. In these computations c is allowed to be an arbitrary function of t. We refer to Lunde's paper for the results.

Havelock (1948, 1949) has considered the accelerated motion of a submerged horizontal circular cylinder in fluid of infinite depth. The complex velocity potential is expanded in a Laurent series about the center, starting with a dipole. In order to satisfy the other boundary conditions, one makes use of (13.54) to obtain singularities of the proper sort. The boundary condition on the circle then yields an infinite set of equations for determining the coefficients in the Laurent series. After finding as many terms as seems necessary for a suitable approximation, one may compute the force on the cylinder. Havelock has carried this out for the first two singularities [a slight inconsistency in the approximation is corrected in MARUO (1957)] and has made numerical computations for an impulsive start and for a constant acceleration. Consider an impulsive start with instantaneous acceleration to constant speed c, and let the cylinder have radius a and its center be submerged to depth h. Then the two leading terms in the resistance are given by R_0, the steady-state resistance given in Eq. (20.52), and by the transient term

$$R_1 = \frac{1}{2}\pi g \varrho a^4 \nu^2 \left(\frac{\pi}{\nu c t}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\nu h} \sin\left(\frac{1}{4}\nu c t - \frac{\pi}{4}\right), \quad \nu = \frac{g}{c^2}. \tag{22.53}$$

Fig. 28, taken from MARUO (1957), shows $(R_0+R_1)/R_0$ plotted against ct/h for $c/\sqrt{gh}=1$.

An exposition of the theory of accelerated motion of submerged bodies is given by MARUO (1957, Chap. 3). Both two- and three-dimensional problems in fluid of infinite or finite depth are considered. We note that the use of KOCHIN'S H-function may be extended with no difficulty to time-dependent motion; this has been done by MARUO and earlier by HASKIND (1946b).

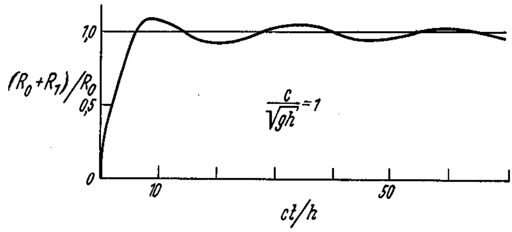

Fig. 28.

An investigation of PALM (1953) also fits into the category of problems under consideration. In considering flow over an uneven bottom in Sect. 20α, it was necessary to impose an upstream boundary condition in order to obtain uniqueness if the velocity is subcritical. In order to avoid this extra condition he formulated an initial-value problem in which the fluid is at rest and the bottom suddenly starts to move. The asymptotic expression for large t in a coordinate system moving with the bottom agrees with the results in Sect. 20α.

Tsunamis and submarine explosions. A tsunami is an ocean wave originating from a sudden upheaval or recession of the ocean floor. If one assumes an ocean of uniform depth h and if the disturbance occurs in a region S of the bottom, one may approximate this situation by the boundary-value problem in which

$$\Phi_y(x, -h, z, t) = \left\{\begin{matrix} V(x, z, t), & 0<t<T, \ (x, z) \text{ in } S, \\ 0, & \text{otherwise}. \end{matrix}\right\} \tag{22.54}$$

If the time-interval of the disturbance is short (i.e., if gT^2/h is small), the solution for Φ is given approximately by distributing over S sources of a form easily derived from (13.53). In fact, in (13.53) let $a=\xi$, $b=-h$, $c=\zeta$, and let $2m(t) = 2m(\xi,\zeta,t) = -\Phi_y(\xi, -h, \zeta, t)/2\pi$; denote the resulting function by $\Phi_s(x, y, z, \xi, -h, \zeta, t)$. Then

$$\Phi(x, y, z, t) = \iint_S \Phi_s(x, y, z, \xi, -h, \zeta, t)\, d\xi\, d\zeta \tag{22.55}$$

is the approximate solution. If one assumes $V(x, z, t) = V(x, z)$ for $0<t<T$, then Φ_s takes the following simple form for $t>T$:

$$\left.\begin{aligned} \Phi_s(x, y, z, \xi, -h, \zeta, t) = -\frac{1}{2\pi} V(\xi, \zeta) \int_0^\infty \frac{\cosh k(y+h)\, J_0(kR)}{\sinh kh \cosh kh} \times \\ \times [\cos\sigma(t-T) - \cos\sigma t]\, dk, \end{aligned}\right\} \tag{22.56}$$

where $\sigma^2 = gk \tanh kh$. If the deformation is assumed to take place so quickly that one may let $T \to 0$ while keeping $VT = L(\xi, \zeta)$ constant (i.e., keeping the same total deformation), (22.56) becomes

$$\Phi_s(x, y, z; \xi, -h, \zeta, t) = \frac{-1}{2\pi} L(\xi, \zeta) \int_0^\infty \frac{\cosh k(y+h)\, J_0(kR)}{\sinh kh \cosh kh}\, \sigma(k) \sin \sigma(k)\, t\, dk, \quad (22.57)$$

and the solution (22.55) is no longer approximate for the formulated problem.

A further approximation may be obtained by assuming the area of disturbance to be so localized that one may assume the whole disturbance to originate at one point, say $(0, -h, 0)$. Then (22.55) becomes simply (22.57) with L replaced by $Q = \iint L\, d\xi\, d\zeta$ and $R^2 = x^2 + z^2$. Although this may be a reasonable approximation to the explosion of a mine on the ocean floor, it is not in general suitable for a tsunami since the diameter of the region of disturbance in the latter may be many times the depth of fluid.

A comparison of (22.55), with (22.57) for Φ_s, with (22.15) shows that one may expect the same qualitative behavior for tsunamis as for waves resulting from an initial deformation of the free surface. In fact, if one makes the substitution (22.19) in the expressions for the surface profiles, they become the following, respectively, for the initially displaced surface and the tsunami:

$$\left.\begin{aligned} \eta(x, z, t) &= \frac{1}{\pi} \iint_{-\infty}^{\infty} \eta(\xi, \zeta, 0)\, \frac{1}{R^2}\, d\xi\, d\zeta \int_0^\infty \varkappa^3 J_0(\varkappa^2) \cos\left(2\omega\varkappa \sqrt{\tanh \varkappa^2 \frac{h}{R}}\right) d\varkappa, \\ \eta(x, z, t) &= \frac{1}{\pi} \iint_{-\infty}^{\infty} L(\xi, \zeta)\, \frac{1}{R^2}\, d\xi\, d\zeta \int_0^\infty \varkappa^3 \frac{J_0(\varkappa^2)}{\cosh^2\left(\varkappa^2 \frac{h}{R}\right)} \cos\left(2\omega\varkappa \sqrt{\tanh \varkappa^2 \frac{h}{R}}\right) d\varkappa. \end{aligned}\right\} \quad (22.58)$$

One may study the development of η along the lines worked out in Sect. 15 for two dimensions.

Many of the investigations of tsunamis have been devoted to an examination of the profile for a given type of initial bottom disturbance. The classical papers on tsunamis are by SANO and HASEGAWA (1915) and SYONO (1936). They have recently been investigated by TAKAHASHI (1942, 1945, 1947), ICHIYE (1950), GAZARYAN (1955) and others. Since the shape of the bottom and the configuration of the shore are of obvious importance in a geophysical application of the theory, much recent attention has been given to this aspect of the propagation of tsunamis. GRIGORASH (1957a) has given a brief survey of the literature together with a substantial bibliography.

The waves resulting from an exploding submerged mine may be represented approximately by using the velocity potential for a source whose strength $m(t)$ has the form of a square pulse of duration T. One may then determine Φ from either (13.49) or (13.53). If one assumes T very small and forms the limit as $T \to 0$ while keeping $mT = Q$ constant, one finds easily the following expressions for Φ:

infinite depth:

$$\Phi = 2Q \int_0^\infty e^{k(y+b)} J_0(kR)\, \sigma(k) \sin \sigma t\, dk, \qquad \sigma^2 = gk; \quad (22.59)$$

depth h:

see errata

$$\Phi = 2Q \int_0^\infty \frac{\cosh k(h+b) \cosh k(y+h)}{\sinh kh \cosh kh}\, \sigma(k) \sin \sigma t\, dk, \qquad \sigma^2 = gk \tanh kh. \quad (22.60)$$

Again one may examine the development of the surface profile by the methods developed in Sect. 15.

Investigations of waves generated by a sudden pulse of the above or similar sort have been made by LAMB (1913, 1922) and TERAZAWA (1915); both took the fluid to be infinitely deep. SRETENSKII (1950, 1949) has made a similar study when the source (two-dimensional) is situated on the bottom of a rectangular basin and within a fluid layer covering a solid sphere. SEZAWA (1929a, b) has included the effect of compressibility of the fluid.

One should recognize that such studies can elucidate only a small part of the phenomena associated with underwater explosions. An investigation of the migration and oscillation of the explosion bubble requires different analytical methods. Furthermore, if the explosion is too violent the linearized boundary condition on the free surface may not be a useful approximation.

Freely floating bodies. The motion of a freely floating body following an initial displacement is of considerable interest and practical importance, but also leads to a difficult mathematical problem. Uniqueness of solution follows from the argument in Sect. 22α. For the sake of perspicuity let us restrict ourselves to motion constrained to be vertical, i.e., heaving motion. Then from (19.59) and (19.62) the boundary conditions to be satisfied on the surface of the body in its equilibrium position, S_0, are

$$\Phi_n(x, y, z, t) = \dot{y}_1(t)\, n_y(x, y, z), \quad (x, y, z) \text{ on } S_0, \tag{22.61}$$

$$M\ddot{y}_1(t) + \varrho g I^A y_1(t) = -\varrho \iint_{S_0} \Phi_t(\xi, \eta, \zeta, t)\, n_y(\xi, \eta, \zeta)\, d\sigma. \tag{22.62}$$

(The notation is explained in Sect. 19β.) In addition Φ must satisfy the free-surface condition

$$\Phi_{tt}(x, 0, z, t) + g\,\Phi_y(x, 0, z, t) = 0 \tag{22.63}$$

and initial conditions, say

$$\Phi_t(x, 0, z, 0) = \Phi_y(x, 0, z, 0) = 0, \tag{22.64}$$

$$\dot{y}_1(0) = \dot{y}_{10}, \qquad y_1(0) = y_{10}. \tag{22.65}$$

As in many previous cases one may reduce the problem to the solution of an integral equation by use of a GREEN's function. In (13.49) replace (a, b, c) by (ξ, η, ζ) and $m(t)$ by $\gamma(\xi, \eta, \zeta, t)$; denote the resulting function by Φ_s:

see errata

$$\left.\begin{aligned} \Phi_s(x, y, z, \xi, \eta, \zeta, t) &= \gamma(\xi, \eta, \zeta, t)\left[\frac{1}{r} - \frac{1}{r_1}\right] + \\ &+ 2\int_0^\infty (g k)^{\frac{1}{2}}\, e^{k(y+\eta)}\, J_0(kR)\, dk \int_0^t \gamma(\xi, \eta, \zeta, \tau) \sin\left[(g k)^{\frac{1}{2}}(t-\tau)\right] d\tau. \end{aligned}\right\} \tag{22.66}$$

We now attempt to express Φ by the integral

$$\Phi(x, y, z, t) = \iint_{S_0} \Phi_s(x, y, z, \xi, \eta, \zeta, t)\, d\sigma, \tag{22.67}$$

for then (22.63) and (22.64) will be satisfied. One should note especially that the relation of Φ to γ is more complicated here than in problems typified by (16.12), for the past history of γ is involved in Φ_s. The conditions (22.61) and (22.62) now become

$$-2\pi\gamma(x, y, z, t) + \iint_{S_0} \Phi_{sn}(x, y, z, \xi, \eta, \zeta, t)\, d\sigma = \dot{y}_1(t)\, n_y(x, y, z), \tag{22.68}$$

$$M\ddot{y}_1(t) + \varrho g I^A y_1(t) = -\varrho \iint_{S_0} d\sigma \iint_{S_0} dS\, \Phi_{st}(x, y, z, \xi, \eta, \zeta, t)\, n_y(\xi, \eta, \zeta), \tag{22.69}$$

where γ also enters into the equations through Φ_s. The two equations form a pair of coupled integro-differential equations for γ and y_1. It is evident that one can probably not hope for an analytic solution even for simple configurations.

see errata

SRETENSKII (1937b), for two dimensions, and later HASKIND (1946b) for three dimensions simplified the problem further by assuming the body to be "thin", i.e., if the surface is given by $z = \pm F(x, y)$, by replacing the boundary condition (22.61) by

$$\Phi_z(x, y, \pm 0, t) = \mp \dot{y}_1(t)\, F_y(x, y) \tag{22.70}$$

and S_0 by the projection of S_0 on the plane $z = 0$ [cf. (20.26) and (20.64)]. With this further assumption one can immediately satisfy (22.68) by taking

$$\gamma(x, y, t) = -\frac{1}{2\pi} \dot{y}_1(t)\, F_y(x, y). \tag{22.71}$$

Eq. (22.69) then becomes an integro-differential equation for $y_1(t)$.

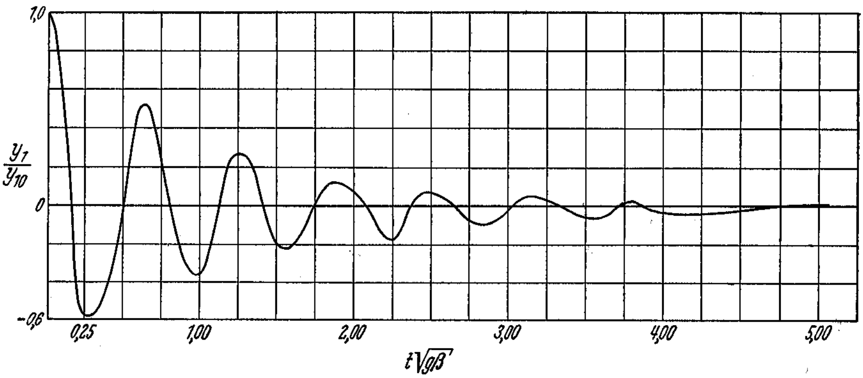

Fig. 29.

The procedure is open to some objection in that the substituted boundary condition (22.70) does not seem to fit into the general perturbation scheme as developed in Sects. 10α, 19α and 20β. It is thus not clear what physical problem really corresponds to the mathematical problem. However, this seems to be the closest anyone has come to reducing the equations to a manageable form. SRETENSKII solved his resulting integro-differential equation numerically for a surface described by $F(y) = l\, e^{\beta|y|}$, where

$$l = \frac{\pi g}{800} = 3.85 \text{ cm}, \qquad \beta = \frac{100}{g} = 0.104 \text{ cm}^{-1}.$$

The resulting graph of y_1/y_{10} is shown in Fig. 29 with a dimensionless abscissa $t\sqrt{g\beta}$. In spite of the questionableness of the formulation of the problem, the graph is instructive in showing the difference between a damped harmonic oscillation and the solution of SRETENSKII'S integro-differential equation. Approximate methods of solution to the problem which assume that the fluid motion at any instant is independent of its past history lead to damped harmonic oscillations.

23. Waves in basins of bounded extent. The study of wave motion in a basin presents no special difficulties not already encountered earlier, and has a particular interest because of the many opportunities of observing such waves. Certain general aspects of the problem may be considered as being contained in earlier sections. For example, the general discussion of initial-value problems in Sect. 22α applies to motion in a basin. However, in order to make use of the results, in particular of Eq. (22.8), in constructing a solution, one must have prior knowledge of the time-dependent GREEN'S function for the geometric boundary. Although the method of images can be used together with (13.49) or (13.53) to construct the GREEN'S function for certain simple configurations, an explicit analytic solution is generally not available.

The time-dependent problem has also been approached in another manner by HADAMARD (1910, 1916), who derived an integro-differental equation for the function $\eta(x, y, t)$ describing the free surface. HADAMARD'S short notes have been worked out by BOULIGAND (1912, 1926, 1927) and developed further. Certain of BOULIGAND'S investigations indicate that singularities which may occur at the intersection of the plane $y=0$ with the basin walls are a result of linearizing the free-surface boundary condition. For an exact statement one should consult the original papers. There is a brief treatment of HADAMARD'S equation in VERGNE (1928, § 10, 14). MOISEEV (1953) has developed a treatment of the time-dependent problem which generalizes somewhat the method used in Sect. 23 α.

In Sect. 23 α we give some general theorems concerning motions periodic in time, and another solution of the initial-value problem. In Sect. 23 β wave motions for several special configurations of the boundary are given. In Sect. 23 γ the theory of wave motion in movable basins is considered.

α) *Periodic waves in basins: general theorems.* If the motion is periodic in time, the velocity potential may be found by solving a Fredholm integral equation, obtained after introduction of an appropriate GREEN'S function. Assume $\Phi(x, y, z, t) = \varphi(x, y, z)\cos(\sigma t + \tau)$; then φ must satisfy the boundary conditions

$$\left.\begin{array}{l}\varphi_y(x, 0, z) - \nu\varphi(x, 0, z) = 0, \qquad (x, z) \text{ in } F, \quad \nu = \sigma^2/g, \\ \varphi_n = 0, \qquad (x, y, z) \text{ on } S,\end{array}\right\} \tag{23.1}$$

where F is the part of the plane $y=0$ occupied by the free surface at rest and S is the surface of the basin. Let $G(x, y, z, \xi, \eta, \zeta)$ be the GREEN'S function for NEUMANN'S problem, i.e.,

$$G = \frac{1}{r} + G_0,$$

where G_0 is regular in the region occupied by fluid and G satisfies the conditions

$$G_n = c \text{ on } S, \qquad G_y(x, 0, z, \xi, \eta, \zeta) = c \text{ on } F, \tag{23.2}$$

where c is an arbitrary nonzero constant. In addition, in order to make the definition of φ unique we require

$$\iint\limits_{S+F} \varphi\, d\sigma = 0.$$

It then follows from GREEN'S theorem that

$$\varphi(x, y, z) = \frac{1}{4\pi}\iint\limits_F G\varphi_y\, d\xi\, d\zeta = \frac{\nu}{4\pi}\iint\limits_F G\varphi(\xi, 0, \zeta)\, d\xi\, d\zeta. \tag{23.3}$$

If one now lets $y \to 0$, one obtains

$$\varphi(x, 0, z) = \frac{\nu}{4\pi}\iint\limits_F G(x, 0, z, \xi, 0, \zeta)\,\varphi(\xi, 0, \zeta)\, d\xi\, d\zeta, \tag{23.4}$$

a homogeneous Fredholm integral equation for $\varphi(x, 0, z)$. If $\varphi(x, 0, z)$ can be found, then $\varphi(x, y, z)$ is determined by (23.2). From the theory of such integral equations there will exist a sequence $\nu_1, \nu_2, \ldots$ of eigenvalues for which (23.4) will yield solutions $\varphi_1, \varphi_2, \ldots$. The functions φ_i corresponding to different ν_i-s are orthogonal on F, as shown in (16.10). If several ν_i-s have the same value, the corresponding φ_i-s can be orthogonalized. The φ_i also form a complete set on F. Each solution φ_i yields a standing wave in the basin.

It is possible to use these solutions to solve the initial-value problem formulated in (22.2), but with $p=0$. Let $\eta(x, z, 0)$ and $\eta_t(x, z, 0)$ be given. We try to express $\Phi(x, 0, z, t)$ in the following form:

$$\Phi(x, 0, z, t) = \sum_{i=1}^{\infty} a_i \varphi_i(x, 0, z) \cos\sigma_i t + b\varphi_i(x, 0, z) \sin\sigma_i t. \tag{23.5}$$

Then

$$\left.\begin{aligned} -g\eta(x, z, 0) &= \Phi_t(x, 0, z, 0) = \sum \sigma_i b_i \varphi_i(x, 0, z), \\ -g\eta_t(x, z, 0) &= \Phi_{tt}(x, 0, z, 0) = -\sum \sigma_i^2 a_i \varphi_i(x, 0, z). \end{aligned}\right\} \tag{23.6}$$

Since the φ_i form a complete set of orthogonal functions over F, the coefficients a_i and b_i can be determined in the usual manner. $\Phi(x, y, z, t)$ is then determined by (23.5) and (23.3).

In order to use the integral equation (23.4) one must first find G, the Green's function to a Neumann problem for a region having a corner along the curve of intersection of the plane $y=0$ and the basin walls. The difficulty with the corner can be overcome in certain cases. If the basin wall intersects the plane perpendicularly, then the basin plus its reflection in the plane $y=0$ has a boundary without this corner. If $\gamma(x, y, z, \xi, \eta, \zeta)$ is a Green's function for the Neumann problem for the extended region, then

$$G(x, y, z, \xi, \eta, \zeta) = \tfrac{1}{2}[\gamma(x, y, z, \xi, \eta, \zeta) + \gamma(x, -y, z, \xi, \eta, \zeta)] \tag{23.7}$$

is a Green's function for the original region. For some other special regions one may construct a Green's function by the method of images, even though the intersection with the plane $y=0$ is not perpendicular.

As mentioned above, each φ_i represents a standing wave of frequency σ_i. It may happen, as we shall see presently, that two or more σ_i-s are equal. Let σ be such an eigenvalue and $\varphi^{(1)}$ and $\varphi^{(2)}$ two of the corresponding potential functions. By forming the standing-wave solution

$$[\lambda_1\varphi^{(1)} + \lambda_2\varphi^{(2)}]\cos\sigma t, \qquad \lambda_1 + \lambda_2 = 1, \tag{23.8}$$

one may vary continuously the position of the nodal curves, say. If n independent φ_i correspond to σ, then the possible nodal curves form an $(n-1)$-parameter family of curves in F. With the two solutions $\varphi^{(1)}$ and $\varphi^{(2)}$ one may also form the solution

$$\Phi(x, y, z, t) = \varphi^{(1)}(x, y, z)\cos\sigma t + \varphi^{(2)}(x, y, z)\sin\sigma t. \tag{23.9}$$

The nodal curves will now migrate from those of $\varphi^{(1)}$ to those of $\varphi^{(2)}$, and then on again to those of $\varphi^{(1)}$. If $\varphi^{(1)}$ and $\varphi^{(2)}$ have a common zero at, say, (x_0, z_0), then a nodal curve for Φ will always pass through (x_0, z_0). Near (x_0, z_0) the waves will appear to progress about (x_0, z_0) like spokes moving about a wheel. There may, of course, be several such centers.

β) Some special boundaries. It is possible to give explicit solutions for standing waves for several particular configurations of the basin. The variety of such configurations, however, is rather small. As a preliminary we note that if the basin has a flat bottom at depth h and if the side walls form a vertical cylinder making a section C with $y=0$ then, from Sect. 13α, we have

$$\left.\begin{aligned} \Phi(x,y,z,t) &= \varphi(x,z)\cosh m_0(y+h)\cos(\sigma t+\tau),\\ m_0\tanh m_0 h - \frac{\sigma^2}{g} &= 0, \end{aligned}\right\} \tag{23.10}$$

where $\varphi(x,z)$ is a solution of

$$\varphi_{xx}+\varphi_{zz}+m_0^2\varphi=0 \tag{23.11}$$

satisfying

$$\varphi_n=0 \text{ on } C. \tag{23.12}$$

The boundary condition (23.12) will limit m_0, and hence σ, to a discrete sequence of eigenvalues

$$m_0^{(1)}, m_0^{(2)}, \ldots; \sigma_1, \sigma_2, \ldots. \tag{23.13}$$

In a coordinate system in which (23.11) can be separated it is usually possible to find the standing waves in basins whose side walls are constant-coordinate surfaces. These statements will be illustrated below for rectangular and cylindrical coordinates.

In connection with the special cases treated below we call attention to papers by HONDA and MATSUSHITA (1913) and SASAKI (1914). The authors investigated experimentally in a systematic way the various modes of motion in rectangular, triangular, circular, ring-shaped, circular-sectorial and ring-sectorial basins and compared measured with calculated periods. In most cases the agreement is with 2%. Photographs showing the various modes were obtained by sprinkling the surface with aluminum powder and exposing a photographic plate for about one period. The nodes then show up as dots, the rest as streaks. In connection see errata with a study of the excitations of waves in a port, McKNOWN (1953) has also investigated experimentally the standing waves in circular and square basins; some striking photographs are included. APTÉ (1957) has studied further the theory of the excitation of standing waves in a square basin and has also given experimental results. Perhaps the first theoretical investigation was by RAYLEIGH (1876, pp. 272—279); he compared his predicted frequencies with observations of his own and of GUTHRIE (1875).

Rectangular basin. Let the basin walls be given by

$$x=0, \quad x=l, \quad z=0, \quad z=b, \quad y=-h.$$

Then from (13.6) one may write down immediately the solution

$$\left.\begin{aligned} &\Phi = A\cosh m_0(y+h)\cos\frac{2\pi}{l}x\cos\frac{p\pi}{b}z\cos(\sigma t+\tau),\\ &m_0^2=\pi^2\left(\frac{q^2}{l^2}+\frac{p^2}{b^2}\right), \quad \frac{\sigma^2}{g}=m_0\tanh m_0 h, \quad p,q=0,1,2,\ldots \end{aligned}\right\} \tag{23.14}$$

Thus the choice of the integers p and q determines m_0 and then σ. If the basin is square, i.e., $l=b$, then the same values of m_0 and σ may correspond to two different solutions obtained by interchanging p and q, assuming $p\neq q$. However, this may also occur for other rectangular basins if b and l are commensurate.

Circular-cylinder basin. Let the basin have radius a. Then from (13.8) we find the solutions

$$\left.\begin{aligned} &\Phi = A\cosh m_0(y+h)J_n(m_0R)\cos(n\alpha+\delta)\cos(\sigma t+\tau), \quad n=0,1,2,\ldots,\\ &J_n'(m_0 a)=0, \quad \frac{\sigma^2}{g}=m_0\tanh m_0 h. \end{aligned}\right\} \tag{23.15}$$

Thus m_0 must be selected so that $m_0 a$ is one of the zeros of J_n'; this then determines σ. For $n=0$ the wave crests and nodes lie on concentric circles, the number of such nodal circles depending upon which zero of J_n' is used to determine m_0. If $n \geqq 1$, then to the same σ there correspond two independent solutions ($\delta=0$, $\delta=\frac{1}{2}\pi$), and the remarks made in connection with (23.8) and (23.9) apply.

The standing waves in a basin shaped like a sector of a circle may be obtained from (13.8). If α_0 is the angle of the sector, then

$$\Phi = A\cosh m_0(y+h)\, J_{\frac{n\pi}{\alpha_0}}(m_0 R)\cos\frac{n\pi}{\alpha_0}\alpha\cos(\sigma t+\tau), \qquad n=0,1,\ldots,$$

$$J'_{\frac{n\pi}{\alpha_0}}(m_0 a)=0, \qquad \frac{\sigma^2}{g}=m_0 h\tanh m_0 h.$$

If the basin is ring-shaped, with inner radius b and outer radius a, then from (13.8) one finds [cf. Sano (1913), Campbell (1953)]:

$$\left.\begin{aligned} \Phi &= A\cosh m_0(y+h)\,[Y_n'(m_0 b)\,J_n(m_0 R) - \\ &\quad - J_n'(m_0 b)\,Y_n(m_0 R)]\cos(n\alpha+\delta)\cos(\sigma t+\tau), \\ &Y_n'(m_0 b)\,J_n'(m_0 a) - J_n'(m_0 b)\,Y_n'(m_0 a) = 0, \\ &\frac{\sigma^2}{g} = m_0 h\tanh m_0 h, \qquad n=0,1,\ldots. \end{aligned}\right\} \tag{23.16}$$

Formulas for sectors of a ring may be obtained and are similar to (23.15).

Basins with sloping side-walls. There are very few explicit solutions known when the sides are not vertical. If the basin is a horizontal cylinder bounded at either end by vertical walls at, say, $z=0$ and $z=l$, the theory of progressive waves in canals, developed in Sect. 18γ, can be carried over with only small changes, namely replacement of $\cos(kz-\sigma t)$ by $\cos kz\cos(\sigma t+\tau)$ where now k is restricted to the values $n\pi/l$ and σ correspondingly. Thus (18.39) and (18.43) give the velocity potentials, after the indicated modifications, for various modes of oscillation of a fluid in a basin of triangular section whose sides form an angle of 45° with the horizontal. However, even though these formulas may be used also for the two-dimensional modes, when $k=0$, they do not give the gravest two-dimensional mode except by a limiting process [described, e.g., in Lamb (1932, p. 443)].

The two-dimensional modes of motion in triangular basins whose sides form an angle $\gamma=m\pi/n$ with the horizontal may also be studied by use of the methods introduced in Sect. 17β for standing waves on beaches. Indeed, it is apparent that Kirchhoff (1879) considered his investigation of waves on beaches as a preliminary to the problem at hand. Because his approach is systematic we shall describe it.

In order to use the results of 17γ we take one side as $y=-x\tan\gamma$, i.e., $z=r\,e^{i\gamma}$; let the other side be given by

$$z = 2a - r\,e^{i\gamma}. \tag{23.17}$$

Then the complex potential $f(z)$ must satisfy not only (17.31) and (17.32), but also

$$f(2a - r\,e^{i\gamma}) = \bar f(2a - r\,e^{-i\gamma}), \tag{23.18}$$

which, taken with (17.31), implies that

$$f(z) = f\big(z\,e^{-i4\gamma} + 2a\,e^{-i2\gamma}(1-e^{-i2\gamma})\big). \tag{23.19}$$

In order to satisfy (17.31), (17.32) and (23.19) KIRCHHOFF first takes

$$f(z) = B_h z^h + \cdots + B_{h+k} z^{h+k}. \tag{23.20}$$

Substitution in (17.32) yields (with $\beta = e^{-2i\gamma}$ as before)

$$1-\beta^h = 0, \quad B_{n+1} = \frac{-i\nu}{n+1}\,\frac{1+\beta^n}{1-\beta^{n+1}}\,B_n, \quad 1+\beta^{h+k} = 0. \tag{23.21}$$

Thus, since $\gamma = m\pi/n$, one must have $h = pn$, $p = 0, 1, \ldots,$ and $k = \frac{1}{2}n = q$, an integer. If one takes $p = 0$, then (23.20) becomes

$$f(z) = B_0\left\{1 + \sum_{s=1}^{q} (-1)^s z^s \frac{\nu^s}{s!}\,\frac{e^{is\gamma}}{\cos s\gamma} \cot\gamma \ldots \cot s\gamma\right\}. \tag{23.22}$$

Conditions (17.31) requires B_0 to be real. Condition (23.18) or (23.19) remains to be satisfied. The function $f(z)$ in its assumed form, is apparently overdetermined, and it is possible to show that for $q > 3$ not all conditions can be satisfied. For $q = 2$, $m = 1$ and $q = 3$, $m = 1$, (23.18) can be satisfied. The potential functions are as follows:

$$\left.\begin{aligned} &\gamma = \pi/4: \\ &f(z) = B_0\left[1-(1+i)\,\nu z + \tfrac{1}{2} i\nu^2 z^2\right] = \tfrac{1}{2} i B_0 (\nu z - 1 + i)^2 \\ &\quad = B_0\left[(1-\nu x)(1+\nu y) - i\nu(x+y)(1-\nu(x-y))\right], \quad a = 1/\nu,\ h = 1/\nu; \end{aligned}\right\} \tag{23.23}$$

$$\left.\begin{aligned} &\gamma = \pi/6: \\ &f(z) = B_0\left[1 - (\sqrt{3}+i)\,\nu z + \tfrac{1}{2}(1+i\sqrt{3})\,\nu^2 z^2 - \tfrac{1}{6} i\nu^3 z^3\right] \\ &\quad = -\tfrac{1}{6} i B_0\left[2 + i(\nu z - \sqrt{3} + i)^3\right] \\ &\quad = -\tfrac{1}{6} B_0\left[2 + (\nu y + 1)\left[(\nu y+1)^2 - 3(\nu x - \sqrt{3})^2\right] + \right. \\ &\quad\quad \left. + i(\nu x + \nu y\sqrt{3})(\nu x - \nu y\sqrt{3} - 2\sqrt{3})(\nu x - \sqrt{3})\right], \quad a = \sqrt{3}/\nu,\ h = 1/\nu. \end{aligned}\right\} \tag{23.24}$$

Here h is the depth of fluid at the deepest point. The surface profile for $\gamma = 45°$ is a straight line, for $\gamma = 30°$ a parabola.

In order to find the higher modes of oscillation KIRCHHOFF returns to the form (17.33) for $f(z)$. It then follows as before that (17.34) must hold and that n must be even, say $2q$. Now, however, instead of taking $\lambda = 1$ it is left to be determined by (23.19). Substitution of (17.33) into (23.19) gives

$$A_{k+2} = A_k \exp\left[i\,2\lambda\nu a\beta^{k+1}(1-\beta)\right], \quad k = 0, 1, \ldots, n-3. \tag{23.25}$$

Altogether there are then $n-1+n-2 = 2n-3$ independent equations to determine $A_1, \ldots, A_{n-1}$ and also λ and νa. Again the conditions can be satisfied for $\gamma = \pi/4$ and $\gamma = \pi/6$.

The solutions for $\gamma = \pi/4$ are as follows, where C is an arbitrary real constant:

$$\left.\begin{aligned} f(z) &= C\left[\cos\lambda\nu(z - a(1-i)) + \cosh\nu\lambda(z - a(1-i))\right], \\ \lambda &= \coth\lambda\nu a = -\cot\lambda\nu a; \\ f(z) &= C\left[\cos\lambda\nu(z - a(1-i)) - \cosh\lambda\nu(z - a(1-i))\right], \\ \lambda &= \tanh\lambda\nu a = \tan\lambda\nu a. \end{aligned}\right\} \tag{23.26}$$

The values of λ and ν can easily be determined graphically. For the first set of solutions the values of $\lambda\nu a$ will be slightly more than $3\pi/4, 7\pi/4, \ldots,$ for the second set slightly less than $7\pi/4$, $11\pi/4, \ldots$ These two sets of solutions correspond, respectively to (18.39) and (18.43) with $k=0$; the eigenvalues λ may be identified with m_i/ν and n_i/ν, respectively. KIRCHHOFF and HANSEMANN (1880) carried out an experimental investigation of the first three antisymmetric modes [Eq. (23.23) gives the first one]; they compare frequencies and positions of maxima and minima. The agreement seems satisfactory, although corrections for surface tension were necessary for the two higher modes.

The solution for $\gamma=30°$ is the following:

$$\left.\begin{aligned} f(z) = C\Big[&\frac{1}{\lambda+1}\,e^{i\lambda\nu[z-a]} + \frac{1}{\lambda-1}\,e^{-i\lambda\nu[z-a]} + \\ &+\frac{1}{\lambda+1}\,e^{i\beta^2\lambda\nu[z-a-\beta^2 a]} + \frac{1}{\lambda-1}\,e^{-i\beta^2\lambda\nu[z-a-\beta^2 a]} + \\ &+\frac{1}{\lambda+1}\,e^{-i\beta\lambda\nu[z-a-\beta a]} + \frac{1}{\lambda-1}\,e^{i\beta\lambda\nu[z-a-\beta a]}\Big], \end{aligned}\right\} \tag{23.27}$$

where C is an arbitrary real constant, $\beta=\frac{1}{2}(1-i\sqrt{3})$, $\beta^2=-\bar{\beta}=-\frac{1}{2}(1+i\sqrt{3})$, and the eigenvalues for λ and ν are determined by the equations

$$\left.\begin{aligned} \frac{\lambda^2-1}{\lambda} = -\sqrt{3}\cot\lambda\nu a, \quad \frac{\lambda^2+\beta}{\lambda} = -i(1+\beta)\cot\beta\lambda\nu a, \\ \frac{\lambda^2+\bar{\beta}}{\lambda} = +i(1+\bar{\beta})\cot\bar{\beta}\lambda\nu a. \end{aligned}\right\} \tag{23.28}$$

If λ is a solution of (23.28), then also $-\lambda$, $\bar{\lambda}$, $\beta\lambda$ and $\bar{\beta}\lambda$ are solutions. There exists a real solution which may be found from the equations

$$\cosh\sqrt{3}\,\lambda\nu a = 2\sec\lambda\nu a - \cos\lambda\nu a, \qquad \lambda = \frac{\sinh\sqrt{3}\,\lambda\nu a - \sqrt{3}\sin\lambda\nu a}{\cosh\sqrt{3}\,\lambda\nu a - \cos\lambda\nu a}. \tag{23.29}$$

The other solutions which may be generated from these do not lead to expressions different from (23.27). The first eigenvalue for $\lambda\nu a$ is a trifle to the right of $3\pi/2$. The form of the free surface corresponding to (23.17) is given by

$$\left.\begin{aligned} \eta(x,t) = \frac{\sigma}{g}\,C\Big\{&\frac{-2}{\lambda^2-1}\cos\lambda\nu(x-a) + \\ &+\left[\frac{1}{\lambda+1}\,e^{\frac{1}{2}\sqrt{3}\lambda\nu x} + \frac{1}{\lambda-1}\,e^{-\frac{1}{2}\sqrt{3}\lambda\nu x}\right]\cos\frac{1}{2}\lambda\nu(x-2a) + \\ &+\left[\frac{1}{\lambda+1}\,e^{-\frac{1}{2}\sqrt{3}\lambda\nu(x-2a)} + \frac{1}{\lambda-1}\,e^{\frac{1}{2}\sqrt{3}\lambda\nu(x-2a)}\right]\cos\frac{1}{2}\lambda\nu x\Big\}\sin(\sigma t+\tau). \end{aligned}\right\} \tag{23.30}$$

Note that (23.24) and (23.27) both give only symmetric modes. MACDONALD (1896) states that antisymmetric modes, if they exist, cannot be represented in the assumed forms (23.20) or (17.33).

VINT (1923) has succeeded in finding an infinite number of modes of motion in an inverted four-sided pyramid, each of whose sides makes a 45° angle with the horizontal. We refer to the original paper for the exact formulas.

Additional solutions have been obtained by inverse methods by SEN (1927) and by STORCHI (1949, 1952). STORCHI's result, although restricted to two-dimensional motion, is neat. Suppose that the form of the free surface is given as $\eta(x,t)=\eta(x)\sin(\sigma t+\tau)=F'(x)\sin(\sigma t+\tau)$, where $F(x)$ is analytic. Then, since $\eta(x)=\sigma g^{-1}\varphi(x,0)$ and $\varphi_y(x,0)=\nu\varphi(x,0)$,

$$f'(x-i0) = \varphi_x(x,0) - i\varphi_y(x,0) = \varphi_x(x,0) - i\nu\varphi(x,0) = \frac{g}{\sigma}\left[F''(x) - i\nu F'(x)\right]$$

and

$$f(x - i0) = \frac{g}{\sigma}[F'(x) - i\nu F(x)] + \text{const}.$$

We may take the constant as zero and write

$$f(x + iy) = \frac{g}{\sigma}[F'(x + iy) - i\nu F(x + iy)], \tag{23.31}$$

where $F(z)$ is the analytic function determined by $F(x)$. From this we have

$$\left.\begin{aligned} \varphi(x, y) &= \frac{g}{2\sigma}\{F'(x + iy) + F'(x - iy) - i\nu[F(x + iy) + F(x - iy)]\}, \\ \psi(x, y) &= \frac{-g}{2\sigma}\{i[F'(x+iy) + F'(x-iy)] - \nu[F(x+iy) + F(x-iy)]\}. \end{aligned}\right\} \tag{23.32}$$

Any streamline, defined by ψ = real const, can now be taken as determining a possible basin shape corresponding to the assumed standing-wave profile. STORCHI applies the procedure to several special choices of F. An obvious disadvantage of this method, as well as of SEN'S, is that only one mode of motion is obtained for a resulting basin shape.

γ) Waves in movable basins. In several preceding sections, especially 19 and 22γ, we considered the wave motion occurring in the presence of an oscillating body when the fluid is exterior to the body. One may attempt analogous problems when the fluid is situated inside the body. Such problems occur in many situations of practical interest, for example, the sloshing of oil in a partly filled compartment of a tanker and the sloshing of fuel in an airplane or rocket. In each of these cases interest centers upon the dynamics of the whole system as well as upon the effect upon the walls of the container. A further interest in such problems arises from the interpretation of the experiments on standing waves, referred to earlier, carried out by HONDA and MATSUSHITA (1913), SASAKI (1914), and KIRCHHOFF and HANSEMANN (1880). The results were intended for comparison with theoretical prediction of standing waves in fixed basins. The waves were actually generated by oscillating the basin and finding the frequencies at which resonance appeared to occur.

We shall not consider the most general motions of the basin consistent with linearization of the free surface conditions, but shall limit ourselves here to a particular problem with small horizontal oscillations. In Sect. 26α small vertical oscillations of the basin will be considered. The general problem of motion of a body containing fluid with a free surface has been treated by MOISEEV (1953) and NARIMANOV (1956, 1957). However, both are primarily concerned with small oscillations. KREIN and MOISEEV (1957) have also considered certain mathematical aspects of this problem. OKHOTSIMSKII (1957) and RABINOVICH (1957) have considered the special case when the fluid is situated in a vertical, or almost vertical cylinder; NARIMANOV also gives special attention to this case. (Publication of the work of these three authors was apparently delayed; it is stated that, for the most part, it was carried out independently of and prior to MOISEEV'S papers.) A particular problem, the one discussed below, was treated by SRETENSKII (1951) and later by MOISEEV (1952a, b, 1953). Two later papers by MOISEEV (1954a, b) apply the theory to engineering problems, especially ships. Waves resulting from a special type of forced oscillation of a rectangular tank have been studied by BINNIE (1941) and TAMIYA (1958). A problem somewhat related to those of this section is the motion of a freely floating body in a fixed bounded basin (there is now no dissipation of energy as in the problem treated at the end of Sect. 22γ). This problem has been dealt with by PERZHNYANKO (1956) and MOISEEV (1958).

Waves in a basin with elastic restoring force. Consider the configuration shown in Fig. 30. The coordinate system OXY is fixed, the system $\bar{O}\bar{X}\bar{Y}$ moves with the carriage. Let $x_0(t) = O\bar{O}$, $u_0 = \dot{x}_0$. The bottom of the fluid is at $\bar{y} = -h$, the side walls at $\bar{x} = \pm a$. The motion will be taken as two-dimensional. Denote the mass of the carriage, per unit width, by m_c, that of the fluid by m_f and the total by $m = m_c + m_f$. Let the spring constant be mk^2. We suppose as usual that the motion may be described by a velocity potential $\Phi(x, y, t)$. Following the notation at the end of Sect. 2, let $\bar{\bar{\Phi}}(\bar{x}, \bar{y}, t)$ describe the motion relative to the basin, i.e.

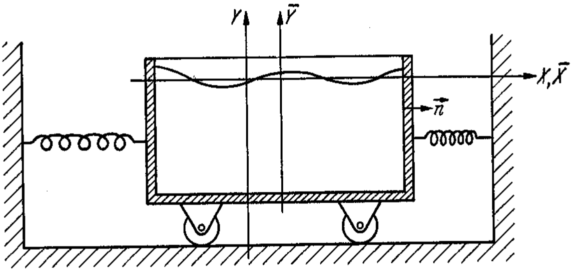

Fig. 30.

$$\left.\begin{aligned} &\Phi(x, y, t) \\ &= \bar{\bar{\Phi}}(\bar{x}, \bar{y}, t) + u_0 \bar{x}. \end{aligned}\right\} \tag{23.33}$$

We shall assume that x_0 and u_0 are both small, and of the same order as $\bar{\bar{\Phi}}$, i.e., in the notation of Sect. 10α, we assume expansions of the form

$$\left.\begin{aligned} &x_0 = \varepsilon x_0^{(1)}, \qquad u_0 = \varepsilon u_0^{(1)}, \\ &\bar{\bar{\Phi}} = \varepsilon \bar{\bar{\Phi}}^{(1)} + \varepsilon^2 \bar{\bar{\Phi}}^{(2)} + \cdots. \end{aligned}\right\} \tag{23.34}$$

We omit the formal details of substitution of the perturbation series in the exact boundary conditions. They lead to the following linearized boundary conditions for $\bar{\bar{\Phi}}$:

see errata

$$\left.\begin{aligned} &\bar{\bar{\Phi}}_{tt}(\bar{x}, 0, t) + g \bar{\bar{\Phi}}_{\bar{y}}(\bar{x}_y, 0, t) + \ddot{u}_0 \bar{x} = 0, \\ &\bar{\bar{\Phi}}_{\bar{x}}(\pm a, \bar{y}, t) = 0, \\ &\bar{\bar{\Phi}}_{\bar{y}}(\bar{x}, -h, t) = 0. \end{aligned}\right\} \tag{23.35}$$

The pressure, after discarding higher-order terms, is given by

$$p = -\varrho \Phi_t = -\varrho [\bar{\bar{\Phi}}_t + \dot{u}_0 \bar{x}]. \tag{23.36}$$

The motion of the carriage is determined by the equation

$$m_c \ddot{x}_0 = \int p \cos(n, \bar{x})\, ds - mk^2 x_0, \tag{23.37}$$

where the integral is taken over the wetted surface when the system is at rest. Substitution of (23.36) gives

$$\left.\begin{aligned} m \ddot{x}_0 &= -\varrho \int \bar{\bar{\Phi}}_t\, ds - mk^2 x_0 \\ &= -\varrho \int_{-h}^{0} \int_{-a}^{a} \bar{\bar{\Phi}}_{tx}\, dx\, dy - mk^2 x_0. \end{aligned}\right\} \tag{23.38}$$

[Eq. (23.38) is also a direct consequence of conservation of momentum.]

The velocity potential $\bar{\bar{\Phi}}$ and the displacement x_0 must be determined together from Eqs. (23.35) and (23.38) and either initial conditions or the further assumption that the motion is harmonic in t.

As a preliminary we shall first suppose that the basin motion, i.e. x_0, is given, so that only (23.35) need be satisfied. One may try separation of variables and express $\bar{\bar{\Phi}}$ in the form

$$\bar{\bar{\Phi}} = \sum T_n(t)\, X_n(\bar{x})\, Y_n(\bar{y}). \tag{23.39}$$

LAPLACE's equation and the last two condition of (23.35) are satisfied by

$$\left.\begin{aligned} X_{2n} Y_{2n} &= \cos \frac{2n}{2a} \pi \bar{x} \cosh \frac{2n\pi}{2a} (\bar{y}+h), \\ X_{2n+1} Y_{2n+1} &= \sin \frac{2n+1}{2a} \pi \bar{x} \cosh \frac{2n+1}{2a} \pi (\bar{y}+h). \end{aligned}\right\} \tag{23.40}$$

In order to find the corresponding T_n, expand x in a Fourier series:

$$x = \sum_{n=0}^{\infty} (-1)^n \frac{8a}{(2n+1)^2 \pi^2} \sin \frac{2n+1}{2a} \pi x \tag{23.41}$$

and substitute (23.39) and (23.41) in the first condition of (23.35):

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} \left[\ddot{T}_{2n} \cosh \frac{2n}{2a} \pi h + T_{2n}\, g \frac{2n\pi}{2a} \sinh \frac{2n}{2a} \pi h\right] \cos \frac{2n}{2a} \pi x + \\ &\quad + \sum_{n=0}^{\infty} \left[\ddot{T}_{2n+1} \cosh \frac{2n+1}{2a} \pi h + T_{2n+1}\, g \frac{2n+1}{2a} \pi \sinh \frac{2n+1}{2a} \pi h + \right. \\ &\quad \left. + \dddot{x}_0 (-1)^n \frac{2a}{(2n+1)^2 \pi^2}\right] \sin \frac{2n+1}{2a} \pi x = 0. \end{aligned}\right\} \tag{23.42}$$

Let us set

$$\sigma_n^2 = g\pi \frac{n}{2a} \tanh \frac{n}{2a} \pi h, \quad b_{2n+1} = -(-1)^n \frac{2a}{(2n+1)^2 \pi^2} \operatorname{sech} \frac{2n+1}{2a} \pi h. \tag{23.43}$$

Then Eq. (23.42) yields the infinite set of differential equations

$$\left.\begin{aligned} &\ddot{T}_{2n} + \sigma_{2n}^2 T_{2n} = 0, \\ &\ddot{T}_{2n+1} + \sigma_{2n+1}^2 T_{2n+1} = b_{2n+1} \dddot{x}_0. \end{aligned}\right\} \tag{23.44}$$

The solution of the first set, $T_{2n} = A_{2n} \cos(\sigma_{2n} t + \tau_{2n})$, is independent of the motion of the basin and yields the symmetric modes of oscillation in a fixed basin. The solution to the second set may also be found by elementary methods, but will not be given here. However, we note that, if x_0 is harmonic, it confirms that resonance occurs at the frequencies of the asymmetric modes for a fixed basin.

We now turn to the joint solution of (23.35) and (23.38). Substitute (23.39) into (23.38). Then, after evaluating the integral, one finds

$$m\ddot{x}_0 + \frac{4a\varrho}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sinh \frac{2n+1}{2a} \pi h\, \dot{T}_{2n+1} + m k^2 x_0 = 0. \tag{23.45}$$

The Eqs. (23.44) and (23.45) taken together may now be used to determine x_0 and the T_n. If we formulate an initial-value problem by requiring, say,

$$x_0(0) = c_0, \quad \dot{x}_0(0) = 0, \quad \bar{\bar{\Phi}}_{\bar{y}}(\bar{x}, \bar{y}, 0) = 0, \quad \bar{\bar{\Phi}}_t(\bar{x}, \bar{y}, 0) = 0, \tag{23.46}$$

then the T_{2n} are all zero and the T_{2n+1} and x_0 must be determined from the differential equations. As usual, one looks for a solution in the form

$$x_0 = c\,e^{-i\omega t}, \qquad T_{2n+1} = d_{2n+1}\,e^{-i\omega t}, \tag{23.47}$$

where both c and d_{2n+1} may, of course, be complex. Substitution in (23.44) and (23.45), followed by elimination of d_{2n+1}, yields the following equation for determining ω:

$$\omega^2 - k^2 = \frac{32 a^2 \varrho}{\pi^3 m}\,\omega^4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \tanh \frac{2n+1}{2a}\,\pi h \,\frac{1}{\omega^2 - \sigma_{2n+1}^2}. \tag{23.48}$$

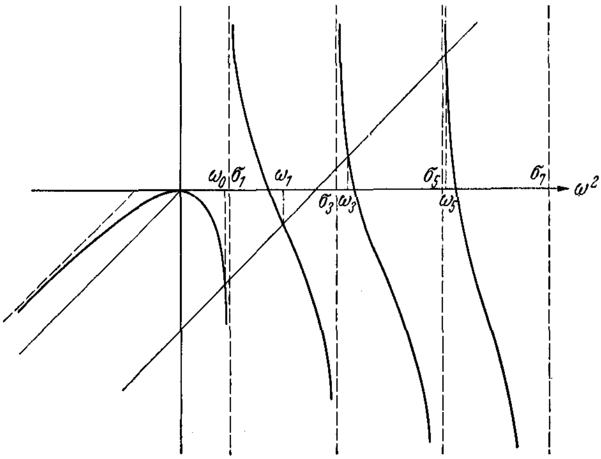

Fig. 31.

One may find the solutions graphically by plotting each side of the equation as function of ω^2. Fig. 31 gives a qualitative idea of the distribution of solutions $\omega_0, \omega_1, \ldots$. As $n\to\infty$, $\omega_{2n+1}^2 - \sigma_{2n+1}^2 \to 0$; this fact, which can be proved analytically and which seems clear from Fig. 31, would not have been so evident if we had divided (23.48) by ω^4 before plotting. A point of importance is that there is

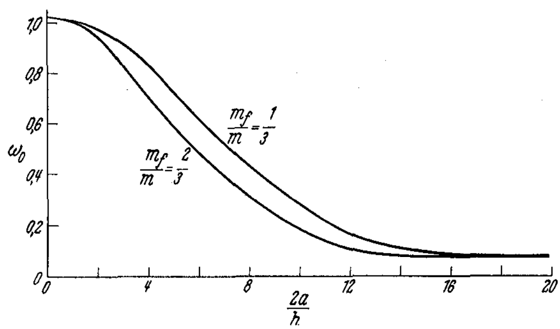

Fig. 32.

no intersection for $\omega^2 < 0$; as a result the motion is stable. This may be proved as follows. Since $x^{-1}\tanh x \leqq 1$, the right hand side of (23.48), for $\omega^2 < 0$, is greater than or equal to

$$\frac{32 a^2 \varrho}{\pi^3 m}\,\omega^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}\,\frac{2n+1}{2a}\,\pi h = \frac{2 a h \varrho}{m}\,\omega^2 = \frac{m_f}{m}\,\omega^2 > \omega^2. \tag{23.49}$$

Hence the line $\omega^2 - k^2$ lies below the left-hand branch of the curve for $k^2 \geqq 0$. The eigenvalues ω_i depend upon the parameters k^2, $2a/h$ and $2\varrho a h/m = m_f/m$. Fig. 32 from MOISEEV (1953) shows the dependence of the fundamental mode ω_0 upon $2a/h$ for two values of m_f/m and $k^2 = 1$.

The general solution for x_0 and T_{2n+1} is

$$x_0(t) = \mathrm{Re} \sum_{s=0}^{\infty} c_s \, e^{-i\omega_s t}, \qquad T_{2n+1}(t) = \mathrm{Re} \sum_{s=0}^{\infty} d_{2n+1,s} \, e^{-i\omega_s t}. \tag{23.50}$$

The solution of the initial-value problem formulated in (23.46) will not be completed. It involves solution of infinite sets of linear equations. Approximate solutions can be obtained by considering only a finite number of equations and variables.

The general theory of stability of such systems is discussed in MOISEEV'S 1953 paper. In an earlier papers (1952b) he studies the special case of a basin containing fluid and serving as the bob of a pendulum. If the suspension is by a parallelogram linkage, so that the container moves parallel to itself, the motion is always stable; if the suspension is by a rod rigidly attached to the container, the motion may be, under certain circumstances, unstable.

The last cited paper by MOISEEV describes briefly the results of an experiment with a pendulum; the measured and computed fundamental frequencies for the see errata two systems of suspension agreed with 0.1%.

24. Gravity waves in the presence of surface tension. Apparently the first one to investigate the theory of waves in a fluid acted upon by both gravity and surface tension was KELVIN (1871 a, b). However, many of the essential features had been discovered earlier through observation by RUSSELL (1844) and others; references may be found in KELVIN'S papers. A good account of the classical researches of KELVIN and RAYLEIGH may be found in LAMB (1932, § 265 to 272). Also, Chap. XX of RAYLEIGH'S *Theory of Sound* (Cambridge 1929; Dover, N. Y., 1945) contains an exposition of many of his own fundamental researches on surface-tension phenomena, including waves.

The chief mathematical complication added by the action of surface tension is a somewhat more elaborate dynamical boundary condition at an interface or free surface. The difference of primary physical interest lies in the existence of a minimum wave velocity and of two wave lengths with the same velocity. Many of the special problems considered in preceding sections can also be solved when surface tension is acting. However, there has been little motivation for carrying through such analyses for wave motion associated with solid boundaries, since it has been recognized that the additional forces would be small. A further difficulty also appears when the solid boundary pierces the surface, for an additional boundary condition is required at the intersection. As a result, most of the investigations have dealt with waves analogous to those considered in Sects. 14α, β and δ, 15, and 22β. In fact, the complex velocity potential for the Cauchy-Poisson initial-value problem including the effect of surface tension, has already been given in Eqs. (22.45) and (22.46). A topic of particular geophysical interest, the stability of an interface, will be dealt with in Sect. 26. Waves on the surface of a viscous fluid, including surface tension, are considered in Sect. 25.

B o u n d a r y c o n d i t i o n s. The linearized conditions to be satisfied at the interface of two fluids have already been given in Eqs. (10.7) and (10.8) (we recall that subscript 1 refers to the lower fluid). If one eliminates η from these two equations and makes use of the fact that LAPLACE'S equation is satisfied on either side of the boundary, one has the following condition:

$$\Delta\Phi_1 = 0 \quad \text{for} \quad y < 0, \qquad \Delta\Phi_2 = 0 \quad \text{for} \quad y > 0. \tag{24.1}$$

$$\eta_t(x, z) = \Phi_{1y}(x, 0, z, t) = \Phi_{2y}(x, 0, z, t), \tag{24.2}$$

$$\varrho_1\left[\Phi_{1tt}(x, 0, z, t) + g\,\Phi_{1y} + \frac{T}{\varrho_1}\Phi_{1yyy}\right] = \varrho_2\left[\Phi_{2tt} + g\,\Phi_{2y}\right]. \tag{24.3}$$

If the upper fluid is absent, one sets ϱ_2 and Φ_2 equal to zero and may, of course, drop the subscript.

If the motion is two-dimensional one may introduce a stream function Ψ and a complex potential $F(z, t) = \Phi + i\Psi$ and express (24.2) and (24.3) as follows:

$$\eta_t(x) = \operatorname{Im} F_1'(x - i\,0) = \operatorname{Im} F_2'(x + i\,0), \tag{24.4}$$

$$\operatorname{Re}\varrho_1\left\{F_{1tt}(x - i\,0) + i\,g\,F_1' - i\frac{T}{\varrho_1}F_1'''\right\} = \operatorname{Re}\varrho_2\left\{F_{2tt}(x + i\,0) + i\,g\,F_2'\right\}. \tag{24.5}$$

If the upper fluid is absent and if the lower fluid is infinitely deep, one may extend the reasoning which led up to LEVI-CIVITA'S differential equation (22.25) to derive the following one which must be satisfied for all z:

$$F_{tt}(z, t) + i\,g\,F' - i\frac{T}{\varrho}F''' = 0. \tag{24.6}$$

Furthermore, if the fluid is of constant depth h, CISOTTI'S equation (22.30) may also be extended to include the effect of surface tension:

$$\left.\begin{aligned} &F_{tt}(z + i\,h, t) + F_{tt}(z - i\,h) + i\,g\,[F'(z + i\,h) - F'(z - i\,h)] - \\ &\quad - i\frac{T}{\varrho}[F'''(z + i\,h) - F'''(z - i\,h)] = 0 \quad \text{for} \quad -2h < y < 0. \end{aligned}\right\} \tag{24.7}$$

E l e m e n t a r y s o l u t i o n s. Let us suppose first that only one fluid is present, and in addition that

$$\Phi(x, y, z, t) = \varphi(x, y, z)\cos(\sigma t + \tau).$$

Then φ must be a potential function satisfying

$$-\sigma^2\varphi(x, 0, z) + g\,\varphi_y + \frac{T}{\varrho}\varphi_{yyy} = 0. \tag{24.8}$$

Just as in Sect. 13α, we may separate out the y-variable and obtain the following expressions:

infinite depth:

$$\varphi(x, y, z) = A\,e^{my}\varphi(x, y) \tag{24.9}$$

where

$$\Delta_2\varphi + m^2\varphi = 0$$

and

$$\sigma^2 = g\,m + \frac{T}{\varrho}m^3;$$

depth h:

$$\varphi(x, y, z) = A\cosh m_0(y + h)\,\varphi(x, z), \tag{24.10}$$

where

$$\Delta_2\varphi + m_0^2\varphi = 0$$

and

$$\sigma^2 = \left(g\,m_0 + \frac{T}{\varrho}m_0^3\right)\tanh m_0 h.$$

One may also with no difficulty construct solutions analogous to (13.3) and (13.4), namely

$$\varphi(x, y, z) = A\left[m\left(1 - \frac{T}{\varrho g}m^2\right)\cos m y + \frac{\sigma^2}{g}\sin m y\right]\varphi(x, z) \tag{24.11}$$

and

$$\varphi(x, y, z) = A\cos m_i(y + h)\,\varphi(x, z) \tag{24.12}$$

for infinite and finite depth, respectively, where m_i in (24.12) must satisfy

$$\sigma^2 = \left(-g m_i + \frac{T}{\varrho} m_i^3\right)\tan m_i h$$

and $\varphi(x, z)$ must be a solution of

$$\Delta_2\varphi - m^2\varphi = 0.$$

Unfortunately, the set of function

$$\{\cosh m_0(y+h),\ \cos m_i(y+h)\}$$

is no longer orthogonal in general, so that the convenience of general solutions like (16.3) is lost.

It is not necessary to repeat the computations of Sect. 13 since they remain unaltered. Essentially the only change is in the relation between the frequency σ and the wave number m. Here the fact of predominant physical interest is that for small values of m the relation is controlled chiefly by the gravitational constant g and for large values of m by T/ϱ.

If one forms two-dimensional progressive waves by superposing the standing-wave solutions obtained from (24.9) and (24.10), a further significant physical fact appears: the wave velocity now has a minimum for some value of $m > 0$, except for very shallow depth. These facts are displayed graphically in Fig. 11 and further information is given in the associated discussion (the curves were computed for water at 20° C and $h = \infty$ or 1 cm). Formulas for the position of the minimum and various associated values are given for infinite depth in the following table; the numerical values are for water at 20° C ($T = 72.8$ dynes/cm, $\varrho = 0.998$ gm/cm^3):

$$\left.\begin{aligned} m_m &= \sqrt{\varrho g/T} = 3.66\ \text{cm}^{-1}, \\ \lambda_m &= 2\pi\sqrt{T/\varrho g} = 1.71\ \text{cm}, \\ c_m &= \sqrt[4]{4 g T/\varrho} = 23.1\ \text{cm/sec}, \\ \sigma_m &= \sqrt[4]{4\varrho g^3/T} = 84.8\ \text{radians/sec} = 13.5\ \text{cycles/sec}. \end{aligned}\right\} \tag{24.13}$$

see errata
When $h \leqq \sqrt{3\,T/2\varrho g}$ there is no longer a minimum value of c for $m > 0$; in this case c increases monotonically with m. The critical depth for water is about 0.33 cm. Except in this latter case every value of c has associated with it waves of two different lengths, each of which travels with velocity c. Kelvin suggested that the shorter waves, whose behavior is controlled chiefly by surface tension be called "ripples". The suggestion has been followed for the most part (French: "rideaux"; German: "Rippeln" or "Kräuselwellen"; Russian: "ryabi"), but they are frequently also called "capillary waves" in contrast with "gravity waves".

The relation between σ and m was subjected to a rather thorough experimental investigation by Matthiessen (1889). He made measurements with water, mercury, alcohol, ethyl ether and carbon disulfide with frequencies ranging from 8.4 to roughly 2000 cycles per second. Agreement between theory and measurement is generally within 5% with the greatest discrepancies occurring near the minima. RAYLEIGH (1890) and MICHIE SMITH (1890) were apparently the first to use the theoretical relation as a means of experimental determination of T, and it has become one of the standard experimental procedures. For more recent developments and further references see BROWN (1936) and TYLER (1941).

Solutions for standing or progressive interfacial waves, analogous to those considered in Sect. 14δ, can be found by application of the same methods. Since the analysis is similar we give only the relation between σ and m. If the two fluids fill the whole space, with their interface at $y=0$, then

$$\sigma^2 = \frac{\varrho_1 - \varrho_2}{\varrho_1 + \varrho_2} g\, m + \frac{T}{\varrho_1 + \varrho_2} m^3. \tag{24.14}$$

If the lower fluid is of depth d_1, the upper of depth d_2 and the interface at $y=0$, then

$$\sigma^2 = \frac{(\varrho_1 - \varrho_2)\, m\, g + T\, m^3}{\varrho_1 \coth m\, d_1 + \varrho_2 \coth m\, d_2}. \tag{24.15}$$

In both (24.14) and (24.15) $\sigma^2 > 0$ if

$$\varrho_2 < \varrho_1 + \frac{T\, m^2}{g}; \tag{24.16}$$

thus the motion may be stable even when the lower fluid is less dense than the upper one. This is not true in the absence of surface tension, as inspection of (14.29) and (14.30) shows.

The analogue of the next example of Sect. 14δ is somewhat more complex, for two surface tensions are necessary. Let T be the surface tension at the free surface $y=0$, and T_{12} that at the interface $y=-d_1$; let the rigid bottom be at $y=-h=-d_1-d_2$. Then the relation analogous to (14.31) is

$$\left.\begin{aligned} &\left(\frac{\sigma^2}{g\,m}\right)^2 [\varrho_1 \coth m\, d_1 \coth m\, d_2 + \varrho_2] - \\ &\quad - \frac{\sigma^2}{g\,m} \varrho_1 \left[\left(1 + \frac{T + T_{12}}{\varrho_1 g} m^2\right) \coth m\, d_2 + \varrho_1 \left(1 + \frac{T}{\varrho_2 g} m^2\right) \coth m\, d_1\right] + \\ &\quad + \left(1 + \frac{T}{\varrho_2 g} m^2\right) \left[\varrho_1 \left(1 + \frac{T + T_{12}}{\varrho_1 g} m^2\right) - \varrho_2 \left(1 + \frac{T}{\varrho_2 g} m^2\right)\right] = 0. \end{aligned}\right\} \tag{24.17}$$

The assumption of $d_1 = \infty$ no longer results in any notable simplification of the equation. However, one may show that the solutions $\sigma^2/g\,m$ are always real, and that they are positive if

$$\varrho_2 < \varrho_1 + \frac{T_{12}}{g} m^2. \tag{24.18}$$

This is the same condition for stability as was found in (24.16) (and is still necessary as well as sufficient). Much of the rest of the pure gravity-wave analysis of Sect. 14δ may be carried through. Thus, if σ_1 is the larger and σ_2 the smaller root of (24.17) for a given m, then one may establish the inequality

$$\left.\begin{aligned} &0 < \frac{\sigma_2^2}{g\,m} < \left(1 + \frac{T}{\varrho_2 g} m^2\right) \tanh m\, d_2 < \frac{\sigma_1^2}{g\,m} < \\ &\quad < \left(1 + \frac{T}{\varrho_2 g} m^2\right)\left(1 + \frac{T + T_{12}}{\varrho_1 g} m^2\right) \min\left\{1,\ \frac{\varrho_1}{\varrho_2} \tanh m\, h\right\}. \end{aligned}\right\} \tag{24.19}$$

If η and η_{12} are the profiles of the free surface and interface, respectively, then one finds, analogously to (14.34),

$$\frac{\eta_{12}}{\eta} = \cosh m\, d_2 - \frac{g\, m}{\sigma_2}\left(1 + \frac{T}{\varrho_2 g} m^2\right) \sinh m\, d_2 ; \tag{24.20}$$

again, it follows from (24.18) that this ratio is positive for the larger and negative for the smaller of the two roots of (24.17). The discussion of the nature of the motion associated with the root σ_2 may be taken directly from Sect. 14δ; however, the upper bound for the velocity c_2 of a progressive wave of wave number m is now given by

$$c_2^2 = \frac{\sigma_2^2}{m^2} = \frac{\sigma_2^2}{g\, m} \cdot \frac{g}{m} < \frac{g}{m}\left(1 + \frac{T}{\varrho_2 g} m^2\right) \tanh m\, d_2 < g\, d_2 \left(1 + \frac{T}{\varrho_2 g} m^2\right). \tag{24.21}$$

Let us turn next to the situation in which the two fluids are moving and look for possible steady motions. Assume each fluid to move to the left with mean velocity c_i and let $F_i(z)$, $i = 1, 2$, be the complex velocity potentials. We again look for solutions in the form [cf. Eq. (14.36)]

$$F_i(z) = -c_i z + f_i(z), \qquad i = 1, 2, \tag{24.22}$$

where f_i is assumed small with respect to $c_i z$. Then the linearized boundary conditions corresponding to (14.37) are

$$\left.\begin{aligned} \eta(x) &= \frac{1}{c_1} \operatorname{Im} f_1(x - i\,0) = \frac{1}{c_2} \operatorname{Im} f_2(x + i\,0), \\ \frac{\varrho_2}{c_2} \operatorname{Re}\{i g f_2(x+i\,0) + c_2^2 f_1'(x+i\,0)\} &= \frac{\varrho_1}{c_1} \operatorname{Re}\Big\{i g f_1(x - i\,0) + c_1^2 f_1'(x - i\,0) - \\ &\qquad - i \frac{T}{\varrho_1} f_1''(x - i\,0)\Big\}. \end{aligned}\right\} \tag{24.23}$$

If we look for a steady motion of the form

$$f_1 = a_1 \mathrm{e}^{-i m z}, \qquad f_2 = a_2 \mathrm{e}^{i m z}, \tag{24.24}$$

then substitution in (24.23) yields

$$\frac{a_1}{c_1} = -\frac{\bar{a}_2}{c_2}$$

and

$$m(\varrho_1 c_1^2 + \varrho_2 c_2^2) = (\varrho_1 - \varrho_2)\, g + T m^2 . \tag{24.25}$$

The last equation will not have a real solution for m, assuming $\varrho_1 > \varrho_2$, unless

$$4 g (\varrho_1 - \varrho_2) T \leqq (\varrho_1 c_1^2 + \varrho_2 c_2^2)^2 . \tag{24.26}$$

There are then two solutions of the form (24.24). The effect of surface tension may be seen more clearly if one graphs each side of (24.25) and finds the intersections, if any. It will be shown in Sect. 26 that this type of motion is unstable if $|c_1 - c_2|$ becomes too large.

Singular solutions. The methods used in Sect. 13γ for finding source-type solutions can generally be extended to take account of surface tension. Aside from the algebraic complications the chief difficulties are associated with selecting the proper boundary conditions at infinity. For a stationary source of pulsating strength one may still impose a radiation condition as in (13.9) and obtain the correct solution. However, for the steadily moving source of constant strength the proper choice is no longer clear. Although it is possible to fall back upon arguments based upon considerations of group velocity, they are not completely convincing, so that it seems safer to formulate first an initial-value problem which can yield either of the two cases mentioned above as a limit when $t\to\infty$. First we give the velocity potential for a source of variable strength $m(t)$, $t\geqq 0$, moving on an arbitrary path $(a(t), b(t), c(t))$. The potential function Φ must satisfy the same conditions given on p. 491 except that 2 is now replaced by

$$\Phi_{tt}(x,0,z,t)+g\,\Phi_y(x,0,z,t)+\frac{T}{\varrho}\,\Phi_{yyy}(x,0,z,t)=0. \tag{24.27}$$

There is no special difficulty involved in finding Φ. For infinite depth, it is as follows:

$$\left.\begin{aligned}\Phi(x,y,z,t)&=\frac{m(t)}{r(t)}-\frac{m(t)}{r_1(t)}+\\&+2\int_0^\infty dk\sqrt{gk+T'k^3}\int_0^t d\tau\, m(\tau)\sin\left[(t-\tau)\sqrt{gk+T'k^3}\right]e^{k[y+b(\tau)]}J_0(kR(\tau)),\end{aligned}\right\} \tag{24.28}$$

where we have written T' for T/ϱ. One may similarly find the function analogous to (13.53) by replacing gk by $gk+T'k^3$. Knowledge of these functions allows one now to repeat, at least in part, the considerations of Sects. 22α and 22β.

For a stationary source at (a, b, c) with strength $m\cos\sigma t$, the velocity potential may be easily derived from (24.28). It is as follows:

$$\left.\begin{aligned}\Phi(x,y,z,t)=&\left[\frac{1}{r}+\frac{1}{r_1}+2\sigma^2\int_0^\infty\frac{1}{T'k^3+gk-\sigma^2}\,e^{k(y+b)}\,J_0(kR)\,dk\right]m\cos\sigma t+\\&+2\pi m\frac{\sigma^2}{g+3T'k_0^2}\,e^{k_0(y+k)}\,J_0(k_0R)\sin\sigma t,\end{aligned}\right\} \tag{24.29}$$

where k_0 is the real solution of $\sigma^2=gk+T'k^3$. If the fluid is of depth h, then

$$\left.\begin{aligned}&\Phi(x,y,z,t)\\&=\left[\frac{1}{r}+\frac{1}{r_2}+2\int_0^\infty\frac{T'k^3+gk+\sigma^2}{T'k^3+gk-\sigma^2\coth kh}\cdot\frac{e^{kh}\cosh k(b+h)\cosh k(y+h)}{\sinh kh}\,J_0(kR)\,dk\right]\times\\&\times m\cos\sigma t+2\pi m\frac{T'k_0^3+gk_0+\sigma^2}{\sigma^2h+(3T'k_0^2+g)\sinh^2k_0h}\times\\&\times e^{k_0h}\sinh k_0h\cosh k_0(b+h)\cosh k_0(y+h)\cdot J_0(k_0R)\sin\sigma t,\end{aligned}\right\} \tag{24.30}$$

where k_0 is the real root of

$$T'k^3+gk-\sigma^2\coth k\cdot h=0.$$

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The velocity potential for a source moving in the direction Ox with constant velocity u_0 may also be obtained from (24.38) by a suitable limiting procedure, although the computation is somewhat more tedious. In a coordinate system moving with the source it is as follows for $h=\infty$:

$$\left.\begin{aligned}\varphi(x,y,z)=\frac{m}{r}-\frac{m}{r_1}+\frac{4m}{\pi}\int_0^{\frac{1}{2}\pi}d\vartheta\,\mathrm{PV}\int_0^\infty dk\,\frac{g+T'k^2}{g+T'k^2-k\,u_0^2\cos^2\vartheta}\times\\\times e^{k(y+b)}\cos[k(x-a)\cos\vartheta]\cos[k(z-c)\sin\vartheta]+\\+4m\sum_{i=1,2}(-1)^{i-1}\int_0^{\vartheta_0}d\vartheta\,k_i(\vartheta)\,\frac{T'k_i^2+g}{T'k_i^2-g}\times\\\times e^{k_i(y+b)}\sin[k_i(x-a)\cos\vartheta]\cos[k_i(z-c)\sin\vartheta],\end{aligned}\right\} \tag{24.31}$$

where

$$\vartheta_0 = \begin{cases} \arccos (4g\,T')^{\frac{1}{4}}/u_0 & \text{if} \quad 4g\,T' \leqq u_0^4 \\ 0 & \text{if} \quad 4g\,T' \geqq u_0^4 \end{cases}$$

and

$$\left.\begin{matrix} k_1(\vartheta) \\ k_2(\vartheta) \end{matrix}\right\} = \frac{u_0^2 \cos^2\vartheta \pm \sqrt{u_0^4 \cos^4\vartheta - 4g\,T'}}{2\,T'}.$$

One may easily show that

$$\nu \sec^2\vartheta < k_1(\vartheta) \leqq \sqrt{g/T'} \leqq k_2(\vartheta) \leqq u_0^2 \cos^2\vartheta / T'.$$

As $T' \to 0$ it is then evident that the integral involving k_2 vanishes and that (24.31) reduces to (13.36).

One may carry out an asymptotic investigation of (24.31), or of φ_x, along the lines of (13.38) and following. However, the analysis is considerably more complicated. The behavior of the wave pattern is roughly as follows. For $u_0^4/4g\,T' \leqq 1$, $\varphi_x(R, \alpha, y)$ is $O(R^{-1})$ for all α, and the disturbance is chiefly local. There is a constant $c > 1$ such that when $1 < u_0^4/4g\,T' < c$ the wave pattern is a superposition of two sets of waves corresponding to the two roots k_1 and k_2. Those corresponding to k_2 are capillary waves which precede the source and bend around it so that their crests eventually make an angle $\frac{1}{2}\pi + \vartheta_0$ with the x-axis. The gravity waves corresponding to k_1 behave similarly except that they follow the source and are longer. If $u_0^4/4g\,T' > c$, a second angle, say ϑ_1, appears, where $\vartheta_1 < \vartheta_0$. There are now three sets of waves. Those associated with k_2 behave as described above. The gravity waves, however, consist of both transverse waves spanning the angle between $\pm(\frac{1}{2}\pi + \vartheta_1)$ and diverging waves which now lie in the wedge bounded by $\frac{1}{2}\pi + \vartheta_1$ and $\frac{1}{2}\pi + \vartheta_0$ and its reflection. One will find a sketch in LAMB's *Hydrodynamics* (1932, p. 470) which was computed for the similar problem of a moving pressure point, the so-called "fishline problem". A precise value for the constant c does not seem to be known. The free surface η may be computed from

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$$\eta(x, z) = \frac{u_0}{g}\left[\varphi_x(x, 0, z) + \frac{T}{\varrho\, u_0^2}\,\varphi_{xy}(x, 0, z)\right].$$

In spite of the general complexity of the asymptotic analysis of (24.31), it is relatively easy to find the asymptotic form of η directly ahead ($\alpha = 0°$) and directly behind ($\alpha = 180°$):

$\alpha = 0$:

$$\left.\begin{aligned} \eta(x, z) = -8m\frac{u_0}{g}\,k_2^2\left[1 + \frac{T'}{u_0^2}\,k_2\right]\sqrt{\frac{\pi}{2k_2 R}}\;\frac{T' k_2^2 + g}{[(T' k_2^2 - g)(3\,T' k_2^2 + g)]^{\frac{1}{2}}} \times \\ \times e^{k_2 b}\cos(k_2 R - \tfrac{1}{4}\pi) + O(R^{-1}); \end{aligned}\right\} \quad (24.32)$$

$\alpha = 180°$:

$$\left.\begin{aligned} \eta(x, z) = 8m\frac{u_0}{g}\,k_1^2\left[1 + \frac{T'}{u_0^2}\,k_1\right]\sqrt{\frac{\pi}{2k_1 R}}\;\frac{T' k_1^2 + g}{[(g - T' k_1^2)(3\,T' k_1^2 + g)]^{\frac{1}{2}}} \times \\ \times e^{k_1 b}\cos(k_1 R - \tfrac{3}{4}\pi) + O(R^{-1}); \end{aligned}\right\} \quad (24.33)$$

here

$$k_1 = k_1(0) = \frac{u_0^2}{2\,T'}\left[1 - \sqrt{1 - 4\,T' g/u_0^4}\right],$$

$$k_2 = k_2(0) = \frac{u_0^2}{2\,T'}\left[1 + \sqrt{1 - 4\,T' g/u_0^4}\right]$$

and we assume $u_0^2 > 4T'g$. One may see rather clearly the effect upon k_1 and k_2 of varying T' and u_0 by finding them as the intersection of the graphs of $T'k^2+g$ and $u_0^2 k$.

There is no special difficulty in finding source solutions for two-dimensional motion, and the asymptotic behavior is of course easier to determine. The related problem of a moving concentrated pressure is treated in Lamb (1932, §§ 270,1). For this problem a paper by DePrima and Wu (1957) is particularly instructive, for they obtain the solution by first formulating the initial-value problem and then finding the limit as $t \to \infty$. In addition, they analyze the form of the surface for large but finite values of t.

25. Waves in a viscous fluid. If one abandons the assumption of a perfect fluid with irrotational motion, one loses at the same time many convenient and powerful mathematical tools from potential theory and the theory of functions of a complex variable. However, the simplifications introduced by the infinitesimal-wave approximation are sufficient to allow obtaining a number of solutions of interest, most of which have been known for many years. However, discovery of errors in early work has resulted in several recent papers. Furthermore, in connection with the theory of stability of interfaces the subject has again attracted attention; this work will be summarized in Sect. 26. One will find general expositions of many of the fundamental results in Lamb (1932, §§ 348 to 351), and Levich (1952, pp. 467—497). Longuet-Higgins (1953b) gives a valuable discussion of the perturbation procedure and carries through certain second-order computations.

Subject to the limitations of the approximation one can find solutions for periodic standing waves in fluid of both infinite and finite depth with a free surface, at the interface of two different fluids in which either may have a fixed horizontal plane as its other boundary, and at the interface and free surface when two different fluids are superposed, the upper one having a free surface. In all cases the presence of surface tension may be admitted. By making use of such solutions together with Fourier analysis one can find the solution to the Cauchy-Poisson initial-value problem [cf. Sretenskii (1941)].

In general, in the investigation of standing waves one is particularly interested in two things, the effect of viscosity upon the relation between wave-length and frequency, and the rate of decay of amplitude. As an alternative to examining the rate of decay, one may instead assume that a space-and time-periodic pressure has been applied to the free surface and determine the rate of transfer of energy necessary to maintain a steady oscillation.

One may still, as for perfect fluids, combine standing-wave solutions which are out of phase in order to form progressive waves. In a coordinate system moving with the waves the wave system will be stationary but the motion will not be steady for, as a result of viscosity, it will decay unless a periodic pressure distribution is moving with the waves and doing work upon the fluid. Fourier analysis may be used to obtain the fluid motion resulting from an arbitrary moving pressure distribution. Indeed, one need not restrict oneself to a pressure distribution but may also include a distribution of shearing stress at the free surface. If a pressure and shearing distribution of localized extent is moving over the fluid the dissipation of wave energy in viscosity will show up in a diminution of amplitude, as one moves away from the pressure area, which is more rapid than for a perfect fluid. Such problems have been investigated by Sretenskii (1941, 1957) and by Wu and Messick (1958). The latter include the effect of surface tension and make a particularly thorough study of the behavior of the solution; they restrict themselves to two-dimensional motion. One should keep in mind that if the fluid is of finite depth it is no longer equivalent to formulate a problem in which the pressure distribution is fixed and the fluid moves with a constant mean velocity.

Instead of attempting to construct a steady progressive-wave solution by means of a moving pressure distribution, one may instead assume that the progressive waves have been somehow initiated and then study their rate of decay with distance from the wave-maker. (This is, of course, closely related to finding the decay with time of an initially given progressive wave.) Studies of this nature have been made by BIESEL (1949) and CARRY (1956), who investigated especially the effect of the bottom, by URSELL (1952), who investigated the effect of side walls for infinite depth, and by HUNT (1952), who combined the two. Dissipation with distance when no walls are present has been treated by DMITRIEV (1953) in connection with the theory of the wave-maker. A point of physical interest in these studies is the relative contribution to dissipation of shearing motion near the surface, near the bottom, near the walls, and within the fluid. CASE and PARKINSON (1957) have studied the damping of standing waves in a circular cylinder of finite depth, making use of the linearized equations of this section; their experimental data seem to confirm the theoretical predictions when the cylinder walls are sufficiently polished. KEULEGAN (1959) has made further measurements with rectangular basins; he finds a striking difference between fluids which wet the container walls and those which do not, but confirms the theory for large enough containers.

The fluid motion resulting from a submerged stationary source of pulsing strength has been derived by DMITRIEV (1953) for two-dimensional motion and infinite depth. SRETENSKII (1957) has carried through the calculations for steady motion of a source in three dimensions. Unfortunately, the source function is not now as useful a tool for constructing solutions to special boundary-value problems as it is for perfect fluids. In particular, one can no longer satisfy the proper boundary condition on a steadily moving body by means of distributions of sources and sinks, as was possible in Sect. 20β. On the other hand, distributions of pulsating sources may still be used to satisfy the linearized boundary conditions on certain types of stationary oscillating bodies. Thus, if the motion is such that the linearized boundary condition specifies the velocity normal to the surface together with zero tangential velocity, then a source distribution may prove useful. For example, the wave-maker problems formulated in (19.26) and (19.31) may be treated in this fashion; DMITRIEV (1953) has done this.

A fundamental assumption of the preceding remarks is that the motion is laminar. Such an assumption seems to be in harmony with the asumption of small motions which is made in deriving the equations of the present section. However, the possible occurrence of turbulent motion in progressive waves has been reported by DMITRIEV and BONCHKOVSKAYA (1953) who found experimental evidence for it near the surface, where the vorticity was highest. The photographs in Fig. 7 do not seem to show any evidence of it, but this may result from special circumstances of the experiments. BOWDEN (1950) has essayed a theory based on VON KÁRMÁN's similarity hypothesis; further references are given there. In the case of steady free-surface flow in a channel the importance of turbulence in modifying the mean-velocity profile is almost obvious. However, investigations have been confined to the necessary modifications of the shallow-water approximation and will be discussed elsewhere.

Finally, we note that much of the theory given below for a constant surface tension T can, in fact, be extended to a more general surface condition. This is indicated in LAMB (1932, §§ 351) and carried out by DORRESTEIN (1951) in some detail for infinite depth. He includes compressibility of the surface film, hysteresis and a "surface viscosity". An earlier investigation of the effect of generalized surface conditions is due to WIEGHARDT (1943).

α) Linearized equations and simple solutions. The linearized equations and boundary conditions have already been derived in Sect. 10. For a stratified fluid with interface at $y=0$ the zeroth-order equations are given in (10.2), the first-order in (10.3). For a single fluid with free surface they are given in (10.4). It is customary and convenient to combine the zeroth- and first-order equations. Thus, if in (10.4) we let $p=p^{(0)}+\varepsilon p^{(1)}$ and $\boldsymbol{v}=\varepsilon \boldsymbol{v}^{(1)}$, then the equations become

see errata

$$\left.\begin{aligned} &u_x+v_y+w_y=0\\ &\boldsymbol{v}_t=-\frac{1}{\varrho}\operatorname{grad}(p+\varrho g y)+\nu\Delta\boldsymbol{v},\\ &u_y+v_x=w_y+v_z=0 && \text{for}\quad y=0,\\ &p-\varrho g\eta-2\mu v_y=-T(\eta_{xx}+\eta_{zz})+\bar{p} && \text{for}\quad y=0,\\ &\eta_t(x,z,t)=v(x,0,z,t). \end{aligned}\right\} \tag{25.1}$$

One may clearly combine (10.2) and (10.3) in the same way. In order to obtain the proper equations in a coordinate system moving to the right with velocity u_0, one need only replace $\partial/\partial t$ by $\partial/\partial t - u_0\,\partial/\partial x$.

The standard procedure for solving the equations is to represent the motion as a potential flow plus a rotational flow and to determine the pressure from the potential part. Thus, let

$$\boldsymbol{v}=\boldsymbol{v}^{(p)}+\boldsymbol{v}^{(r)} \tag{25.2}$$

where

$$\boldsymbol{v}^{(p)}=\operatorname{grad}\Phi \tag{25.3}$$

and let

$$p=-\varrho\Phi_t-\varrho g y. \tag{25.4}$$

It then follows from the second equation in (25.1) that $\boldsymbol{v}^{(r)}$ must satisfy

$$\frac{\partial}{\partial t}\boldsymbol{v}^{(r)}=\nu\Delta\boldsymbol{v}^{(r)}. \tag{25.5}$$

The relation between $\boldsymbol{v}^{(p)}$ and $\boldsymbol{v}^{(r)}$ is established through the boundary conditions. In the several examples treated below the motion is two-dimensional. However, there is no difficulty in principle and not much additional algebraic complexity in solving the analogous three-dimensional problems. The essential simplification in two dimensions is that the components of $\boldsymbol{v}^{(r)}$ may be expressed, as a consequence of the continuity equation, in terms of a single function Ψ:

$$u^{(r)}=\frac{\partial\Psi}{\partial y},\qquad v^{(r)}=-\frac{\partial\Psi}{\partial x}. \tag{25.6}$$

It then follows easily from (25.5) that

$$\frac{\partial\Psi}{\partial t}=\nu\Delta\Psi. \tag{25.7}$$

Standing waves-infinite depth. We shall try to find a solution to the equations which has a profile of the form

$$\eta(x,t)=A(t)\cos(mx+\alpha). \tag{25.8}$$

If such a solution exists, the nature of $A(t)$ will, of course, be of especial interest. We take Φ and Ψ of the form

$$\Phi = F(y,t)\cos(mx+\alpha), \qquad \Psi = G(y,t)\sin(mx+\alpha). \tag{25.9}$$

Eq. (25.7) then implies that

$$\Psi = (c\,e^{ly} + d\,e^{-ly})\,e^{\omega t}\sin(mx+\alpha), \tag{25.10}$$

see errata

wheren

$$l^2 = m^2 + \frac{\omega}{\nu}. \tag{25.11}$$

Neither l nor ω need be real. The form of Φ is further determined by $\Delta\Phi = 0$ and its relation to Ψ through the third boundary condition in (25.1). It must be

$$\Phi = (a\,e^{my} + b\,e^{-my})\,e^{\omega t}\cos(mx+\alpha). \tag{25.12}$$

If, as usual, we require the motion to remain bounded as $y \to -\infty$, we must take $b = 0$. If l has a non-vanishing real part, which we assume for the present, we may without loss of generality take it to be positive. Hence one must have $d = 0$. It follows from the third condition of (25.1) that

$$a = c\,\frac{l^2+m^2}{2m^2}. \tag{25.13}$$

Substitution in the formula for η_t and integration with respect to t yield

$$\eta = c\,\frac{1}{2\nu m}\,e^{\omega t}\cos(mx+\alpha) = A_0\,e^{\omega t}\cos(mx+\alpha). \tag{25.14}$$

Finally, one must substitute into the dynamical boundary condition in (25.1). There p is computed from (25.4) with $y = 0$. For future use we retain the external pressure distribution $\bar{p}$, which we take in the form

$$\bar{p} = p_0\,e^{\omega t}\cos(mx+\alpha), \tag{25.15}$$

where p_0 may be complex. The boundary condition yields an equation relating l and m:

$$\nu^2(l^2+m^2)^2 - 4\nu^2 m^3 l + g m + T' m^3 = -m\,\frac{p_0}{\varrho}\,\frac{2m\nu}{c} = -m\,\frac{p_0}{\varrho A_0}, \tag{25.16}$$

or, by making use of (25.11), an equation relating ω and m:

$$(\omega + 2m^2\nu)^2 - 4\nu^2 m^3\sqrt{m^2+\frac{\omega}{\nu}} + g m + T' m^3 = -m\,\frac{p_0}{\varrho A_0}. \tag{25.17}$$

Consider first Eq. (25.16) with $p_0 = 0$ and let

$$z = \frac{l}{m}, \qquad K = \frac{g m + T' m^3}{\nu^2 m^4}. \tag{25.18}$$

Then (25.16) takes the dimensionless form

$$(z^2+1)^2 - 4z + K = 0.$$

An examination of this equation shows that two of its roots are always complex with negative real parts. These roots are discarded since the corresponding motion would not die out as $y \to -\infty$; in fact, we explicitly assumed earlier that l has a positive real part. [Note that if we had made the other possible assumption, i.e., that l had a negative real part, the resulting equation corresponding to (25.16) would have had roots with positive real part, again to be discarded.]

The other two roots have positive real part. Whether or not there is an imaginary part depends upon the value of K. There is a critical value $K_c \approx 0.581$ such that if $K < K_c$ the two allowable solutions are both real. If $K > K_c$, the solutions are complex conjugates. Let the two complex roots of positive real part be denoted by $l_1 \pm i l_2$. Then one may establish that $l_1/m > 0.683$. When the two admissible roots are real, both of them lie between 0 and m.

One may find the values of ω associated with the two admissible roots from (25.11). If they are both real ($K < K_c$), then $\omega = -\nu(m^2 - l^2) < 0$. In this case the motion is critically damped and the initial configuration of the surface gradually subsides. This occurs for a given m if ν is large enough. On the other hand, no matter how small ν is, it also occurs when m is large enough, i.e., for very small wavelength. If the two admissible roots are complex ($K > K_c$), then

$$\left.\begin{aligned} \omega &= -\nu m^2\left(1 - \frac{l_1^2}{m^2} + \frac{l_2^2}{m^2} \pm 2i\,\frac{l_1 l_2}{m^2}\right) \\ \text{and}\qquad e^{\omega t} &= 2\,e^{-\nu m^2\left(1 - \frac{l_1^2}{m^2} + \frac{l_2^2}{m^2}\right)t} \cos 2\,\frac{l_1 l_2}{m^2}\,t. \end{aligned}\right\} \tag{25.19}$$

One may establish that $1 - l_1^2/m^2 + l_2^2/m^2 > 0.534$, so that this motion consists of damped standing-wave oscillations. The larger m is, the more quickly it is damped.

Because of the relative complexity of Eqs. (25.16) and (25.17), it is convenient and leads to more perspicuous results to find the relation between ω and m in the two limiting cases of small and large viscosity. First consider the case of small viscosity. If in (25.17) one lets $\nu \to 0$, one regains the relation $\omega^2 = -gm - T'm^3$ of (24.9); let $\sigma_0^2 = gm + T'm^3$. However, if one retains all terms of the first power in ν, (25.17) becomes

$$\omega^2 + 4\nu m^2 \omega + gm + T'm^3 = 0, \tag{25.20}$$

which has roots

$$-2m^2\nu \pm \sqrt{4m^4\nu^2 - gm - T'm^3} \approx -2m^2\nu \pm i\sigma_0 \tag{25.21}$$

if $4m^4\nu^2 \ll gm + T'm^3$. Hence the surface profile can be described by

$$\eta = A_0\, e^{-2m^2\nu t} \cos(\sigma_0 t + \tau)\cos(mx + \alpha). \tag{25.22}$$

To this order of approximation, the frequency σ_0 is related to m as in a perfect fluid, but the amplitude is gradually damped. To have some idea of the orders of magnitude involved in the damping, one should consult the table on p. 645 where the row τ_0 gives computations relevant to this.

In order to find the behavior for large ν, divide equation (25.17) by $4m^4\nu^2$ and expand the term $[1 + \omega/m^2\nu]^{\frac{1}{2}}$ in a series. If one retains only terms in ν^{-1} and ν^{-2}, the resulting equation leads to

see errata

$$3\omega^2 + 4m^2\nu\omega + 2(gm + T'\omega^3) = 0. \tag{25.23}$$

The two solutions, both real and negative, are approximately, if $4m^4\nu^2 \gg gm + T'm^3$,

$$\omega_1 = -\frac{gm + T'm^3}{2m^2\nu}, \qquad \omega_2 = -\frac{4}{3}m^2\nu. \tag{25.24}$$

Here $|\omega_1| < |\omega_2|$ and hence ω_1 is the more important root inasmuch as it represents a slower damping of the motion. As is pointed out by LAMB (1932, p. 628), the root ω_1 corresponds to a value of l only slightly less than m, so that the motion is nearly irrotational. It should also be noted that by different methods of analyzing (25.17) for large ν one may obtain somewhat different coefficients for ω_2.

In the preceding analysis it was assumed explicitly that l had a non-vanishing real part. If l is pure imaginary, $l = il'$, another family of solutions exists. It is now convenient two write Φ and Ψ in the forms

$$\left.\begin{aligned}\Phi &= a\, e^{my}\, e^{\omega t} \cos(mx+\alpha),\\ \Psi &= (c\cos l'y + d\sin l'y)\, e^{\omega t} \sin(mx+\alpha),\end{aligned}\right\} \tag{25.25}$$

where

$$\omega = -\nu(l'^2 + m^2) < 0. \tag{25.26}$$

The motion is thus a purely subsiding one. The boundary conditions determine the following relations between a, c, and d:

$$a = c\frac{m^2 - l'^2}{2m^2}, \qquad d = c\frac{1}{4\nu^2 m^3 l}\left[\nu^2(m^2 - l'^2)^2 + gm + T'm^3\right]. \tag{25.27}$$

All real values of l' are now admissible. The surface profile is given by

$$\eta = c\frac{1}{2m\nu}\, e^{\omega t} \cos(mx+\alpha). \tag{25.28}$$

The two sets of solutions may now be used to investigate the development of an initial disturbance [cf. SRETENSKII (1941)].

Forced standing waves. We may apply Eq. (25.16) or (25.17) to answer the following question. Suppose that m is given. Can we determine p_0 in such a way that a steady standing wave

$$\eta = A_0\, e^{-i\sigma t} \cos(mx+\alpha) \tag{25.29}$$

of prescribed frequency σ is maintained? From (25.17) p_0 is then determined by

$$-m\frac{p_0}{\varrho A_0} = (2m^2\nu - i\sigma)^2 - 4\nu^2 m^3\sqrt{m^2 - i\frac{\sigma}{\nu}} + gm + T'm^3. \tag{25.30}$$

If, for small viscosity, one discards terms higher than the first in ν, one obtains

$$p_0 = 4i\sigma\mu m A_0 - \sigma^2 + gm + T'm^3. \tag{25.31}$$

If we take $\sigma^2 = gm + T'm^3$, the frequency obtained from perfect-fluid theory, the necessary pressure distribution becomes

$$\bar{p} = 4\sigma\mu m A_0 i\, e^{-i\sigma t} \cos(mx+\alpha). \tag{25.32}$$

Thus the pressure must lead the surface displacement by a quarter of a period.

Standing waves-finite depth. If the fluid is of depth h, the analysis is similar to that above, but yields expressions of much greater complexity. The functions Φ and Ψ may be shown to have the forms

$$\left.\begin{aligned}\Phi &= \frac{1}{m}\left[dl\cosh m(y+h) + cm\sinh m(y+h)\right] e^{\omega t}\cos(mx+\alpha),\\ \Psi &= \left[c\cosh m(y+h) + d\sinh m(y+h)\right] e^{\omega t}\sin(mx+\alpha),\end{aligned}\right\} \tag{25.33}$$

where again

$$\omega = \nu(l^2 - m^2). \tag{25.34}$$

Let

$$L = \cosh lh, \qquad L' = \sinh lh, \qquad M = \cosh mh, \qquad M' = \sinh mh.$$

Then c and d are related by

$$2m(cmM + dlM') - (l^2 + m^2)(cL + dL') = 0. \tag{25.35}$$

The relation between l and m corresponding to (25.16) becomes

$$\left.\begin{aligned} &\nu^2 (l^2+m^2)^2 \frac{(l^2+m^2)\,(l\,L\,M - m\,L'\,M') - 2m^2 l}{(l^2+m^2)\,(l\,L\,M' - m\,L'\,M)} - \\ &\quad - 4\nu^2 m^3 l \frac{2m\,(m\,M L - l\,M'\,L') - (l^2+m^2)}{2m\,(m\,M\,L' - l\,M'\,L)} + g\,m + T'm^3 = -\,m\frac{p_0}{\varrho\, A_0} \end{aligned}\right\} \tag{25.36}$$

and the surface profile is

$$\eta = \frac{1}{2\nu\, m}\,(c\,L + d\,L')\,e^{\omega t}\cos(m\,x+\alpha) = A_0\,e^{\omega t}\cos(m\,x+\alpha). \tag{25.37}$$

The formulas become more perspicuous in the case of small viscosity and no external pressure and exhibit the importance of the presence of the bottom. If in (25.36) one sets $p_0 = 0$ and retains only terms of order ν^0, $\nu^{\frac{1}{2}}$ and ν, the following equation results:

$$\left.\begin{aligned} &\omega^3 - m\sqrt{\nu}\tanh m h\,\omega^{\frac{5}{2}} + \tfrac{9}{2} m^2\nu\,\omega^2 + (g m + T'm^3)\tanh m h\,\omega - \\ &\quad - (g m + T'm^3)\,m\sqrt{\nu}\,\omega^{\frac{1}{2}} + \tfrac{1}{2}(g m + T'm^3)\,m^2\nu\tanh m h = 0. \end{aligned}\right\} \tag{25.38}$$

One may solve this equation by expanding ω in powers of $\nu^{\frac{1}{2}}$,

$$\omega = \omega_0 + \omega_1\sqrt{\nu} + \omega_2\nu + \cdots,$$

substituting in (25.38) and keeping only terms in ν^0, $\nu^{\frac{1}{2}}$ and ν. The term independent of ν yields $\omega_0 = \pm i\sigma_0$, where σ_0 is given in (24.10) and is the frequency for an inviscid fluid. To the order of accuracy consistent with (25.38), one finds

$$\omega = \pm i\sigma_0 - (1\pm i)\,\frac{1}{2} m\sqrt{2\sigma_0\nu}\,\operatorname{cosech} 2m h - 2m^2\nu\,\frac{\cosh 4m h + \cosh 2m h - 1}{\cosh 4m h - 1}. \tag{25.39}$$

The first two terms were given by HOUGH (1897). The correct expression (25.39) was first given by BIESEL (1949); HOUGH had given $-2m^2\nu$ for the last term
see errata
but the apparently made an error in calculation, for (25.39) was derived independently of BIESEL's work and has also been checked by CARRY (1956) [BASSET's analysis (1888, p. 314) overlooks the terms in $\nu^{\frac{1}{2}}$].

The formula (25.39) should be compared with (25.21), the corresponding formula for infinite depth. There the effect of viscosity enters only with the first power of ν. The dissipation of energy in the body of the fluid is evidently of less importance than in the vicinity of the bottom. When two fluids are superposed, a similar phenomenon occurs in the neighborhood of the interface [cf. (25.44)].

Standing waves-stratified fluids. Consider now the situation in which a fluid typified by ϱ_1 and μ_1 fills the space $y<0$ and another typified by $\varrho_2<\varrho_1$ and μ_2 the space $y>0$. The equations to be satisfied in the two fluids and at their interface are given in (10.3). The method of solution is analogous to that used for a single fluid. However, separate functions Φ_1, Ψ_1, and Φ_2, Ψ_2 are needed for the lower and upper fluids. For a standing-wave solution they may be taken in the form

$$\left.\begin{aligned} \Phi_1 &= a_1\,e^{\omega t}\,e^{m y}\cos(m\,x+\alpha), & \Psi_1 &= b_1\,e^{\omega t}\,e^{l_1 y}\sin(m\,x+\alpha), \\ \Phi_2 &= a_2\,e^{\omega t}\,e^{-m y}\cos(m\,x+\alpha), & \Psi_2 &= b_2\,e^{\omega t}\,e^{-l_2 y}\sin(m\,x+\alpha), \end{aligned}\right\} \tag{25.40}$$

where we assume both l_1 and l_2 to have positive real parts. ω, l_1, l_2 and m are related by the equation

$$\omega = \nu_1(l_1^2 - m^2) = \nu_2(l_2^2 - m^2). \tag{25.41}$$

see errata

Substitution of (25.40) in the various boundary conditions at $y=0$ gives four homogeneous equations relating a_1, a_2, b_1, and b_2. The determinant of the coefficients set equal to zero yields another relation between ω_1, l_1, l_2 and m:

$$\left.\begin{aligned}[(\varrho_1+\varrho_2)\,\omega^2+(\varrho_1-\varrho_2)\,g\,m+T\,m^3]\,[\mu_1\,m+\mu_2\,l_2+\mu_2\,m+\mu_1\,l_1]+\\+4\omega\,m\,(\mu_1\,m+\mu_2\,l_2)\,(\mu_2\,m+\mu_1\,l_1)=0.\end{aligned}\right\}\tag{25.42}$$

In the limiting case of small viscosity, (25.42) gives

$$\omega^2+\frac{4m}{\varrho_1+\varrho_2}\,\frac{\sqrt{\varrho_1\,\varrho_2\,\mu_1\,\mu_2}}{\sqrt{\mu_1\,\varrho_1}+\sqrt{\mu_2\,\varrho_2}}\,\omega^{\frac{3}{2}}+\frac{(\varrho_1-\varrho_2)\,g\,m+T\,m^3}{\varrho_1+\varrho_2}=0.\tag{25.43}$$

This has the approximate solutions, when the coefficient of $\omega^{\frac{3}{2}}$ is small relative to the last term,

$$\omega=\pm i\,\sigma_0-\frac{1\pm i}{\sqrt{2}}\sqrt{\sigma_0}\,\frac{2m}{\varrho_1+\varrho_2}\cdot\frac{\sqrt{\varrho_1\,\varrho_2\,\mu_1\,\mu_2}}{\sqrt{\varrho_1\,\mu_1}+\sqrt{\varrho_2\,\mu_2}}-\frac{2m^2}{\varrho_1\,\varrho_2}\cdot\frac{\varrho_1\,\mu_1^{\frac{3}{2}}+\varrho_2\,\mu_2^{\frac{3}{2}}}{(\sqrt{\varrho_1\,\mu_1}+\sqrt{\varrho_2\,\mu_2})^2}\tag{25.44}$$

where σ_0 is the perfect-fluid frequency given in Eq. (24.14). This solution was first given by HARRISON (1908). The most significant physical fact about (25.44) when compared with (25.21) is that, to the order of approximation considered, the latter shows a rate of decay proportional to $m^2\nu$ and no influence of viscosity on the frequency, whereas (25.44) shows a rate of decay and an alteration of the frequency proportional to $m\sqrt{\nu}$ (in a dimensional sense). The greater importance of viscosity for stratified fluids may be ascribed to the different boundary condition at the interface. HARRISON computed the wave velocity and modulus of decay (time required for the amplitude to decrease by a factor e^{-1}) for an air-water interface at 17° C ($\varrho_1=1$, $\varrho_2=0.00129$, $\nu_1=0.0109$, $\nu_2=0.139$, $T=74$ in c.g.s. units). In the following table reproduced from HARRISON's paper v_0, v_c

Wavelength (cm)	1	10	100	1 000
v_0 (cm/sec)	12.48	39.46	124.79	394.62
v_c	24.90	40.05	124.81	394.62
v	24.89	40.04	124.81	394.62
τ_0	$1^{\mathrm{s}}.162$	$1^{\mathrm{m}}56^{\mathrm{s}}.2$	$3^{\mathrm{h}}12^{\mathrm{m}}39^{\mathrm{s}}.4$	$321^{\mathrm{h}}\ 5^{\mathrm{m}}40^{\mathrm{s}}$
τ	$1^{\mathrm{s}}.125$	$1^{\mathrm{m}}34^{\mathrm{s}}.1$	$1^{\mathrm{h}}21^{\mathrm{m}}40^{\mathrm{s}}.6$	$36^{\mathrm{h}}50^{\mathrm{m}}36^{\mathrm{s}}$
τ_c	$1^{\mathrm{s}}.106$	$1^{\mathrm{m}}34^{\mathrm{s}}.0$	$1^{\mathrm{h}}21^{\mathrm{m}}40^{\mathrm{s}}.3$	$36^{\mathrm{h}}50^{\mathrm{m}}34^{\mathrm{s}}$

and v are the wave velocities neglecting, respectively, both surface tension and viscosity, viscosity, and neither; τ_0, τ, τ_c are the moduli of decay taking account of the water viscosity only, a water-air interface without surface tension and a water-air interface with surface tension. A striking aspect is the apparent importance of the air-water interface in damping long waves and almost total lack of influence on wave velocity [the latter fact is obvious from (25.44)].

For very large viscosities the results are analogous to those for a single fluid. The two values of ω analogous to those in (25.24) are

$$\omega_1=-\frac{(\varrho_1-\varrho_2)\,g\,m+T\,m^3}{\varrho_1+\varrho_2}\,\frac{1}{2m^2}\,\frac{\varrho_1+\varrho_2}{\mu_1+\mu_2},\qquad \omega_2=-m^2\,\frac{\mu_1+\mu_2}{\varrho_1+\varrho_2}.\tag{25.45}$$

The analysis of the roots of (25.42) for general values of ν_1 and ν_2 is difficult. However, it has been carried through by CHANDRASEKHAR (1955, especially pp. 170—173) for the special situation $\nu_1=\nu_2$ and $T=0$. In this case $l_1=l_2$.

The behavior is similar to that described for (25.17) except that the critical value K_c separating a steadily decaying motion from an oscillatory decaying one is now a function of $(\varrho_1 - \varrho_2)/(\varrho_1 + \varrho_2)$. This value (actually a different one since he chooses a different parameter) is tabulated for a variety of density combinations. Further analysis of (25.17) may be found in a paper by HIDE (1955) and TCHEN (1956b).

KUSAKOV (1944) has carried through an analysis similar to HARRISON's when the upper fluid is of depth h_2, the lower of depth h_1. However, the results do not seem to be consistent with HARRISON's (or those above) when h_1 and h_2 become large. This problem has also been considered by HIDE (1955), but with an approximation that neglects the viscous boundary conditions on the walls. HARRISON, in the same paper, has treated also the problem when the upper fluid is of finite depth and with a free surface. We shall not reproduce the results except to remark that his computations show that a thin layer of fluid of slightly different density exerts a very marked influence on the damping. The effect of a variable surface tension upon wave motion is investigated briefly in LAMB (1932, § 351) and at some length in LEVICH (1952, pp. 477—490).

see errata

Pulsing stationary source. DMITRIEV (1953) has derived the form of the functions Φ and Ψ and the surface profile in the presence of a submerged source of pulsating intensity $-Q\cos\sigma t$. We shall give here only his expression for the surface profile and an asymptotic expression for large distances from the source. Let the source be located at $(0, -h_0)$ and let

$$h = h_0\sqrt{\frac{\sigma}{\nu}}, \qquad \bar{x} = x\sqrt{\frac{\sigma}{\nu}}, \qquad \bar{y} = y\sqrt{\frac{\sigma}{\nu}}, \qquad \varepsilon = \frac{\sigma^2}{g}\sqrt{\frac{\nu}{\sigma}}.$$

The surface profile is then represented by

$$\left.\begin{aligned} \eta &= \mathrm{Re}\,\frac{Q\,e^{i\sigma t}}{\pi}\,\frac{\sigma}{g}\int\limits_0^\infty \frac{1-2i\chi^2}{4\varepsilon\left[i\chi^3\left(\chi-(i+\chi^2)^{\frac{1}{2}}\right)-\chi^2\right]+i(\chi-\varepsilon)}\,e^{-h\chi}\cos\bar{x}\chi\,d\chi \\ &= Q\frac{\sigma}{g}\,(1+100\,\varepsilon^4)^{\frac{1}{2}}\,e^{-h\varepsilon-4\varepsilon^3\xi\bar{x}}\cos\left(\sigma t-\varepsilon\bar{x}+4\varepsilon^3 h-\arctan 10\,\varepsilon^2\right)+\cdots. \end{aligned}\right\} \quad (25.46)$$

26. Stability of free surfaces and interfaces. In this section we wish to examine the circumstances under which a small disturbance of a free surface or of an interface between two fluids will increase in magnitude with time. The energy for this increase may come either from available potential energy, e.g. if the lower fluid is lighter than the upper one, available kinetic energy in the case of flowing fluids, from forced motion of solid boundaries, or possibly some other source such as a given pressure distribution over a free surface. Surface tension and viscosity may be expected to have a stabilizing effect, so that special interest attaches to the study of their influence. We shall use the nature of the energy source as a convenient one for separating classes of problems, even though not every situation falls clearly into one of them.

Since the boundary conditions and equations which we shall use for the mathematical analysis have been linearized, following the assumption that the disturbances are small, one cannot expect the predictions of the theory to be valid quantitatively much beyond the initiation of an unstable motion. However, a great advantage in the use of linearized theory is that an arbitrary initial disturbance can be analyzed into Fourier components and the behavior of individual components examined separately.

α) Interface between stationary superposed fluids. Following our earlier notation, let us identify quantities referring to the lower fluid by the subscript 1 and to the upper fluid by 2. Let a sinusoidal disturbance of wave number m exist at the interface. Consider first the case of perfect fluids with no surface tension. Then, if both fluids are infinitely deep, the relation (14.28) must hold. If $\varrho_1 > \varrho_2$, the standing-wave solution of Sect. 14δ obtains. However, if $\varrho_1 < \varrho_2$, then $\sigma^2 < 0$ and σ is imaginary. Let $\omega^2 = -\sigma^2$, i.e.

$$\omega^2 = \frac{\varrho_2 - \varrho_1}{\varrho_2 + \varrho_1} g\, m. \tag{26.1}$$

Then one must replace $\cos(\sigma t + \tau)$ in the Φ_i of that section by, say, $\sinh \omega t$. The profile of the free surface is then, according to (10.8), given by

$$\eta = A \sin m\, x \cosh \omega\, t. \tag{26.2}$$

The amplitude of the initial corrugations of the surface evidently increases very rapidly with time, and the solution is a valid approximation for only a limited time interval. The nature of the disturbance need not have been restricted to $\sin m x$; any function $\varphi(x, z)$ satisfying $\Delta \varphi + m^2 \varphi = 0$ would have yielded the same behavior. Eq. (26.1) still holds if the two fluids are bounded below and above, respectively, by $y = -h_1$ and $y = h_2$ except that ω is given by

see errata

$$\omega^2 = \frac{\varrho_2 - \varrho_1}{\varrho_2 \coth m\, h_2 + \varrho_1 \coth m\, h_1} g\, m < \frac{\varrho_2 - \varrho_1}{\varrho_2 + \varrho_1} g\, m. \tag{26.3}$$

The surface is still unstable, but the rate of growth of the amplitude is slower.

Effect of surface tension. Let us now suppose that surface tension acts at the interface. Then the relation between σ and m given in (24.14) or (24.15) must hold, and a standing-wave solution is possible even if $\varrho_2 > \varrho_1$, provided that (24.16) holds, i.e.

$$\varrho_2 < \varrho_1 + \frac{T\, m^2}{g}. \tag{26.4}$$

Thus the interface is stable under small disturbances of sufficiently small wave length. If the unequality in (26.4) is reversed and we again set $\omega^2 = -\sigma^2$, then (26.2) holds once more and the solution is unstable. However, the value of ω^2 is less than that when $T = 0$, so that the rate of growth of the disturbance is retarded. It is also clear from the form of the relationship between ω^2 and m that there is a wave number for which ω^2, that is the rate of growth of the disturbance, is a maximum. If both fluids are of infinite depth this mode of maximum instability occurs when

$$m^2 = (\varrho_2 - \varrho_1)\, g/3\, T. \tag{26.5}$$

The effect of finite depth of the fluids is to displace the position of the maximum to higher values of m (smaller wavelengths) but a precise calculation requires solving a transcendental equation.

Effect of viscosity. The influence of viscosity in stabilizing interfacial disturbances has been the subject of a number of recent papers, in particular Bellman and Pennington (1954), Chandrasekhar (1955), Hide (1955) and Tchen (1956). The relevant equation relating ω and m is now (25.42). Because of the high degree of this equation it is not easy to give a complete discussion of its admissible roots. However, it is easy to establish that if

$$(\varrho_1 - \varrho_2)\, g + T\, m^2 < 0, \tag{26.6}$$

then (25.42) has a positive real root ω_0 satisfying

$$0 < \omega_0 < \sqrt{(\varrho_2 - \varrho_1)\, g\, m - T\, m^3}. \tag{26.7}$$

Thus the presence of viscosity does not alter the conditions for instability, as the presence of surface tension did, but it does have a stabilizing effect in that the rate of growth of a disturbance is slower.

In order to show the existence of a positive root under condition (26.6), one can write (25.42) in the form

$$(\varrho_1+\varrho_2)\,\omega^2+(\varrho_1-\varrho_2)\,g\,m+T\,m^3=-4\omega\,m\,\frac{(\mu_1 m+\mu_2 l_2)\,(\mu_2 m+\mu_1 l_1)}{\mu_1 m+\mu_2 l_2+\mu_2 m+\mu_1 l_1} \tag{26.8}$$

and sketch as functions of ω the curves represented by the two sides of the equation (remembering that l_1 and l_2 are functions of ω). The statement above then follows easily from the fact that both curves are continuous and the one represented by the right-hand function starts at the origin like

$$-2\,m^2\,(\mu_1+\mu_2)\,\omega$$

and goes to $-\infty$ in the fourth quadrant, behaving as $\omega\to\infty$ like

$$-4\,\omega^{\frac{3}{2}}\,m\,\frac{\sqrt{\varrho_1\,\varrho_2\,\mu_1\,\mu_2}}{\sqrt{\varrho_1\,\mu_1}+\sqrt{\varrho_2\,\mu_2}}\,.$$

A more elaborate discussion of the roots is given by BELLMAN and PENNINGTON (1954).

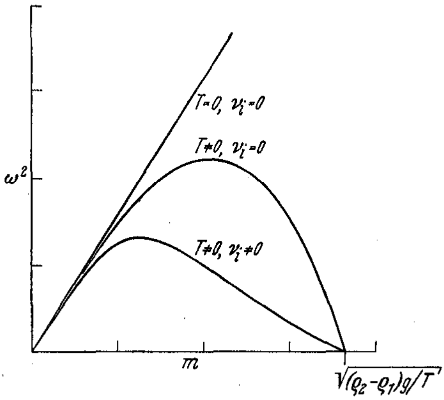

Fig. 33.

The behavior of ω_0 as a function of m in the interval defined by (26.7) and in particular the mode of maximum instability has been investigated by the authors cited earlier. CHANDRASEKHAR has computed the curves $\omega_0(m)$ for $\nu_1=\nu_2$, $T=0$ and a number of values of $(\varrho_2-\varrho_1)/(\varrho_2+\varrho_1)$. HIDE has recomputed these by an approximate method and then applied the method further to a fluid of finite depth with a continuous density variation $\varrho_0\,e^{\beta y}$. TCHEN has devised a different method of approximate computation and includes the effect of surface tension. Fig. 33, which is chiefly qualitative, shows the variation of ω^2 as a function of m in the interval of instability.

Accelerated fluid. If the whole system of fluid is being accelerated in the y-direction by a constant amount $\dot{v}_0=g_1$, then the relative motion in a moving coordinate system is the same as if the system were at rest and g had been replaced by $g+g_1$, as is immediately evident from Eq. (2.15). With this change the reasoning of the preceding paragraphs still applies. This fact was pointed out by G.I. TAYLOR (1950) who, on the basis of it, formulated the following rule (neglecting the influence of surface tension): If the fluids are being accelerated in a direction from the more to the less dense fluid, the interface is stable; in the converse case it is unstable. Experiments carried out by LEWIS (1950) for large accelerations, about $50\,g$, confirm TAYLOR'S observation and the predicted initial rate of growth. TAYLOR'S paper gave rise to a number of others treating various aspects of the instability of accelerated interfaces. In addition to those cited in the last paragraph, we mention INGRAHAM (1954), PLESSET (1954), BIRKHOFF (1954), KELLER and KOLODNER (1954), and LAYZER (1955) but shall not summarize the contents. The effect of an imposed acceleration oscillating in magnitude will be discussed in Sect. 26γ.

β) Interface between moving fluids. Consider the situation in which the fluid occupying the region $y<0$ $(y>0)$ is moving to the left with velocity $-c_1$ $(-c_2)$, and suppose that a small disturbance exists near the interface. If we suppose that the fluid is perfect and the motion in each fluid irrotational, then we may describe it by the velocity potentials

$$\Phi_i(x, y, z, t) = -c_i x + \phi_i(x, y, z, t). \tag{26.9}$$

We shall assume $c_1 \neq c_2$.

The kinematic boundary condition at the interface may be written, after linearization appropriate to the assumption of a small disturbance, in the form:

$$\eta_t(x, y, t) = c_1 \eta_x + \phi_{1y}(x, 0, z, t) = c_2 \eta_x + \phi_{2y}(x, 0, z, t). \tag{26.10}$$

The dynamical boundary condition (3.9) yields the following generalization of (10.8):

$$\varrho_1(\phi_{1t} - c_1\phi_{1x}) - \varrho_2(\phi_{2t} - c_2\phi_{2x}) + (\varrho_1 - \varrho_2) g\eta = T(\eta_{xx} + \eta_{zz}) \quad \text{for} \quad y = 0. \tag{26.11}$$

If η is eliminated between (26.10) and (26.11), one finds

$$\left.\begin{aligned} \varrho_1(\phi_{1tx} - c_1\phi_{1xx}) - \varrho_2(\phi_{2tx} - c_2\phi_{2xx}) + \frac{\varrho_1 - \varrho_2}{c_1 - c_2} g(\phi_{2y} - \phi_{1y}) + \\ + \frac{1}{c_1 - c_2} T(\phi_{2yyy} - \phi_{1yyy}) = 0. \end{aligned}\right\} \tag{26.12}$$

Let us now restrict our attention to two-dimensional motion of fluids bounded above by $y = h_2$ and below by $y = -h_1$, and let the initial displacement be $\eta(x, 0)$. Then from (15.2) we know that the subsequent motion may be resolved into harmonic progressive waves moving to the right and left. It will be sufficient for our purpose to examine a single component of the spectrum. Hence, we look for a solution in the form

$$\left.\begin{aligned} \phi_1 &= a_1 \cosh m(y + h_1)\, e^{i(mx - \sigma t)}, \\ \phi_2 &= a_2 \cosh m(y - h_2)\, e^{i(mx - \sigma t)}. \end{aligned}\right\} \tag{26.13}$$

It follows from (26.10) that $(c_1 - c_2)\eta_x = -\phi_{1y} + \phi_{2y}$. Hence

$$\eta = \frac{-i}{c_1 - c_2}[a_1 \sinh m h_1 + a_2 \sinh m h_2]\, e^{i(mx - \sigma t)}. \tag{26.14}$$

It then follows from (26.10) that

$$\frac{a_1 \sinh m h_1 + a_2 \sinh m h_2}{c_1 - c_2} = \frac{a_1 m}{\sigma + c_1 m} \sinh m h_1 = -\frac{a_2 m}{\sigma + c_2 m} \sinh m h_2. \tag{26.15}$$

Substitution of (26.13) in (26.12) and use of (26.15) yield the following relation between σ and m:

$$\varrho_1(\sigma + c_1 m)^2 \coth m h_1 + \varrho_2(\sigma + c_2 m)^2 \coth m h_2 - (\varrho_1 - \varrho_2) g m - T m^3 = 0. \tag{26.16}$$

The solution may be expressed as follows:

$$\left.\begin{aligned} \frac{\sigma}{m} = -\frac{c_1\varrho_1 \coth m h_1 + c_2 \varrho_2 \coth m h_2}{\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2} \pm \\ \pm \sqrt{\frac{(\varrho_1 - \varrho_2)\frac{g}{m} + T m}{\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2} - (c_1 - c_2)^2 \frac{\varrho_1 \varrho_2 \coth m h_1 \coth m h_2}{(\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2)^2}}. \end{aligned}\right\} \tag{26.16}$$

It is evident from the form of the term under the radical that σ cannot be real unless

$$(\varrho_1 - \varrho_2)\frac{g}{m} + T m > (c_1 - c_2)^2 \frac{\varrho_1 \varrho_2 \coth m h_1 \coth m h_2}{\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2}. \tag{26.17}$$

It is thus evident that there are no real solutions unless the left-hand side is positive and that there may even then exist an interval of wave numbers for which the disturbance is unstable (if both g and T are zero, such a velocity discontinuity is always unstable). If one assumes $\varrho_1 > \varrho_2$, the minimum value of the left-hand side is

$$2\sqrt{(\varrho_1 - \varrho_2)\, g\, T} \tag{26.18}$$

and occurs for $m^2 = (\varrho_1 - \varrho_2)\, g/T$. Since

$$\frac{\varrho_1 \varrho_2 \coth m h_1 \coth m h_2}{\varrho_1 \coth m h_1 + \varrho_2 \coth m h_2} > \frac{\varrho_1 \varrho_2}{\varrho_1 + \varrho_2}, \tag{26.19}$$

the disturbance will be unstable for some wave numbers whenever

$$(c_1 - c_2)^2 > 2\,\frac{\varrho_1 + \varrho_2}{\varrho_1 - \varrho_2} \sqrt{(\varrho_1 - \varrho_2)\, g\, T}. \tag{26.20}$$

One may conclude from (26.19) that the horizontal walls have a destabilizing effect in the sense that wave numbers which are stable for infinitely deep fluids may become unstable modes in the presence of walls. For an air-water interface the right side of (26.20) is about (646 cm/sec)^2. The corresponding wavelength is 1.71 cm; if the water is at rest ($c_1 = 0$), then the wave velocity is 0.84 cm/sec in the direction of the wind.

Let us suppose that c_1 and c_2 are both positive, i.e. that both fluids really do flow to the left. Then it follows from (26.16) that, if the roots are real, one of them is always negative and thus, from (26.13), represents a wave moving along the interface in the direction of the stream. The other will propagate upstream if

$$(\varrho_1 - \varrho_2)\,\frac{g}{m} + T m > \varrho_1 c_1^2 \coth m h_1 + \varrho_2 c_2^2 \coth m h_2, \tag{26.21}$$

otherwise also downstream.

An investigation along the above lines of the stability of an interface between flowing fluids was first given by Kelvin (1871). Similar treatments with additional information may be found in many texts, especially Lamb (1932, §§ 232, 268) and Rayleigh's *Theory of Sound* (Cambridge 1929, § 365). Kelvin's intention was to try to predict the minimum wind velocity which will cause a small disturbance on smooth water to increase in amplitude, and to find the unstable wave lengths. The predicted minimum velocity, roughly 650 cm/sec, is much higher than the observed minimum which is about 100 cm/sec. An evident objection to the analysis above is that viscosity of both air and water has been neglected. Since this alters in an essential way the behavior of the fluids near the interface, it is not surprising that the prediction is not accurate. One should not expect confirmation except in circumstances in which it is possible to show that the effect of viscosity is confined to a neighborhood of the interface small with respect to the minimum wave lengths considered. The subject of wind generation of waves is still in an unsettled state. One will find summaries of the present status in the article by H.U. Roll in Vol. XLVIII of this Encyclopedia, especially pp. 703—717, and also in a critical exposition by Ursell (1956). A summary of some of the work in the USSR on wave generation is included in Shuleikin (1956).

The inclusion of viscosity in the analysis above leads to a somewhat more difficult development than in the case of standing waves. An exposition of the present achievements in this theory will be omitted; they consist chiefly of papers by Wuest (1949) and Lock (1951, 1954).

γ) Vertically oscillated basins. Let S denote the wetted surface of a basin and F the water surface when the basin is at rest. We shall suppose that the basin is being oscillated in the y-direction according to some given law, which may be specified by giving $v_0(t)$, the velocity of a point of the basin. It will be most convenient to describe the motion of the fluid in coordinates fixed in the basin; these will be denoted by x, y, z. We shall assume the oscillations and the resulting motion to be of small amplitude so that we may linearize the equations and boundary conditions.

If Φ is the velocity potential for the motion relative to the basin and η the profile of the surface, both in coordinates fixed in the basin, then it follows easily from (2.17) that the only necessary change is to replace g by $g+\dot{v}_0$ in the boundary conditions at the free surface. They become:

$$\eta_t(x, z, t) = \Phi_y(x, 0, z, t), \tag{26.22}$$

$$(g+\dot{v}_0)\,\eta + \Phi_t(x, 0, z, t) = T'(\eta_{xx}+\eta_{zz}), \qquad T' = T/\varrho. \tag{26.23}$$

On the basin walls one must have

$$\Phi_n = 0 \quad \text{on} \quad S. \tag{26.24}$$

We wish, as usual, to investigate the character of the motion of the fluid.

The problem formulated above is clearly related to the problem considered in Sect. 23γ. However, the resulting motions are quite different. RAYLEIGH (1883) appears to have made the first theoretical investigation of this problem. More recently it has been studied by MOISEEV (1953, 1954), BENJAMIN and URSELL (1954), SCHULTZ-GRUNOW (1955) and BOLOTIN (1956). MOISEEV's analysis is the most general in that the only restriction upon the basin shape is that it should allow construction of a GREEN'S function for the Neumann problem; surface tension is not taken into account. BENJAMIN and URSELL restrict themselves to basins in the form of a vertical cylinder with horizontal bottom, but include the effect of surface tension. However, at the intersection with the walls they assume a 90° angle of contact with the free surface. This is in contradiction with the observed behavior of fluids but simplifies the mathematical treatment. In spite of this shortcoming it seems desirable to include the effect of surface tension, and this will be done below. BOLOTIN's paper considers a modification for viscous damping. The treatment below follows closely that of BENJAMIN and URSELL.

Let the basin be of depth h, let C denote the intersection of the walls with the plane $y=0$, and let $\boldsymbol{n}$ be a normal to the wall at a point of C. Then, from (26.22) and (26.24) it follows that $\eta_{tn}=\Phi_{yn}=0$, or $\eta_n=$const at each point of C; we take this constant to be zero, thus assuming a 90° contact angle with the wall. It then follows from (26.23) that $(\eta_{xx}+\eta_{zz})_n=0$.

Let $\varphi_k(x, y, z)$ be a set of functions harmonic in the region bounded by the basin and the plane $y=0$ and satisfying (26.24), and such that $\varphi_k(x, 0, z)$ form a complete set of orthonormal functions in the area of the (x, z)-plane bounded by C. Then $\Phi(x, 0, z, t)$, $\eta(x, z, t)$ and $\eta_{xx}+\eta_{zz}$ can each be expanded in series in $\varphi_k(x, 0, z)$. The expansion of $\Phi(x, 0, z, t)$ determines $\Phi(x, y, z, t)$ as as eries in $\varphi_k(x, y, z)$. In the case at hand, when the basin is a vertical cylinder, one may separate variables as in Sect. 12α and construct a set φ_k in the form

$$\varphi_k(x, y, z) = \frac{\cosh m_k(y+h)\,\varphi_k(x, z)}{\cosh m_k h}, \tag{26.25}$$

where

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \varphi_k(x, z) + m_k^2 \varphi_k(x, z) = 0. \tag{26.26}$$

The eigenvalues m_k^2 are determined by the boundary condition on the contour C, namely $(\partial/\partial n)\, \varphi_k = 0$.

Let the expansion for η be written in the form

$$\eta(x, z, t) = \sum_{n=1}^{\infty} a_k(t)\, \varphi_k(x, z). \tag{26.27}$$

Then, by differentiating (26.27) and using (26.26) one gets

$$\eta_{xx} + \eta_{zz} = -\sum a_k(t)\, m_k^2 \varphi_k(x, z). \tag{26.28}$$

If

$$\Phi(x, 0, z, t) = \sum b_k(t)\, \varphi_k(x, z),$$

then

$$\Phi_y(x, y, z, t) = \sum b_k(t)\, m_k \frac{\sinh m_k(y+h)}{\cosh m_k h} \varphi_k(x, z)$$

and, from (26.22),

$$b_k(t)\, m_k \tanh m_k h = \dot{a}_k(t).$$

Hence

$$\Phi(x, y, z, t) = \sum \dot{a}_k(t) \frac{\cosh m_k(y+h)}{m_k \sinh m_k h} \varphi_k(x, z). \tag{26.29}$$

Now substitute (26.27) to (26.29) in the remaining boundary condition (26.23):

$$\sum \left[(g + \dot{v}_0)\, a_k + T' m_k^2 a_k + \frac{1}{m_k} \ddot{a}_k \coth m_k h\right] \varphi_k = 0.$$

Since the φ_k are orthogonal, we may set each coefficient of φ_k equal to zero. With the special choice

$$\dot{v}_0 = c \cos \sigma t \tag{26.30}$$

the following set of differential equations determine the a_k:

$$\ddot{a}_k(t) + [(g m_k + T' m_k^3) \tanh m_k h + c m_k \tanh m_k h \cos \sigma t]\, a_k(t) = 0. \tag{26.31}$$

If we set

$$\left.\begin{aligned} &\tau = \tfrac{1}{2}\sigma t, \qquad p_k = \frac{4}{\sigma^2}(g m_k + T' m_k^3) \tanh m_k h = 4\frac{\sigma_k^2}{\sigma^2}, \\ &q_k = -\frac{2}{\sigma^2} c m_k \tanh m_k h, \end{aligned}\right\} \tag{26.32}$$

where σ_k is the frequency of free oscillations in the mode m_k when the basin is fixed, then (26.31) takes one of the standard forms for the Mathieu equation:

$$\frac{d^2}{d\tau^2} a_k + [p_k - 2 q_k \cos 2\tau]\, a_k = 0. \tag{26.33}$$

Of particular interest in the present context is the behavior of the solutions a_k as τ, or t, becomes large. It is known from the theory of differential equations with periodic coefficients that a pair of fundamental solutions can be given in the form

$$e^{\mu\tau} Q(\tau), \qquad e^{-\mu\tau} Q(-\tau), \tag{26.34}$$

where Q is of period π, unless $i\mu$ is an integer. In the latter case there exists a periodic solution, of period π if $i\mu$ is even and of period 2π if odd, and another independent nonperiodic solution. The coefficient μ will be a function of the parameters p_k, q_k and it is particularly pertinent to the present investigation to know for what regions in the (p, q)-plane μ has a nonzero real part. These regions have been investigated for other purposes and may be found, for example, in N. W. McLachlan's *Theory and application of Mathieu functions* (Oxford, 1947, pp. 40, 41). In Fig. 34, reproduced from Benjamin and Ursell, the shaded regions represent the unstable regions of the (p, q)-plane where μ has a nonzero real part. In the unshaded regions μ is pure imaginary (but not an integer) and the two solutions (26.34) are bounded for all τ. The boundaries between regions correspond to the periodic solutions occurring when $i\mu$ is an integer. In the unstable regions the periodicity behavior of the solutions is of two types. In the second, fourth, ... regions μ is real and the solutions (26.34) are functions of period π multiplied by exponentials. In the first, third, ... regions $\mu = \mu_1 + i$, μ_1 real, and the solutions (26.34) now become functions of period 2π multiplied by exponentials. In terms of t the two sets of regions correspond, respectively, to frequencies σ and $\frac{1}{2}\sigma$.

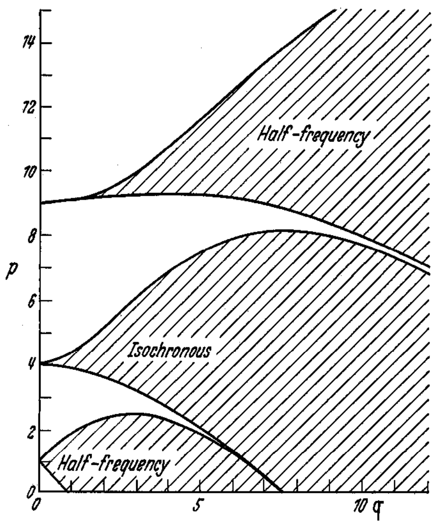

Fig. 34.

For a given mode of oscillation m_k one must compute p_k and q_k and plot (p_k, q_k) on the stability chart in order to find out whether the mode is stable or not. It seems likely, and, in fact, has been proved by Moiseev (1954, p. 44), that for any given values of σ and c some of the possible modes will be unstable. However, the analysis above has neglected the damping effect of viscosity and it may be supposed that the only unstable modes which actually occur are those associated with the smaller values of m_k. In any case, as has been emphasized earlier, the analysis is only suitable for describing the initial stages of the motion.

If the half-frequency of oscillation $\frac{1}{2}\sigma$ is equal, or nearly so, to one of the frequencies σ_k for free oscillation of the fluid, or to a subharmonic of σ_k, i.e. $\frac{1}{2}\sigma = \sigma_k/n$, then $p_k = 1$, or n^2, and it is evident from Fig 34 that (p_k, q_k) will be in an unstable region. If $\frac{1}{2}\sigma = \sigma_k$, (p_k, q_k) will lie in the lowest region and standing waves with half the frequency of the basin will be generated. If $\sigma = \sigma_k$, (p_k, q_k) will lie in the second region and the generated standing waves will have the same frequency as the basin. Thus the mode σ_k can be excited by oscillating the basin with frequency either σ_k or $2\sigma_k$. It is pointed out by Benjamin and Ursell that an apparent discrepancy between experimental observations of Faraday and Rayleigh and of Matthiessen can be explained by the above remarks.

Benjamin and Ursell made an experimental investigation with a circular cylinder in order to determine by experiment the boundaries of the lowest region of instability. The measurements provide a surprisingly good confirmation within certain limitations.

27. Higher-order theory of infinitesimal waves. It is implicit in the theory of infinitesimal waves developed in the preceding sections of this chapter that the approximation given by first-order theory to the solution of a particular problem, assuming that one exists, can be improved by including further terms in the perturbation series. The solution of the resulting boundary-value problems, at least in the simplest cases, can be carried through in a manner similar to that of the first-order theory, although the computations become more and more tedious the higher the order of approximation. Nevertheless, in view of the interest of the results, the computations have been carried through by a number of persons and by a variety of methods.

STOKES (1849) was apparently the first to make the calculation for progressive waves; in fact, the method used below in Sect. 27α is not essentially different from STOKES' first method. Later, in connection with the publication of his collected papers, STOKES (1880) added a supplement describing a different procedure. RAYLEIGH turned to the problem several times (1876, 1911, 1915, 1917) and introduced still another method of approximation. It should be noted, however, that both STOKES' second method and RAYLEIGH's method are limited to two-dimensional irrotational progressive waves. RAYLEIGH (1915) seems to be the first to have given an adequate treatment of the higher-order theory of standing waves. In addition to these classical papers there have been many others extending or improving the earlier theory; some of these will be noted below.

In all such computations, and indeed in the numerous first-order computations carried out in the earlier sections of this chapter, there is the tacit assumption that there exists an "exact solution" which is being approximated and which can be approached more and more closely by pursuing the selected method of approximation. Unfortunately, it is seldom that one is able to prove the existence of an exact solution or of convergence of the method of approximation, and, in fact, BURNSIDE (1916) cast doubt upon the usefulness of the Stokes-Rayleigh type of approximation of periodic progressive waves of permanent type. BURNSIDE's objection was later met by NEKRASOV's (1921, 1922, 1951), LEVI-CIVITA's (1925) and STRUIK's (1926) proofs of the existence of such waves for both infinite and finite depth. However, the existence of a standing wave satisfying the exact boundary conditions has not been demonstrated as yet. The same is true of the more complicated problems considered in earlier sections. However, this mathematical shortcoming is possibly of no more importance than the neglect in many problems of relevant physical parameters such as viscosity.

One should bear in mind that the higher-order infinitesimal waves considered below are not the only higher-order approximations. The solitary and cnoidal waves of the next chapter bear a similar relation to the first-order shallow-water theory. In addition, in the last chapter another method of approximating exact waves, due to HAVELOCK (1919a), will be described.

α) *Periodic progressive waves.* In the following we shall be seeking a wave which moves without change of form, i.e. a progressive wave in the sense of Sect. 7. Hence we shall expect to be able to represent Φ and η in the form

$$\Phi(x, y, z, t) = \varphi(x - ct, y, z), \qquad y = \eta(x - ct, z), \tag{27.1}$$

where c is the velocity of the wave. It will be convenient to represent the motion in a moving coordinate system, say $\bar{x} = x - ct$. However, we shall henceforth drop the bar over the x. The boundary conditions at the free surface are then the following:

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$$\eta_x(x, z)\,\varphi_x(x, \eta(x, z)\, z) - \varphi_y + \eta_z \varphi_z - c\varphi_x = 0, \tag{27.2}$$

$$-c\varphi_x(x, \eta(x, z), z) + \tfrac{1}{2}(\operatorname{grad}\varphi)^2 + g\eta - T'(R_1^{-1} + R_2^{-1}) = 0, \tag{27.3}$$

where $R_1^{-1}+R_2^{-1}$ is given by (3.5′) and, as usual, $T'=T/\varrho$. Surface tension is being taken into account both for the intrinsic interest of the results and because of an interesting phenomenon which occurs in the higher-order approximations. We shall suppose that the wave length $\lambda=2\pi/m$ of the wave system has been given, so that c is still an unknown of the problem.

Let us now, as in Sect. 10α, assume that φ, η and c may all be expanded in a perturbation series in some parameter ε:

$$\left.\begin{aligned} \varphi &= \varepsilon\varphi^{(1)}+\varepsilon^2\varphi^{(2)}+\cdots,\\ \eta &= \varepsilon\eta^{(1)}+\varepsilon^2\eta^{(2)}+\cdots,\\ c &= c_0+\varepsilon c_1+\varepsilon^2 c_2+\cdots. \end{aligned}\right\} \tag{27.4}$$

After substituting in (27.2) and (27.3) and collecting terms in the manner of Sect. 10α, one obtains the following boundary conditions which must be satisfied successively by $\varphi^{(1)}, \eta^{(1)}, c_0$; $\varphi^{(2)}, \eta^{(2)}, c_1$; $\varphi^{(3)}, \eta^{(3)}, c_2$:

$$c_0\eta^{(1)}_x+\varphi^{(1)}_y=0,\qquad g\eta^{(1)}-c_0\varphi^{(1)}_x-T'(\eta^{(1)}_{xx}+\eta^{(1)}_{zz})=0; \tag{27.5}$$

$$\left.\begin{aligned} &c_0\eta^{(2)}_x+\varphi^{(2)}_y=\varphi^{(1)}_x\eta^{(1)}_x+\varphi^{(1)}_z\eta^{(1)}_z-\eta^{(1)}\varphi^{(1)}_{yy}-c_1\eta^{(1)}_x,\\ &g\eta^{(2)}-c_0\varphi^{(2)}_x-T'(\eta^{(2)}_{xx}+\eta^{(2)}_{zz})=c_1\varphi^{(1)}_x-\tfrac{1}{2}(\operatorname{grad}\varphi^{(1)})^2+c_0\eta^{(1)}\varphi^{(1)}_{xy}; \end{aligned}\right\} \tag{27.6}$$

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$$\left.\begin{aligned} c_0\eta^{(3)}_x+\varphi^{(3)}_y={}&\varphi^{(2)}_x\eta^{(1)}_x+\varphi^{(2)}_z\eta^{(1)}_x-\varphi^{(2)}_{yy}\eta^{(1)}-c_2\eta^{(1)}_x+\varphi^{(1)}_x\eta^{(2)}_x+\varphi^{(1)}_z\eta^{(2)}_z-\\ &-\varphi^{(1)}_{yy}\eta^{(2)}-c_1\eta^{(2)}_x+\eta^{(1)}\left[\varphi^{(1)}_{xy}\eta^{(1)}_x+\varphi^{(1)}_{zy}\eta^{(1)}_z\right]-\tfrac{1}{2}\varphi^{(1)}_{yyy}\eta^{(1)2},\\ g\eta^{(3)}-c_0\varphi^{(3)}_x-T'(\eta^{(3)}_{xx}+\eta^{(3)}_{zz})={}&c_2\varphi^{(1)}_x+c_1\varphi^{(2)}_x+c_1\varphi^{(1)}_{xy}\eta^{(1)}+c_0\varphi^{(2)}_{xy}\eta^{(1)}+\\ &+c_0\varphi^{(1)}_{xy}\eta^{(2)}+\tfrac{1}{2}c_0\varphi^{(1)}_{xyy}\eta^{(1)2}+\operatorname{grad}\varphi^{(1)}\cdot\operatorname{grad}\varphi^{(2)}+\\ &+\eta^{(1)}\operatorname{grad}\varphi^{(1)}\cdot\operatorname{grad}\varphi^{(1)}_y-T'\big[\eta^{(1)}_{xx}\eta^{(1)2}_z+\eta^{(1)}_{zz}\eta^{(1)2}_x-\\ &-2\eta^{(1)}_{xz}\eta^{(1)}_x\eta^{(1)}_z-\tfrac{3}{2}(\eta^{(1)}_{xx}+\eta^{(1)}_{zz})(\eta^{(1)2}_x+\eta^{(1)2}_z)\big], \end{aligned}\right\} \tag{27.7}$$

where all conditions are to be satisfied on the plane $y=0$. It is possible, of course, to carry the approximations further, but three steps are ample to illustrate the procedures. The solution will be carried through in outline through the third order for infinite depth and through the second order for finite depth. As an expansion parameter we may take $\varepsilon=Am$, where A is a length determining the amplitude of the waves. The motion will be restricted to be two-dimensional.

Infinite depth. The solutions of (27.5) are already known from (13.5). We take them in the following form

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$$\varphi^{(1)}=\frac{c_0}{m}e^{my}\sin mx,\qquad \eta^{(1)}=\frac{1}{m}\cos mx,\qquad c_0^2 m=g+m^2T'. \tag{27.8}$$

After substitution in (27.9), one finds

$$\left.\begin{aligned} &\varphi^{(2)}_y+c_0\eta^{(2)}_x=c_1\sin mx-c_0\sin 2mx,\\ &c_0\varphi^{(2)}_x-g\eta^{(2)}+T'\eta^{(2)}_{xx}=-c_1c_0\cos mx-\tfrac{1}{2}c_0^2\cos 2mx. \end{aligned}\right\} \tag{27.9}$$

Elimination of $\eta^{(2)}$ yields

$$c_0^2\varphi^{(2)}_{xx}+g\varphi^{(2)}_y-T'\varphi^{(2)}_{yxx}=2c_1c_0^2 m\sin mx-3c_0m^2T'\sin 2mx \tag{27.10}$$

as the boundary condition to be satisfied by $\varphi^{(2)}$. If $c_1\neq 0$, one cannot find a periodic potential function satisfying (27.10). Hence we set

$$c_1=0, \tag{27.11}$$

A solution of Laplace's equation satisfying (27.10) with $c_1=0$ and vanishing as $y\to-\infty$ is easily found to be

$$\varphi^{(2)}=\frac{3}{2}\,\frac{c_0}{m}\,\frac{m^2\,T'}{g-2m^2\,T'}\,e^{2my}\sin 2mx, \tag{27.12}$$

providing $m^2 \neq g/2T'$. The corresponding expression for $\eta^{(2)}$ is

$$\eta^{(2)}=\frac{1}{2}\,\frac{1}{m}\,\frac{g+m^2\,T'}{g-2m^2\,T'}\cos 2mx. \tag{27.13}$$

One could, of course, add terms of the form given in (27.8) but with arbitrary multipliers. However, such solutions are discarded since we wish to allow only first-order terms of this form.

Two striking facts show up in (27.12) and (27.13): First, if surface tension is neglected, $\varphi^{(2)}$ vanishes and $\varphi^{(1)}$ gives the velocity potential correctly to at least the second order. The second fact is the zero in the denominator in both $\varphi^{(2)}$ and $\eta^{(2)}$, which shows that $\varphi^{(2)}$ and $\lambda^{(2)}$ become unbounded as m approaches $\sqrt{g/2T'}$. One may argue, of course, that this simply shows that validity of the perturbation method is limited to smaller and smaller values of Am the closer one comes to $\sqrt{g/2T'}$. However, it seems also to be an indication that near $m=\sqrt{g/2T'}$ the mode represented by $\varphi^{(2)}$ is of the same order of magnitude as that represented by $\varphi^{(1)}$. That this is indeed the case is clear from an examination of the equation determining $\varphi^{(1)}$ and $\varphi^{(2)}$ when $m=\sqrt{g/2T'}$. In fact, $\varphi^{(2)}$ was not determined by (27.10) for this value of m and, furthermore, (27.8) does not give the complete solution of (27.5). The solution with which we must start in this case is

$$\varphi^{(1)}=\frac{c_0}{m}\left[e^{my}\sin mx + a\,e^{2my}\sin 2mx + b\,e^{2my}\cos 2mx\right], \tag{27.14}$$

where a and b are as yet undetermined constants. Thus these two modes of motion are of the same order for $m=\sqrt{g/2T'}$. One may now substitute (27.14) and the corresponding $\eta^{(1)}$ into (27.9). By reasoning similar to that used earlier in setting $c_1=0$, we now find

$$a=\pm\tfrac{1}{2},\qquad b=0,\qquad c_1=\pm\tfrac{1}{4}c_0. \tag{27.15}$$

There are thus two possible first-order modes depending upon the sign of a. $\varphi^{(2)}$ is now a sum of terms with modes $\sin 3mx$ and $\sin 4mx$, but will not be given here. The wave profile, including modes through $\cos 2mx$, may be written as follows:

$$\eta=A\left[\cos mx+\frac{1}{2}Am\frac{g+m^2\,T'}{g-2m^2\,T'}\cos 2mx\right],\qquad m\neq\sqrt{\frac{g}{2T'}}, \tag{27.16}$$

$$\eta=A\left[\cos mx\pm\frac{1}{2}\cos 2mx\right],\qquad m=\sqrt{\frac{g}{2T'}}. \tag{27.17}$$

The two signs in the second solution correspond roughly to the change of sign occurring in the first when k passes through $\sqrt{g/2T'}$. Comparison of the two cases also gives an indication of the limitations upon Am necessary in the first solution, namely,

$$|Am|<\left|\frac{g-2m^2\,T'}{g+m^2\,T'}\right|. \tag{27.18}$$

A reversal of curvature at the center of the wave trough for $m<\sqrt{g/2T'}$, or of the crest for $m>\sqrt{g/2T'}$, will occur when

$$|Am|>\frac{1}{2}\left|\frac{g-2m^2\,T'}{g+m^2\,T'}\right|. \tag{27.19}$$

The existence of the singularity in the expressions for $\eta^{(2)}$ and $\varphi^{(2)}$ was first noticed by HARRISON (1909). WILTON (1915) examined the matter more carefully, found the solutions (27.17) and, in fact, carried all approximations further. Some of WILTON's computed profiles are shown in Fig. 35. Although WILTON casts doubt upon the existence of the solution (27.17) with $+\frac{1}{2}$, such profiles seen to have been observed by KAMESVARA RAV (1920). However, the matter apparently still awaits a thorough experimental investigation, as do also similar higher modes mentioned below.

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Let us now turn to the next order, assuming $m \neq \sqrt{g/2T'}$. Substitution of (27.8) and (27.11) to (27.13) into (27.7) and elimination of $\eta^{(3)}$ yield the following boundary condition to be satisfied by $\varphi^{(3)}$ on $y=0$:

$$\left.\begin{aligned} c_0^2 \varphi_{xx}^{(3)} + g\varphi_y^3 - T'\varphi_{yxx}^{(3)} &= c_0^2 m\left[2c_2 - \frac{1}{2}c_0\frac{2g-m^2T'}{g-2m^2T'} + \frac{3}{8}c_0\frac{m^2T'}{g+m^2T'}\right]\sin mx + \\ &+ \frac{9}{8}c_0^3 m\left[\frac{4m^2T'}{g-2m^2T'} - \frac{m^2T'}{g+m^2T'}\right]\sin 3mx. \end{aligned}\right\} \tag{27.20}$$

Again in order to avoid an unbounded solution we must set the coefficient of $\sin mx$ equal to zero. This yields a value for c_2:

$$c_2 = \frac{1}{2}c_0\left[1 + \frac{\frac{3}{2}m^2T'}{g-2m^2T'} - \frac{3}{8}\frac{m^2T'}{g+m^2T'}\right]. \tag{27.21}$$

One may now find a potential function satisfying (27.20) and vanishing as $y \to -\infty$. The solutions for $\varphi^{(3)}$ and $\eta^{(3)}$ are as follows:

$$\varphi^{(3)} = -\frac{9}{16}\frac{c_0}{m}\frac{m^2T'(g+2m^2T')}{(g-2m^2T')(g-3m^2T')}e^{3my}\sin 3mx; \tag{27.22}$$

$$\left.\begin{aligned} \eta^{(3)} &= \frac{1}{m}\left[\frac{1}{8} + \frac{3}{16}\frac{m^2T'(5g+2m^2T')}{(g+m^2T')(g-2m^2T')}\right]\cos mx + \\ &+ \frac{3}{16}\frac{1}{m}\frac{2g^2 - gT'm^2 - 30(m^2T')^2}{(g-2m^2T')(g-3m^2T')}\cos 3mx, \end{aligned}\right\} \tag{27.23}$$

for $m \neq \sqrt{g/2T'}, \sqrt{g/3T'}$. From (27.22) one sees again that $\varphi^{(3)}$ would vanish if surface tension were neglected. Although we shall not carry through the computation, this does not happen for $\varphi^{(4)}$. It is also evident that another singularity has appeared at $m = \sqrt{g/3T'}$. In fact, when one examines the reason for the appearance of the singularities, it is evident that a mode of the form $\cos nmx$ will always show a singularity at $m = \sqrt{g/nT'}$. In each such case the reason is the same as in the situation discussed earlier with $n=2$: for $m = \sqrt{g/nT'} \equiv m_n$ the proper first-order solution is of the form

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$$\varphi^{(1)} = \frac{c_0}{m}\left[e^{my}\sin mx + a_n e^{nmy}\sin mx\right],$$

with a_n to be determined subsequently (according to WILTON only a_2 is not unique). Thus (27.8) should be qualified by $m^2 \neq g/nT'$. One should note that, although m_n is getting small (and hence λ_n large) as n increases, the wave number of the second first-order mode is $\sqrt{ng/T'}$. Hence, on the basis of the results in Sect. 25, one will expect this mode to be quickly damped for large values of n. However, one may presume the first few to be observable. We remark that these special associated pairs of first-order waves always straddle the wave number for minimum c_0, namely m_1.

The wave profile, velocity potential and wave velocity are now given by

$$\left.\begin{aligned}\eta &= A m \eta^{(1)} + A^2 m^2 \eta^{(2)} + A^3 m^3 \eta^{(3)} + \cdots,\\ \varphi &= A m \varphi^{(1)} + A^2 m^2 \varphi^{(2)} + A^3 m^3 \varphi^{(3)} + \cdots,\\ c &= c_0 + A m c_1 + A^2 m^2 c_2 + \cdots.\end{aligned}\right\} \tag{27.24}$$

To the third order the profile for pure gravity waves ($T'=0$) is represented by the following function:

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$$\left.\begin{aligned}\eta &= A\left\{\left[1 + \tfrac{1}{8} A^2 m^2\right] \cos m x + \tfrac{1}{2} A m \cos 2 m x + \tfrac{3}{8} A^2 m^2 \cos 3 m x + \cdots\right\}\\ &= A'\left\{\cos m x + \tfrac{1}{2} A' m \cos 2 m x + \tfrac{3}{8} A'^2 m^2 \cos 3 m x + \cdots\right\},\end{aligned}\right\} \tag{27.25}$$

where $A' = A\left[1 + \frac{1}{8} A^2 m^2\right]$; the velocity becomes

$$c = \sqrt{\frac{g}{m}}\left(1 + \frac{1}{2} A^2 m^2 + \cdots\right). \tag{27.26}$$

The velocity potential to the third order is

$$\varphi = A \sqrt{\frac{g}{m}}\, e^{m y} \sin m x. \tag{27.27}$$

If one sets $g=0$, then the wave profile for pure capillary waves becomes

$$\eta = A\left\{\left[1 - \tfrac{1}{16} A^2 m^2\right] \cos m x - \tfrac{1}{4} A m \cos 2 m x - \tfrac{15}{16} A^2 m^2 \cos 3 m x + \cdots\right\} \tag{27.28}$$

and the velocity

$$c = \sqrt{T' m}\left[1 - \tfrac{1}{16} A^2 m^2 + \cdots\right]. \tag{27.29}$$

For pure gravity waves the approximations were carried to the fifth order by STOKES, RAYLEIGH (1917) and others.

It is of interest to compare the profiles represented in (27.25) and (27.28). The effect of including higher-order terms in pure gravity waves is to sharpen and raise the crests and to broaden and raise the troughs. For pure capillary waves the effect is just the reverse. For combined gravity-capillary waves the increasing importance of the second-order term near $m=\sqrt{g/2T'}$ will first show up as a reversal of curvature at the middle of the flattened part of the wave; formula (27.19) gives the condition for the first occurrence. In Fig. 35 are shown a pure gravity wave as computed by WILTON (1914) for $Am=0.86$ (here A is the amplitude), and five gravity-capillary waves, the last two corresponding to the solutions (27.17), also computed by WILTON (1915) for a value of $T/\varrho g = 0.075$. It should be remarked that the value of $Am=0.86$ is much larger than any for which it is possible to prove convergence of the perturbation series and is, in fact, very close to the value of Am for the highest possible irrotational wave of permanent type (see Sect. 33 α), namely 0.891.

Finite depth. When a solid bottom is present at $y=-h$, the only necessary modification of the preceding analysis is substitution of the boundary condition $\varphi_y^{(i)}(x,-h)=0$ for $\varphi_y^{(i)}\to 0$ as $y\to-\infty$. This increases the computational labor by a substantial amount, but otherwise introduces no difficulties. However, we call attention to the remarks on the definition of wave velocity in Sect. 7; the velocity c below is the one defined there also as c.

The wave profile, velocity potential and wave velocity, including the effect of surface tension, are as follows, to the second order:

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$$\eta = A\left\{\cos m + \frac{1}{2} A m \frac{(2+\cosh 2mh)\operatorname{cosech} 2mh}{\tanh^2 mh - 3T'm^2(g+T'm^2)^{-1}} \cos 2mx\right\}, \tag{27.30}$$

$$\left.\begin{aligned} \varphi = A c_0 \Bigg\{ & \frac{\cosh m(y+h)}{\sinh mh}\sin mx + \\ & + \frac{3}{4} A m \frac{(g+3T'm^2)\coth mh - (g+T'm^2)\tanh mh}{(g+T'm^2)\tanh^2 mh - 3T'm^2}\,\frac{\cosh 2m(y+h)}{\sinh 2mh}\sin 2mx\Bigg\}, \end{aligned}\right\} \tag{27.31}$$

$$c^2 = c_0^2 = \left(\frac{g}{m} + T'm\right)\tanh mh. \tag{27.32}$$

The velocity is the same as in the first-order theory; this occurred also for infinite depth. In contrast to the case of infinite depth, the term $\varphi^{(2)}$ does not vanish

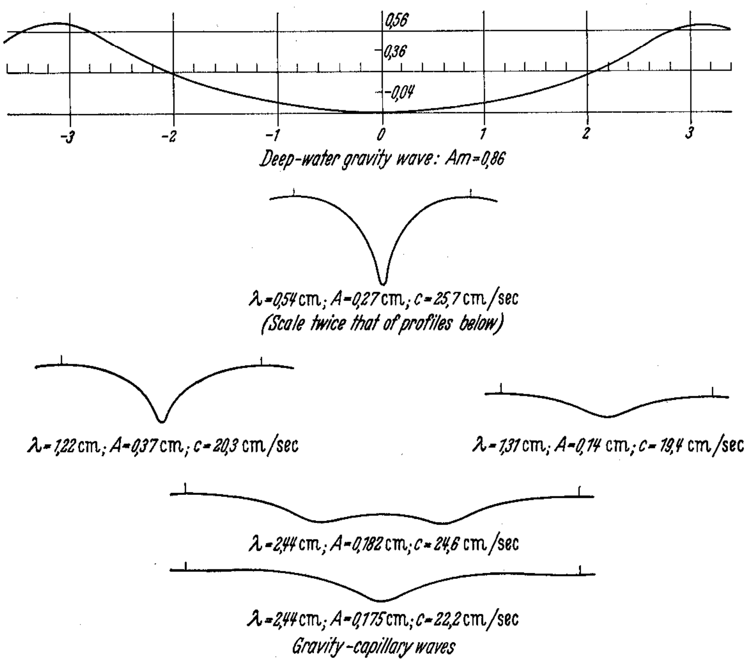

Fig. 35.

when $T'=0$. The singularity in the coefficient of $\cos 2mx$ still persists provided that $h > \sqrt{3T'/g}$. The earlier discussion of this phenomenon is still relevant, and a detailed one will be omitted here. However, even if surface tension is neglected in (27.30), the second-order term may still become large for small values of mh, as has been emphasized by MICHE (1944). If one again takes as an indication of increasing predominance of the second-order term a reversal of curvature at the bottom of the trough, one finds that this occurs for

$$A m > \frac{1}{2}\,\frac{\tanh^2 mh \sinh 2mh}{2+\cosh 2mh}, \tag{27.33}$$

or approximately

$$A m > \tfrac{1}{3}\tanh mh \sinh^2 mh$$

as given by MICHE. The occurrence of this secondary crest when mh is small has frequently been observed. It has been investigated experimentally by MORISON and CROOKE (1953) and by HORIKAWA and WIEGEL (1959).

The wave profile and velocity computations were carried by STOKES to the third order, and by DE (1955) to the fifth order, for pure gravity waves in fluid of finite depth. The following expressions are taken from a report by SKJELBREIA (1959):

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$$\left.\begin{aligned} \eta &= A\left\{\cos m\,x + \frac{1}{4}\,A\,m\,\frac{\cosh m\,h\,(2+\cosh 2m\,h)}{\sinh^3 m\,h}\cos 2m\,x + \right.\\ &\qquad\left. + \frac{3}{64}\,A^2 m^2\,\frac{8\cosh^6 m\,h+1}{\sinh^6 m\,h}\cos 3m\,x + \cdots\right\}, \\ c^2 &= \frac{g}{m}\tanh m\,h\left[1 + A^2 m^2\,\frac{8+\cosh 4m\,h}{8\sinh^4 m\,h} + \cdots\right]. \end{aligned}\right\} \tag{27.34}$$

SKJELBREIA has provided comprehensive tables allowing easy computation of η, φ and many other quantities of interest, all to the third order.

Particle orbits. A particularly interesting phenomenon occurs when higher-order approximations are used in the computation of the paths of individual particles. The equations which the coordinates of a particle must satisfy are

$$\frac{dx}{dt} = \varphi_x(x - c\,t,\,y), \qquad \frac{dy}{dt} = \varphi_y(x - c\,t,\,y). \tag{27.35}$$

Since φ depends upon the parameter ε, the solutions x and y also will. We assume then that x and y may be expanded into series of the form

$$x(t) = x_0 + \varepsilon\,x_1(t) + \cdots, \qquad y(t) = y_0 + \varepsilon\,y_1(t) + \cdots, \tag{27.36}$$

substitute them into (27.35) together with the appropriate expansion of φ in powers of ε, and then equate the several powers of ε separately. This results in a sequence of equations of which the first two are as follows

$$\frac{dx_1}{dt} = \varphi_x^{(1)}(x_0 - c_0\,t,\,y_0), \qquad \frac{dy_1}{dt} = \varphi_y^{(1)}(x_0 - c_0\,t,\,y_0); \tag{27.37}$$

$$\left.\begin{aligned} \frac{dx_2}{dt} &= x_1(t)\,\varphi_{xx}^{(1)}(x_0 - c_0\,t,\,y_0) + y_1\,\varphi_{xy}^{(1)} + \varphi_x^{(2)}, \\ \frac{dy_2}{dt} &= x_1(t)\,\varphi_{xy}^{(1)}(x_0 - c_0\,t,\,y_0) + y_1\,\varphi_{yy}^{(1)} + \varphi_y^{(2)}. \end{aligned}\right\} \tag{27.38}$$

The first set, (27.37), was already solved in (14.17) and (14.18) and to the first order of approximation gave circular or elliptical orbits. The solution for higher orders is facilitated by neglecting surface tension and assuming $h = \infty$, for then $\varphi^{(2)}$ and $\varphi^{(3)}$ both vanish. From (27.8) one finds easily the orbit to the second order:

$$\left.\begin{aligned} x(t) &= x_0 - A\,e^{m y_0}\sin m(x_0 - c_0\,t) + A^2 m^2 c_0\,e^{2m y_0}\,t, \\ y(t) &= y_0 + A\,e^{m y_0}\cos m(x_0 - c_0\,t). \end{aligned}\right\} \tag{27.39}$$

The circular orbits of first-order theory are now modified by a general drift in the direction of wave motion. The total amount of fluid transported per unit time (and width) is $\frac{1}{2}A^2 m c_0$. As the formula shows, this additional flow is concentrated chiefly near the surface.

When the depth is finite, or when surface tension is taken into account, the orbits become more complicated. Let

$$K = \frac{(g + 3\,T'\,m^2)\coth m\,h - (g + T'\,m^2)\tanh m\,h}{(g + T'\,m^2)\tanh^2 m\,h - 3\,m^2\,T'}. \tag{27.40}$$

The particle orbits, accurate to the second order, are as follows:

$$\left.\begin{aligned} x(t) &= x_0 - A\,\frac{\cosh m\,(y_0 + h)}{\sinh m\,h}\sin m\,(x_0 - c_0\,t) + \frac{1}{2}\,A^2\,m^2\,c_0\,t\,\frac{\cosh 2m\,(y_0 + h)}{\sinh^2 m\,h} + \\ &\quad + \frac{1}{4}\,A^2\,m\left[\operatorname{cosech}^2 m\,h - 3\,K\,\frac{\cosh 2m\,(y_0 + h)}{\sinh 2m\,h}\right]\sin 2m\,(x_0 - c_0\,t), \\ y(t) &= y_0 + A\,\frac{\sinh m\,(y_0 + h)}{\sinh m\,h}\cos m\,(x_0 - c_0\,t) + \\ &\quad + \frac{3}{4}\,A^2\,m\,K\,\frac{\sinh 2m\,(y_0 + h)}{\sinh 2m\,h}\cos 2m\,(x_0 - c_0\,t). \end{aligned}\right\} \tag{27.41}$$

The mass-transport term in $x(t)$ is still present, and in fact, persists to the very bottom. The elliptical orbits of the first-order theory are now modified not only by the forward drift at all levels, but also by another superposed cyclic motion of twice the frequency. The effect of this is to make the orbits approximately epitrochoidal (neglecting for a moment the drift) with a small hump at the bottom which in extreme cases can become a cusp or a loop. This behavior has, in fact, been observed by MORISON and CROOKE (1953). For capillary waves the situation is reversed and a dimple appears at the top.

The existence of mass transport will be reconsidered in the last chapter, where it will be demonstrated that it is a general consequence of irrotational motion when the exact boundary conditions are satisfied. The theoretically predicted monotonically decreasing forward drift with increasing depth is not confirmed experimentally for small values of $m h$, say $m h < 2$. Instead, with respect to a coordinate system moving with the mean velocity of the fluid, there is an observed forward flow near the bottom and top and a backward flow in the middle portions. It is not surprising that the perfect-fluid model does not give a good prediction for small $m h$, for the high shear rate near the bottom indicates that viscosity should not be neglected. LONGUET-HIGGINS (1953b) has, in fact, devoted a long monograph to development of the higher-order theory of waves in a viscous fluid and finds theoretical drift curves agreeing qualitatively with observed ones. We shall not carry through the details here and refer to LONGUET-HIGGINS' paper.

Wave energy. One of the striking facts about progressive first-order pure gravity waves is that the kinetic and potential energy per wave length are equal (see Sect. 15β). This equal division of energy no longer holds when higher-order terms are taken into account. It is particularly easy to show this for $h = \infty$, for then we may use (27.25) and (27.27). The average potential energy in a wavelength is

$$\mathscr{V}_{\mathrm{av}} = \frac{m}{2\pi}\int_0^{2\pi/m} dx \int_0^{\eta} \varrho\,g\,y\,dy = \frac{m}{2\pi}\int_0^{2\pi/m} \frac{1}{2}\,\varrho\,g\,\eta^2\,dx = \frac{1}{4}\,\varrho\,g\,A^2\left[1 + \frac{1}{2}\,A^2\,m^2\right]. \tag{27.42}$$

The average kinetic energy is

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$$\left.\begin{aligned} \mathscr{T}_{\mathrm{av}} &= \frac{m}{2\pi}\int_0^{2\pi/m} dx \int_0^{\infty} \frac{1}{2}\,\varrho\,(\varphi_x^2 + \varphi_y^2)\,dy = \frac{m}{2\pi}\int_0^{2\pi/m} \frac{1}{4}\,\varrho\,A^2\,c_0^2\,m\,e^{2m\eta}\,dx \\ &= \frac{1}{4}\,\varrho\,A^2\,g\,[1 + A^2\,m^2]. \end{aligned}\right\} \tag{27.43}$$

Composite waves. Previously in this section we have been discussing a wave of permanent type whose prototype is the first-order progressive wave of the form $\eta = A \cos m(x - ct)$. It is natural to inquire into the behavior of higher-order waves whose first-order prototype is composite, say

$$\eta = A_1 \cos m_1 (x - c_1 t) + A_2 \cos m_2 (x - c_2 t). \qquad (27.44)$$

To find the corresponding second-order terms one may use Eqs. (10.11) and (10.12); the computations are tedious but not difficult. The third order would introduce modifications of both c_1 and c_2 and lead to a much longer computation. As might be expected in analogy with the theory of sound, the second-order terms introduce waves of wave numbers $m_1 - m_2$ and $m_1 + m_2$, as well as $2m_1$ and $2m_2$. The velocity potential to the second order is given by

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$$\left.\begin{aligned}\Phi = {} & A_1 c_1 e^{m_1 y} \sin m_1 (x - c_1 t) + A_2 c_2 e^{m_2 y} \sin m_2 (x - c_2 t) + \\ & + 2 A_1 A_2 \frac{m_1 m_2 (c_1 - c_2) g}{g (m_1 - m_2) - (m_1 c_1 - m_2 c_2)^2} e^{(m_1 - m_2) y} \sin [(m_1 - m_2) x - (m_1 c_1 - m_2 c_2) t].\end{aligned}\right\} \quad (27.4$$

The profile is then computed from BERNOULLI's law

$$\eta = -\frac{1}{g}\left[\Phi_t(x, \eta, t) + \frac{1}{2}(\Phi_x^2 + \Phi_y^2)\right]$$

with retention of only terms of first or second order [cf. Eqs. (10.9) and (10.11)]. We omit the rather long expression.

BIESEL (1952) has derived formulas for a composite wave with a finite number of components and for h finite. He computes a number of quantities of interest. However, the formulas are very long and will not be reproduced here.

Three-dimensional waves. By using the full three-dimensional equations as given in (27.5) to (27.7) one may develop a higher-order theory of doubly modulated waves analogous to those considered in Sect. 14γ by first-order theory. This has been done by FUCHS (1952) and SRETENSKII (1954) to whose papers we refer for the resulting motion.

Further references. Development of systematic methods of computation of higher-order approximations has recently attracted the attention of several persons. Among these are SRETENSKII (1952), BORGMAN and CHAPPELEAR (1957), DAUBERT (1957, 1958) in a series of notes, JOLAS (1958) and NORMANDIN (1957). SRETENSKII (1953, 1955) has investigated the higher-order theory of wave motion resulting from a moving pressure distribution and waves in a circular canal.

β) *Standing waves.* As will be evident below, the formulation of a higher-order theory of standing waves is somewhat clumsier than that for progressive waves of permanent type. Part of the difficulty stems from the fact that one necessarily must deal with one more variable, namely t. The type of motion we are seeking will be represented by a profile $\eta(x, t)$ and a velocity potential $\Phi(x, y, t)$ periodic in both x and t:

$$\eta(x + r\lambda, t + s\tau) = \eta(x, t), \quad \Phi(x + r\lambda, y, t + s\tau) = \Phi(x, y, t). \qquad (27.46)$$

If we fix the wave length $\lambda = 2\pi/m$, then the period $\tau = 2\pi/\sigma$ will have to be determined as one of the unknowns of the problem. In addition, we wish to have the first-order standing wave $\eta = A \cos mx \cos \sigma t$ of Sect. 14α serve as a prototype and first-order solution of the more general problem. As a further condition, we shall suppose the motion to be symmetric with respect to a vertical line through a crest.

RAYLEIGH (1915) was apparently the first to consider this problem. It was later attacked in an entirely different way, using Lagrangian coordinates, by SEKERZH-ZENKOVICH (1947, 1951a, b, 1952), who treated both two- and three-dimensional waves for infinite depth, two-dimensional waves for finite depth, and composite waves for infinite depth. PENNEY and PRICE (1952), following approximately RAYLEIGH's method, carried the approximation for two-dimensional motion and $h=\infty$ to the fifth order, and to the second order for h finite and for doubly modulated standing waves. The method used below is a modification of theirs. The two-dimensional problem has recently been studied in a series of notes by CHABERT D'HIÈRES (1957, 1958). CARRY (1953) has carried to the second-order the superposition of two standing waves of the same wave length but 90° out of phase and of differing first-order amplitudes. INGRAHAM (1954) has carried to the second order the stability analysis of superposed two-fluid systems discussed at the beginning of Sect. 26α.

Since η and Φ are periodic in both x and t, we may expand each in a double Fourier series. However, it is also necessary to bring into the form of the series some indications of orders of magnitudes of the components, and in such a way that the first-order term is of the desired sort. We assume the following expansions for an infinitely deep fluid:

$$\left.\begin{aligned}\sigma &= \sigma_0+\varepsilon\sigma_1+\varepsilon^2\sigma_2+\cdots,\\ \eta(x,t) &= \sum_{r=1}^{\infty}\varepsilon^r\eta^{(r)}=\sum_{r=1}^{\infty}\varepsilon^r\sum_{p,q=0}^{r}[a_{pq}^{(r)}\cos q\sigma t+b_{pq}^{(r)}\sin q\sigma t]\cos pmx,\\ \Phi(x,y,t) &= \sum_{r=1}^{\infty}\varepsilon^r\Phi^{(r)}=\sum_{r=1}^{\infty}\varepsilon^r\sum_{p,q=1}^{r}[c_{pq}^{(r)}\cos q\sigma t+d_{pq}^{(r)}\sin q\sigma t]\,e^{pmy}\cos pmx.\end{aligned}\right\}\tag{27.47}$$

We may immediately set $d_{p0}^{(r)}=0$, $b_{p0}^{(r)}=0$ and with no loss of generality also $c_{00}^{(r)}=0$. Since the mean water level has been fixed at $y=0$, we must also have $a_{00}^{(r)}=0$. We shall again take $\varepsilon=Am$, where A is the amplitude of the first-order term.

Substitution of (27.47) into the exact kinematic and dynamic boundary conditions,

$$\left.\begin{aligned}&\Phi_x(x,\eta,t)\,\eta_x-\Phi_y+\eta_t=0\\ &\Phi_t+\tfrac{1}{2}(\Phi_x^2+\Phi_y^2)+g\eta-T'(R_1^{-1}+R_2^{-1})=0,\end{aligned}\right\}\tag{27.48}$$

results, as in Sects. 10α and 27α, in a series of equations for successive determination of the coefficients $a_{pq}^{(r)},\ldots,d_{pq}^{(r)}$ and $\sigma_0,\sigma_1,\ldots$. Because of the assumed form of the solution, the equations are now always linear equations between the coefficients. The boundary conditions for $\Phi^{(1)}$ and $\eta^{(1)}$, namely,

$$\left.\begin{aligned}&\Phi_y^{(1)}-\eta_t^{(1)}=0,\\ &\Phi_t^{(1)}+g\eta^{(1)}-T'\eta_{xx}^{(1)}=0,\end{aligned}\right\}\tag{27.49}$$

yield

$$\left.\begin{aligned}&-\sigma_0a_{01}^{(1)}\sin\sigma_0t+\sigma_0b_{01}^{(1)}\cos\sigma_0t=0,\\ &-\sigma_0a_{11}^{(1)}\sin\sigma_0t+\sigma_0b_{11}^{(1)}\cos\sigma_0t-k\,[c_{10}^{(1)}+c_{11}^{(1)}\cos\sigma_0t+d_{11}^{(1)}\sin\sigma_0t]=0;\end{aligned}\right\}\tag{27.50}$$

$$\left.\begin{aligned}&g\,[a_{01}^{(1)}\cos\sigma_0t+b_{01}^{(1)}\sin\sigma_0t]+[-\sigma_0c_{01}^{(1)}\sin\sigma_0t+\sigma_0d_{01}^{(1)}\cos\sigma_0t]=0,\\ &(g+m^2T')\,[a_{10}^{(1)}+a_{11}^{(1)}\cos\sigma_0t+b_{11}^{(1)}\sin\sigma_0t]+\\ &\qquad+[-\sigma_0c_{11}^{(1)}\sin\sigma_0t+\sigma_0d_{11}^{(1)}\cos\sigma_0t]=0.\end{aligned}\right\}\tag{27.51}$$

From these follow immediately

$$a_{01}^{(1)} = b_{01}^{(1)} = c_{10}^{(1)} = a_{10}^{(1)} = 0, \qquad d_{11}^{(1)} = -\frac{\sigma_0}{m} a_{11}^{(1)}, \qquad c_{11}^{(1)} = -\frac{\sigma_0}{m} b_{11}^{(1)}, \tag{27.52}$$

and

$$\sigma_0^2 = g\,m + m^3\,T'. \tag{27.53}$$

We shall in addition fix the phase by making the arbitrary choice

$$a_{11}^{(1)} = \frac{1}{m}, \qquad b_{11}^{(1)} = 0, \tag{27.54}$$

so that

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$$\eta^{(1)} = \frac{1}{m} \cos m\,x \cos \sigma_0\,t, \qquad \Phi^{(1)} = -\frac{c_0}{m^2} \cos m\,x \sin \sigma_0\,t. \tag{27.55}$$

This is a rather clumsy way to derive a first-order solution which was found much more directly earlier in Sect. 14α. However, it provides a caricature of the procedure necessary at each new stage of approximation. Since the higher-order approximations lead to extremely tedious calculations, they will be completely omitted and only the results given.

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The profile and velocity potential through the second order are given by

$$\left.\begin{aligned}
\eta &= A \cos\sigma_0\,t \cos m\,x + \frac{1}{4} A^2\,m \frac{g + m^2\,T'}{g + 4m^2\,T'} \cos 2m\,x + \\
&\quad + \frac{1}{4} A^2\,m \frac{g + m^2\,T'}{g - 2m^2\,T'} \cos 2\sigma_0\,t \cos 2m\,x, \\
\Phi &= -A \frac{\sigma_0}{m} \sin\sigma_0\,t\, e^{m y} \cos m\,x + \frac{1}{4} A^2\,\sigma_0 \sin 2\sigma_0\,t - \\
&\quad - \frac{1}{4} A^2\,\sigma_0 \frac{3m^2\,T'}{g - 2m^2\,T'} \sin 2\sigma_0\,t\, e^{2m y} \cos 2m\,y,
\end{aligned}\right\} \tag{27.56}$$

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for $m^2 \neq g/2T'$; here $\sigma_1 = 0$. If $m^2 = g/2T'$, the situation is similar to that discussed in Sect. 27α following (27.13). For this value of m we must start with a first-order solution of the form:

$$\left.\begin{aligned}
\Phi^{(1)} &= -\frac{\sigma_0}{m^2} [\sin\sigma_0\,t\, e^{m y} \cos m\,x + (b_1 \sin 2\sigma_0\,t - b_2 \cos 2\sigma_0\,t)\, e^{2m y} \cos 2m x], \\
\eta^{(1)} &= \frac{1}{m} [\cos\sigma_0\,t \cos m\,x + (b_1 \cos 2\sigma_0\,t + b_2 \sin 2\sigma_0\,t) \cos 2m\,x].
\end{aligned}\right\} \tag{27.57}$$

The values of b_1, b_2 and σ_1 are now determined by the second-order equations and are

$$b_1 = \pm \tfrac{1}{2} \qquad b_2 = 0, \qquad \sigma_1 = \pm \tfrac{1}{8} \sigma_0. \tag{27.58}$$

Thus the first-order profile for $m^2 = g/2T'$ is

$$\eta = A \cos\sigma_0\,t \cos m\,x \pm \tfrac{1}{2} A \cos 2\sigma_0\,t \cos 2m\,x. \tag{27.59}$$

The amplitude relation between the two first-order modes is the same as for progressive waves of this length.

The expression for the third-order standing wave is very clumsy if T' is retained. Also, as might be expected from analogy with the progressive wave, another apparent singularity appears for $m^2 = g/3\,T'$. If one sets $T' = 0$, the expressions for η and Φ become much simpler and are as follows:

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$$\left.\begin{aligned}
\eta &= A\cos\sigma t\cos m x + \tfrac{1}{4}A^2 m\cos 2m x + \tfrac{1}{4}A^2 m\cos 2\sigma t\cos 2m x + \\
&\quad + \tfrac{1}{32}A^3 m^2\,[-2\cos 3\sigma t\cos m x + 9\cos\sigma t\cos 3m x + \\
&\qquad\qquad + 3\cos 3\sigma t\cos 3m x; \\
\Phi &= -\frac{\sigma A}{m}\sin\sigma t\, e^{m y}\cos m x + \tfrac{1}{4}\sigma A^2\sin 2\sigma t + \\
&\quad + \tfrac{5}{32}\sigma m A^3\sin 3\sigma t\, e^{m y}\cos m x + \tfrac{3}{16}\sigma m A^3\sin 3\sigma t\, e^{3m y}\cos 3m x; \\
\sigma &= \sqrt{g m}\left(1 - \tfrac{1}{8}A^2 m^2\right).
\end{aligned}\right\}\qquad(27.60)$$

As has been mentioned earlier, the approximation has been carried to the fifth order by PENNEY and PRICE (1952). However, it is not necessary to carry the approximation so far in order to see some important features of the motion, namely the sharpening of the crests and flattening of the troughs, the absence of any nodal points and the decrease of frequency with amplitude. One interesting feature does require carrying the approximation to at least the fourth order: this is the absence of any time during a period when the surface is completely flat. In connection with an experimental test of a predicted standing wave of greatest amplitude-length ratio by PENNEY and PRICE, G.I. TAYLOR (1953) has also provided an experimental verification of the correctness of the theory in an extreme case.

Orbits. The method of computation of orbits including higher-order terms is the same as that outlined at the end of Sect. 27α and we omit a detailed exposition. For infinite depth and $T' = 0$ the orbits to the second order are given by

$$\left.\begin{aligned}
x &= x_0 - A\, e^{m y_0}\sin m x_0\cos\sigma_0 t, \\
y &= y_0 + A\, e^{m y_0}\cos m x_0\cos\sigma_0 t + \tfrac{1}{4}A^2 m\, e^{2m y_0}\cos 2\sigma_0 t.
\end{aligned}\right\}\qquad(27.61)$$

The effect of the last term in y is easily seen to be a small wiggle superposed on the first-order straight-line trajectories discussed in Sect. 14α, except directly beneath the crests where the trajectory is still vertical but with the midpoint somewhat above the equilibrium position.

Pressure distribution. A particularly interesting consequence of keeping second-order terms appears in the behavior of the pressure distribution. From (27.56) and BERNOULLI's theorem one finds for the average pressure over a wave length at depth y

$$\left.\begin{aligned}
\overline{p - p_0} &= \frac{1}{\lambda}\int_0^\lambda (p - p_0)\,dx = -\varrho g y - \tfrac{1}{4}\varrho A^2\sigma_0^2 e^{2m y} + \\
&\quad + \tfrac{1}{4}\varrho A^2\sigma_0^2 e^{2m y}\cos 2\sigma_0 t - \tfrac{1}{2}\varrho A^2\sigma_0^2\cos 2\sigma_0 t.
\end{aligned}\right\}\qquad(27.62)$$

The terms with e^{2my} as a factor drop off quickly. However, the last term is independent of y and at all depths yields a fluctuation about the hydrostatic pressure with double the frequency of the standing waves. The existence of this depth-independent fluctuation, deriving from the term Φ_t in BERNOULLI's theorem and the purely time-dependent term in Φ, was pointed out by MICHE (1944, p. 73). The matter has been investigated more intensively by LONGUET-HIGGINS (1950) who has extended the theory to include a more general wave motion and compressibility of the fluid. He has further applied the theory to give a plausible explanation of recorded microseisms. KIERSTEAD (1952) has extended LONGUET-HIGGINS' analysis to include two-fluid systems. COOPER and LONGUET-HIGGINS (1951) have carried out laboratory experiments showing excellent agreement with the predicted pressure distribution for both progressive and standing waves.

Finite depth. Computations of the surface profile, particle orbits and other quantities for finite depth have been carried to the third order by SEKERZH-ZENKOVICH (1951) and CARRY and CHABERT D'HIÈRES (1957). We reproduce here the results only to the second order (for pure gravity waves):

$$\left.\begin{aligned} \eta &= A\cos\sigma t\cos m x + \tfrac{1}{8}A^2 m\tanh m h \times \\ &\quad\times\left[1+\coth^2 m h - \coth^2 m h\,(3\coth^2 m h - 1)\cos 2\sigma t\right]\cos 2 m x; \\ \Phi &= -A\frac{\sigma}{m}\frac{\cosh m(y+h)}{\sinh m h}\sin\sigma t\cos m x + \frac{1}{16}A^2\sigma(3+\coth^2 m h)\sin 2\sigma t + \\ &\quad + \frac{3}{8}A^2\sigma\frac{\coth m h}{\sinh^2 m h}\frac{\cosh 2m(y+h)}{\sinh 2 m h}\sin 2\sigma t\cos 2 m x, \\ \sigma^2 &= \sigma_0^2 = g m\tanh m h, \qquad \sigma_1 = 0. \end{aligned}\right\} \quad (27.63)$$

The pressure averaged over a wave length [cf. (27.62)] is

$$\left.\begin{aligned} \overline{p-p_0} = -\varrho g y + \frac{1}{8}\frac{A^2\sigma^2}{\sinh^2 m h}\left[1-\cosh 2m(y+h) - \right. \\ \left. -(2\cosh 2 m h - \cosh 2m(y+h) - 1)\cos 2\sigma t\right]. \end{aligned}\right\} \quad (27.64)$$

On the bottom, $y=-h$, one finds

$$\overline{p-p_0} = \varrho g h - \tfrac{1}{2}\varrho A^2\sigma^2\cos 2\sigma t. \quad (27.65)$$

We note that here also, as in the case of progressive waves, the importance of the second-order terms in η and Φ increases as $m h\to 0$.

γ) Waves in a viscous fluid. The Eqs. (10.2) to (10.4), used in Sect. 25 in developing the first-order theory of waves in a viscous fluid, may be considered as the first in a sequence for the determination of higher-order approximations. Although the formulation of the equations appears to be straight forward, if laborious, the higher-order theory does not seem to have attracted many investigators. HARRISON (1909) made a second-order investigation of progressive waves and LONGUET-HIGGINS (1953) has recently made an elaborate study of both progressive and standing waves in an attempt to explain certain observed features of mass transport velocities. We shall not attempt to summarize either paper. However, the following results, taken from HARRISON, may be of interest. For the wave profile to the second order he gives the following expression when ν is small [cf. Eq. (25.22)]:

$$\left.\begin{aligned} \eta &= A\,e^{-2\nu m^2 t}\cos(m x - \sigma_0 t) + \\ &\quad + A^2 e^{-4\nu m^2 t}\left[\frac{1}{2}m\cos 2(m x-\sigma_0 t) - m^2\left(\frac{\nu^2}{4 g m}\right)^{\frac{1}{4}}\sin 2(m x - \sigma_0 t)\right], \end{aligned}\right\} \quad (27.66)$$

where $\sigma_0^2 = g m$. The effect of viscosity, besides damping, is to make the leading side of the crest steeper than the trailing side. According to HARRISON the average horizontal velocity of a particle, again for small ν, is

$$\left.\begin{aligned} & A^2\sigma_0 m\,e^{2 m y - 4\nu m^2 t} - A^2 m^2\sqrt{\tfrac{1}{2}\sigma_0\nu}\times \\ &\quad\times\left[(4\cos l_2 y + \sin l_2 y)\,e^{(m+l_1)y} + \sin 2 m y\right]e^{-4\nu m^2 t} + \\ &\quad + A^2 m^3\nu\left[4\,e^{(m+l_1)y}\sin l_2 y + 3\,e^{2 l_1 y}\right]e^{-4\nu m^2 t}, \end{aligned}\right\} \quad (27.67)$$

where, as in (25.19), $l = l_1 + i l_2$ and $\nu(l^2 - m^2) = \omega \approx -2\nu m^2 + i\sigma_0$. This formula should be compared with $A^2 m^2 c_0 e^{2my}$ computed from (27.39), to which it reduces when $\nu = 0$.

E. Shallow-water waves

This chapter will deal with special solutions based on the shallow-water approximation, following the method of FRIEDRICHS (1948) as presented in subsection 10β. The shallow-water approximation for the waves over a rigid bottom yields a set of nonlinear equations [cf. (10.32)] even in the first approximation. If these equations are then linearized, they result in a hyperbolic-type equation which reduces to the simple wave equation for a flat horizontal bottom. Consequently, the solutions resulting from the shallow-water approximation are completely different in character from those resulting from the infinitesimal-wave approximation of subsection 10α and Chap. D, which resulted in linear equations and linear boundary conditions. That is, the shallow-water approximation leads to nonlinear hyperbolic-type equations, whereas the infinitesimal-wave approximation leads to a set of linear equations satisfying the boundary conditions and having each successive approximation to the velocity potential satisfy the simplest elliptic equation, namely the Laplace equation.

After the first-order shallow-water approximation (10.32) has been applied to several problems, the method of FRIEDRICHS (1948) and KELLER (1948) will be extended to obtain the second and third approximations of the shallow-water theory and thereby present, for the first time, the exact second approximation to the cnoidal wave of KORTEWEG and DE VRIES (1895), and the solitary wave of BOUSSINESQ (1871), and RAYLEIGH (1876). These higher-order approximations lead directly to relations predicting the maximum heights of cnoidal waves and solitary waves.

28. The fundamental equations for the first approximation. The shallow-water expansion method introduced by FRIEDRICHS (1948) is discussed in Sect. 10. For this application the expansion parameter ε was selected so that the first approximation would be identical to the nonlinear equations of the classical shallow-water theory, which is based on the assumption of hydrostatic pressure variation throughout and neglect of the variation of the horizontal velocity components with depth, so that the complicated boundary-value problem is greatly simplified to the following nonlinear equations:

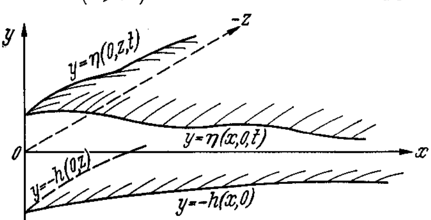

Fig. 36.

$$\left.\begin{aligned} &u_t + u\,u_x + w\,u_z = -g\,\eta_x,\\ &w_t + u\,w_x + w\,w_z = -g\,\eta_z,\\ &\eta_t + [u(\eta + h)]_x + [w(\eta + h)]_z = 0 \end{aligned}\right\} \tag{28.1}$$

[see LAMB (1932, p. 254) or STOKER (1957, p. 23)]. The coordinates and notation are shown in Fig. 36.

The set of nonlinear equations (28.1) is identical to (10.32) and is the first approximation in FRIEDRICHS' (1948) shallow-water expansion method as discussed in Sect. 10; this provides some mathematical justification for these classical equations. It is evident that the higher-order approximations following (10.23) and (10.33) also require that ε be sufficiently small; consequently, as will be shown, this expansion method is applicable if the product of water depth and surface curvature is small. Therefore, in some cases, this shallow-water theory is applicable to extremely large water depths as long as the wave length is sufficiently long, the most common application being to tidal waves, that is, the oceanic tides produced by the gravitational action of the sun and the moon [see, e.g., LAMB (1932) or DEFANT'S article in Vol. XLVIII of this Encyclopedia].

The mathematical justification for this shallow-water expansion method, at least for special cases, lies in the existence proof of FRIEDRICHS and HYERS (1954) for the solitary wave, and the existence proof of LITTMAN (1957) for the more general cnoidal waves. Both of these proofs demonstrate that this expansion method converges to the exact solutions for these particular problems. see errata

The nonlinear first approximation given by (28.1) is considerably simplified if the rigid bottom surface $h(x, z)$ is flat and horizontal, as may be seen by letting $h = \text{const}$ so that (28.1) may be written as

$$\left.\begin{aligned} u_t + u u_x + w u_z &= -g(\eta + h)_x, \\ w_t + u w_x + w w_z &= -g(\eta + h)_z, \\ (\eta + h)_t + [u(\eta + h)]_x + [w(\eta + h)]_z &= 0. \end{aligned}\right\} \tag{28.2}$$

This is identical to the well known two-dimensional gas-dynamics equation [see, e.g., LAMB (1932)] if we write

$$\left.\begin{aligned} \varrho(x, z, t) &= [\eta(x, z, t) + h], \\ \frac{\gamma p}{\varrho^2} &= \frac{c^2}{\varrho} = \frac{c^2}{\eta + h} = g = \text{const}. \end{aligned}\right\} \tag{28.3}$$

Since the isentropic gas relationship is $p = \text{const} \times \varrho^{\gamma}$, the first-order nonlinear shallow-water approximation for a flat horizontal bottom is identical to the isentropic two-dimensional gas flow having a specific heat ratio of $\gamma = 2$. This is the basis of the so called hydraulic analogy which has been used for many experimental investigations [see, e.g., STOKER (1957)].

It must be noted, however, that this hydraulic analogy is only valid for a flat horizontal bottom, as may be seen by comparing (28.1) and (28.2), and even more important, it is valid only as a first approximation even for the nonlinear case. It will be shown in Sect. 31 that the second approximation to shallow-water theory yields finite-amplitude waves (the solitary wave or cnoidal waves) which can be propagated without a change in shape or form, a fact which completely invalidates the hydraulic analogy to compressible gas flow since (28.2), or the gas dynamics equation, predicts that any finite disturbance quickly forms a finite discontinuity, e.g. [see, e.g., LAMB (1932), pp. 278, 481)].

In Sect. 29, immediately following, it will be shown that even for the linearized first approximation the hydraulic analogy to compressible gas flow is limited to a flat horizontal bottom.

29. The linearized shallow-water theory. The first approximation to shallow-water theory can now be linearized by two different methods, each suitable for various problems. We shall assume that $u_z = w_x$, so that a velocity potential $\Phi(x, z, t)$ exists. The first method is more appropriate for investigating steady water flow in canals or rivers and consists of the following approximations for carrying out the linearization:

$$u(x, z) = U + \varphi_x \approx U, \qquad w(x, z) = \varphi_z \ll U, \tag{29.1}$$

$$\eta(x, z) \ll h(x, z), \tag{29.2}$$

so that (28.1) is linearized to

$$\left(1-\frac{U^2}{c^2}\right)\varphi_{xx}+\varphi_{zz}+\frac{(U+\varphi_x)\,h_x}{h}+\frac{\varphi_z h_z}{h}=0, \tag{29.3}$$

$$c^2=g\,h(x,z). \tag{29.4}$$

In agreement with the previous discussion, (29.3) corresponds to the linearized gas dynamics equation only if the bottom is flat and horizontal, i.e. if h is constant.

The second method of linearization corresponds to the classical tidal-wave theory, or long-wave theory [see, e.g., LAMB (1932, p. 254) or Eqs. (10.36)] and can be obtained by writing

$$u(x,z,t)=\Phi_x\ll 1,\qquad w(x,z,t)=\Phi_z\ll 1, \tag{29.5}$$

$$\eta(x,z,t)\ll h(x,z), \tag{29.6}$$

so that (28.1) is linearized to

$$(\Phi_{xx}+\Phi_{zz})+\left(\frac{h_x}{h}\Phi_x+\frac{h_z}{h}\Phi_z\right)=\frac{1}{g\,h}\Phi_{tt}. \tag{29.7}$$

Again, as before, (29.7) corresponds to the linearized gas dynamic case, or the simple acoustic wave-propagation equation, only if the bottom is flat and horizontal. In this case the general solution of (29.7) for one-dimensional flow is the well known d'Alembert solution of the simple wave equation,

$$\Phi(x,t)=F(x-c\,t)+f(x+c\,t)\qquad c=\sqrt{g\,h}=\text{const}, \tag{29.8}$$

which is used to study long-wave-length oscillations in canals when the water is either at rest or moving with a velocity $U\ll c$. The limitation to small perturbations and constant h for one-dimensional flow allows (28.1) to be linearized to

$$\left.\begin{aligned}\Phi_{xt}&=u_t=-g\,\eta_x,\\ \eta_{xx}&=\frac{1}{gh}\eta_{tt}=\frac{1}{c^2}\eta_{tt};\end{aligned}\right\} \tag{29.9}$$

various applications of this, including the canal theory of tides, are given in LAMB (1932, pp. 254—273) and DEFANT (1957).

For the case of a canal having a non-rectangular but constant cross-section, we may generalize (29.9) by defining the mean depth h as the undisturbed cross-sectional area S divided by the width b of the canal at the undisturbed free water surface [see LAMB (1932), p. 256]. When the canal has a variable depth $h(x)$ and the disturbance may be considered one-dimensional, then (29.7) may be written in terms of the varying cross-sectional area $S(x)$ for constant width b as follows

$$\frac{1}{g\,h}\Phi_{tt}=\frac{1}{h}(h\,\Phi_x)_x=\frac{1}{S}(S\,\Phi_x)_x,\qquad S(x)=b\,h(x),\qquad b=\text{const}. \tag{29.10}$$

Then from (29.9) we obtain

$$\frac{1}{S}(S\,\eta_x)_x=\frac{1}{g\,h}\eta_{tt}, \tag{29.11}$$

which is the same as the expression derived by GREEN (1838) for a canal that is varying in both width b and depth h so that

$$S(x)=h(x)\,b(x).$$

However, the exact linearized first order approximation is (29.7), and the form of this equation indicates that large values of $b'(x)$ would invalidate the one-dimensional assumption, especially if h_z is relatively large. This is also indicated by LAMB (1932, p. 274). However, (29.7) provides the rigorous proof that (29.10) is applicable to one-dimensional, long-wavelength, small-amplitude disturbances in a canal of rectangular cross-section having a constant width and a varying depth.

If we now limit our analysis to long wave lengths having a simple harmonic oscillation of frequency $\sigma/2\pi$, so that we may write

$$\eta(x,t) = \eta(x)\sin(\sigma t + \tau), \qquad \Phi(x,t) = \varphi(x)\cos(\sigma t + \tau),$$

Eqs. (29.10) and (29.11) reduce to

$$\frac{1}{S}(S\varphi_x)_x + \frac{\sigma^2}{g h}\varphi = 0, \qquad \frac{1}{S}(S\eta_x)_x + \frac{\sigma^2}{g h}\eta = 0. \tag{29.12}$$

If we solve these equations in order to determine the harmonic oscillations in long canals with various special choices of varying cross-section, boundary conditions at the ends of the canal or finiteness conditions may further limit the allowable values of the frequency to a sequence of eigenvalues or fundamental frequencies, $\sigma_1, \sigma_2, \ldots$. Associated with each σ_i there is an η_i and Φ_i. The general solution of the Eqs. (29.12) is then a superposition of these characteristic solutions,

$$\eta(x,t) = \sum A_n \eta_n(x)\sin(\sigma_n t + \tau_n), \qquad \Phi(x,t) = \sum A_n \varphi_n(x)\cos(\sigma_n t + \tau_n),$$

where A_n and τ_n are arbitrary. Emphasis, however, is usually upon finding the fundamental mode σ_0, η_0 and the first few higher modes. We consider two special problems in order to illustrate the procedure. Other more complex situations are analyzed in LAMB (1932, p. 275 ff.) or in DEFANT (1957).

Let the canal be of rectangular cross-section with $h = h_0$, $b = \beta x$. We shall suppose it to be bounded at the ends by vertical walls at $x = x_1 > 0$, $x = x_2 > x_1$. The Eq. (29.12) for φ now becomes

$$\varphi_{xx} + \frac{1}{x}\varphi_x + \frac{\sigma^2}{g h}\varphi = 0,$$

BESSEL's equation of order zero. The general solution is of the form

$$c\, J_0(\sigma x/c) + D\, Y_0(\sigma x/c), \qquad c^2 = g h.$$

The boundary conditions at the ends, $\phi'(x_1) = \phi'(x_2) = 0$, can be satisfied only if

$$J_1(\sigma x_1/c)\, Y_1(\sigma x_2/c) - J_1(\sigma x_2/c)\, Y_1(\sigma x_1/c) = 0.$$

This equation determines the eigenvalues $\sigma_1, \sigma_2, \ldots$. The various modes of motion are then of the form

$$\left.\begin{aligned} \Phi_n = A_n\left[Y_1(\sigma_n x_2/c)\, J_0(\sigma_n x/c) - J_1(\sigma_n x_2/c)\, Y_0(\sigma_n x/c)\right]\cos(\sigma_n t + \tau_n),\\ n = 1, 2, \ldots\end{aligned}\right\} \tag{29.13}$$

If $x_1 = 0$, the solution Y_0 must be excluded because of its singularity at the origin and the eigenvalues are determined simply from $J_1(\sigma_n x_2/c) = 0$, $n = 1, 2, \ldots$.

A solvable case in which h is variable is the canal of rectangular cross-section with $b = b_0$ and

$$h(x) = h_0\left(1 - \frac{x^2}{L^2}\right).$$

Eq. (29.12) now becomes

$$\left(1-\frac{x^2}{L^2}\right)\varphi_{xx}-2\frac{x}{L^2}\varphi_x+\frac{\sigma^2}{g\,h_0}\varphi=0, \tag{29.14}$$

the equation for the spherical harmonics $P_\nu(x/L)$, $Q_\nu(x/L)$ with $(\sigma L)^2/g h_0=\nu(\nu+1)$. The condition that the solution should be finite on $|x|\leqq L$ requires one to discard Q_ν and further restricts ν to integers, thus determining the fundamental frequencies:

$$\sigma_n^2=\frac{g\,h_0}{L^2}\,n(n+1).$$

The fundamental solutions are then formed with Legendre polynomials:

$$\Phi_n=A_n P_n(x/L)\cos(\sigma_n t+\tau_n). \tag{29.15}$$

Motions of the type considered above may be identified with the long period oscillations called seiches which occur in certain lakes or canals throughout the world. Many applications are presented by CHRYSTAL (1905, 1906) and the periods observed in several lochs and lakes seem to correspond to those calculated by the linear shallow-water theory. The linear shallow-water equation (29.7) should be very suitable for the study of seiches because of their long period and relatively small amplitude. Usually the complete Eq. (29.7) must be solved numerically by the method of finite differences because the contour of the body of water is quite irregular and the depth variation is important.

When the motion cannot be considered one-dimensional, one must use the complete two-dimensional equations (29.7). If the motion is harmonic with frequency $\sigma/2\pi$, so that

$$\eta(x,z,t)=\eta(x,z)\sin(\sigma t+\tau),\qquad \Phi(x,z,t)=\varphi(x,z)\cos(\sigma t+\tau),$$

then the right-hand side of (29.7) is replaced by $-(\sigma^2/g h)\Phi$. However, just as in the one-dimensional case, the allowable values of σ may be restricted by the boundary conditions or finiteness conditions to a sequence of eigenvalues σ_1, $\sigma_2, \ldots$ with associated functions $\eta_1, \eta_2, \ldots, \Phi_1, \Phi_2, \ldots$. The general solution is again a superposition. We illustrate with several typical examples, but refer again to LAMB (1932, p. 282ff.) or DEFANT (1957) for a more comprehensive treatment.

Consider first a rectangular basin of constant depth h bounded by $x=0$, $x=x_0$, $z=0$, $z=z_0$. Then (29.7) becomes

$$\Phi_{xx}+\Phi_{zz}+\frac{\sigma^2}{g\,h}\Phi=0$$

and the boundary conditions are

$$\Phi_x(0,z,t)=\Phi_x(x_0,z,t)=\Phi_z(x,0,t)=\Phi_z(x,z_0,t)=0.$$

It is easy to verify that the fundamental solutions are

$$\Phi_{mn}=A_{mn}\cos\frac{m\pi x}{x_0}\cos\frac{n\pi z}{z_0}\cos(\sigma_{mn}t+\tau_{mn}) \tag{29.16}$$

where the eigenvalues σ_{mn} are given by

$$\sigma_{mn}=\pi\sqrt{g\,h}\,\sqrt{\left(\frac{m}{x_0}\right)^2+\left(\frac{n}{z_0}\right)^2}.$$

The result should be compared with (23.14) which reduces to this when $m_0 h$ is small enough so that $\tanh m_0 h \cong m_0 h$.

As another example consider a basin of circular planform of radius a and depth h. In polar coordinates, $x = r\cos\vartheta$, $y = r\sin\vartheta$, Eq. (29.7) becomes

$$\Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\vartheta\vartheta} + \frac{\sigma^2}{g h}\Phi = 0$$

and Φ must satisfy $\Phi_r(a, \vartheta, t) = 0$. The fundamental solutions are easily found by separation of variables to be

$$\Phi_{mn} = A_{mn} J_n(\sigma_{mn} r/c)\cos(n\vartheta + \delta_{mn})\cos(\sigma_{mn} t + \tau_{mn}), \qquad c^2 = g h, \tag{29.17}$$

where the fundamental frequencies σ_{mn} are the roots of

$$J_n'(\sigma_{mn} a/c) = 0, \qquad m = 1, 2, \ldots.$$

The solution (23.15) again reduces to this if $\tanh m_0 h \cong m_0 h$.

If the planform is ring-shaped with the rings having radii a and $b < a$, then one needs also the solution Y_n in order to satisfy the boundary condition on $r = b$. (The singularity of Y_n at the origin obviously causes no difficulty, for it is not in the fluid). The fundamental solutions now become

$$\left.\begin{aligned}\Phi_{mn} = A_{mn}\left[Y_n'\left(\frac{\sigma_{mn}}{c} b\right) J_n\left(\frac{\sigma_{mn}}{c} r\right) - J_n'\left(\frac{\sigma_{mn}}{c} b\right) Y_n\left(\frac{\sigma_{mn}}{c} r\right)\right] \times \\ \times \cos(n\vartheta + \delta_{mn})\cos(\sigma_{mn} t + \tau_{mn}),\end{aligned}\right\} \tag{29.18}$$

where the fundamental frequencies σ_{mn} are determined from the equation

$$J_n'\left(\frac{\sigma a}{c}\right) Y_n'\left(\frac{\sigma b}{c}\right) - J_n'\left(\frac{\sigma b}{c}\right) Y_n'\left(\frac{\sigma a}{c}\right) = 0. \tag{29.19}$$

As before, the solution (23.16) reduces to this one for small $m_0 h$.

As a final example of two-dimensional seiches we consider the long-period simple harmonic oscillation in a shallow circular basin with depth variation depending only on r. Then, in polar coordinates (29.7) becomes

$$\Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\vartheta\vartheta} + \frac{h_r}{h}\Phi_r + \frac{\sigma^2}{g h}\Phi = 0. \tag{29.20}$$

If the depth variation is parabolic,

$$h(r) = h_0\left(1 - \frac{r^2}{a^2}\right),$$

LAMB (1932, p. 291) has shown that the fundamental solutions are given by

$$\Phi_{mn} = A_{mn}\left(\frac{r}{a}\right)^m \cos(m\vartheta + \delta_{mn}) F\left(\alpha, \beta, \gamma; \frac{r^2}{a^2}\right)\cos(\sigma_{mn} t + \tau_{mn}), \tag{29.21}$$

where F is the hypergeometric series

$$F\left(\alpha, \beta, \gamma; \frac{r}{a}\right) = 1 + \frac{\alpha\beta}{1\cdot\gamma}\left(\frac{r}{a}\right)^2 + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}\left(\frac{r}{a}\right)^4 + \cdots,$$

and

$$\alpha = m + n, \qquad \beta = 1 - n, \qquad \gamma = m + 1.$$

The fundamental frequencies σ_{mn} are determined from

$$\frac{\sigma_{mn}^2 a^2}{g h_0} = 2m(2n-1) + 4n(n-1).$$

Both m and n must be integers in the above formulas. They simplify in an obvious fashion for the symmetric mode $m=0$.

It has been pointed out above in connection with several of the examples that the results obtained by analyzing the problem by means of the infinitesimal-wave approximation reduce to those obtained by the linearized shallow-water approximation if $mh=2\pi h/\lambda$ is small enough so that $\tanh mh \cong mh$. One should note that this holds also for the velocity of propagation of a periodic wave:

$$c=\sqrt{\frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}}=\sqrt{gh}\left[1-\frac{1}{6}\left(\frac{2\pi h}{\lambda}\right)^2+\cdots\right]=\sqrt{gh}\left[1+O\left(\frac{h}{\lambda}\right)^2\right]. \quad (29.22)$$

The remainder $O(h/\lambda)^2$ confirms the appropriateness of the term "long-wave approximation" sometimes applied to the shallow-water theory.

This exact agreement of the linearized results in the limiting case is encouraging justification for both the shallow-water approximation and the infinitesimal-wave approximation since they originate not only from different physical considerations, but also by entirely different types of mathematical approximation, as discussed in Sect. 10. The shallow-water approximation leads to hyperbolic type nonlinear equations, while the infinitesimal-wave approximation deals with linear elliptic equations. STOKER (1947, p. 32) gives a detailed comparison of the two linearized approximations for the case of wave motion over a flat bottom at a 6° slope.

α) Linearized shallow-water theory applied to two-dimensional steady flow. The first method of linearizing the shallow-water theory, as given by (29.3), is applicable to the determination of the variation in water depth for the steady flow in a shallow open channel or river. However, in practically all cases (29.3) must be solved numerically, so that it does not entail a prohibitive amount of extra labor to solve directly the more exact original nonlinear first-order equations (28.1) using the methods discussed in the next section (30) on nonlinear first-order theory. As a matter of fact, for supercritical flow, defined by $U>\sqrt{gh}$, the method of characteristics is very easy to use in the numerical solution for a nearly horizontal open channel having a flat bottom and varying width, as shown in Sect. 30. The subcritical case, having a flow velocity everywhere less than $\sqrt{gh}$, can be satisfactorily approximated by the one-dimensional hydraulic theory which assumes that the velocity at each cross-section $S(x)$ is independent of y and z. This method would yield, of course, a constant depth over a given cross-section and would therefore not be satisfactory for predicting the rise in water level about an island, or a jetty, or a pile in a swiftly moving relatively wide stream. For this particular application the linearized form of (29.3) is very useful, especially for subcritical flow, i.e. for $U^2/gh<1$.

We now consider the application of (29.3) to the problem of determining the water depth variation about a two-dimensional cylinder that is perpendicular to the bottom and has a narrow cross-section parallel to the flow as shown in Fig. 37. If the bottom is approximately flat and horizontal everywhere near the vertical cylinder, then we may consider h as constant and, providing that $U^2/gh<1$, write (29.3) as

$$\left.\begin{aligned} \beta^2\varphi_{xx}+\varphi_{zz}=0 \quad \text{or} \quad & \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial(\beta z)^2}=0,\\ \beta^2=1-F^2=1-\frac{U^2}{gh} & =\text{const}>0. \end{aligned}\right\} \quad (29.23)$$

The fundamental solution of (29.23), in view of (29.1), for two-dimensional profiles which may be considered symmetrical about the z-axis as shown in Fig. 37, is

$$\left.\begin{aligned} \varphi(x,z) &= +\frac{1}{2\pi}\int_0^L f(\xi)\ln\sqrt{(x-\xi)^2+(\beta z)^2}\,d\xi,\\ u &= U+\varphi_x = U+\frac{1}{2\pi}\int_0^L \frac{(x-\xi)\,f(\xi)\,d\xi}{(x-\xi)^2+(\beta z)^2},\\ w &= \varphi_z = \frac{\beta^2 z}{2\pi}\int_0^L \frac{f(\xi)\,d\xi}{(x-\xi)^2+(\beta z)^2}. \end{aligned}\right\} \tag{29.24}$$

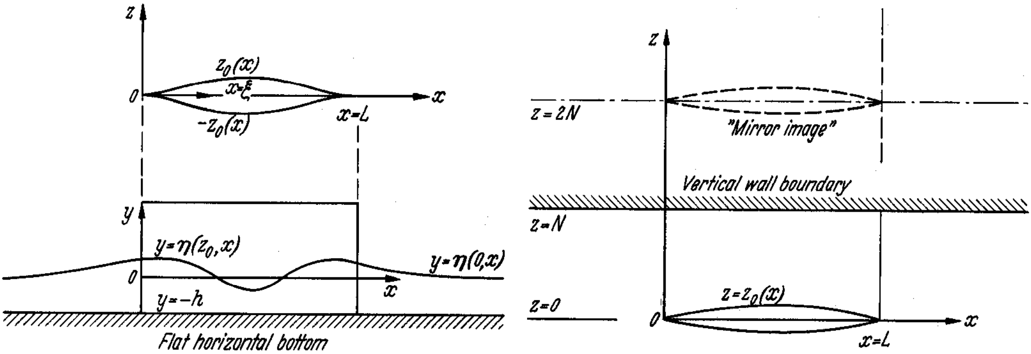

Fig. 37. Shallow-water flow about a two-dimensional symmetrical cylinder perpendicular to the flat horizontal bottom.

The boundary condition for the two-dimensional shape (see Fig. 37) is

$$\frac{dz_0}{dx} = \frac{w}{u} = \frac{\varphi_z(x,z_0)}{U+\varphi_x(x,z_0)} = \frac{\varphi_z}{U}\left[1+O\left(\frac{\varphi_x}{U}\right)\right], \tag{29.25}$$

where the same linearization procedure has been applied to the boundary condition as was used in deriving (29.3). Therefore (29.24) may also be similarly linearized by writing

$$U\frac{dz_0}{dx} = \varphi_z(x,z_0) = \frac{\beta}{2\pi}\int_0^L \frac{f(\xi)\left(\frac{d\xi}{\beta z_0}\right)}{1+\left(\frac{x-\xi}{\beta z_0}\right)^2}.$$

Hence if we let $\frac{(x-\xi)}{\beta z_0}=p$, then for $z_0\geqq 0$,

$$U\frac{dz_0}{dx} = \frac{\beta}{2\pi}\int_{-\frac{L-x}{\beta z_0}}^{\frac{x}{\beta z_0}} \frac{f(x-\beta z_0 p)\,dp}{1+p^2} = \frac{1}{2}\beta f(x) + O(z_0^2).$$

Therefore

$$f(\xi) = \frac{2U}{\beta} z_0'(\xi) + O(z_0^2), \tag{29.26}$$

so that the linearized form of (29.24) is

$$\left.\begin{aligned} \frac{u(x,z)}{U} &= 1+\frac{1}{\pi\beta}\int_0^L \frac{(x-\xi)\,z_0'(\xi)\,d\xi}{(x-\xi)^2+(\beta z)^2} = 1+\frac{\varphi_x}{U},\\ \frac{w(x,z)}{U} &= \frac{z\beta}{\pi}\int_0^L \frac{z_0'(\xi)\,d\xi}{(x-\xi)^2+(\beta z)^2} = \frac{\varphi_z}{U}. \end{aligned}\right\} \tag{29.27}$$

On the actual surface of the two-dimensional profile (29.27) may be further linearized to

$$\left.\begin{aligned}\frac{u(x,z_0)}{U} &= 1+\frac{1}{\pi\beta}\left[\lim_{\varepsilon\to 0}\int\limits_0^{x-\varepsilon}+\int\limits_{x+\varepsilon}^{L}\frac{z_0'(\xi)\,d\xi}{x-\xi}\right] = 1+\frac{1}{\pi\beta}\,\mathrm{PV}\int\limits_0^L\frac{z_0'(\xi)}{x-\xi}\,d\xi,\\ \frac{w(x,z_0)}{U} &= z_0'(x).\end{aligned}\right\}\qquad(29.28)$$

On the other hand, for large values of z we may write

$$\frac{u(x,z)}{U}\approx 1+\frac{1}{\pi\beta^3 z^2}\int\limits_0^L(x-\xi)\,z_0'(\xi)\,d\xi,\qquad \frac{w(x,z)}{U}\approx\frac{1}{\pi\beta z}\int\limits_0^L z_0'(\xi)\,d\xi.\qquad(29.29)$$

The change $\eta(x,z)$ in the original constant water depth h can then be determined by the linearized relations corresponding to (29.1) and (29.2) as

$$\frac{\eta(x,z)}{h}+O\left(\frac{\eta}{h}\right)^2=-F^2\frac{\varphi_x}{U}+O\left(\frac{\varphi_x^2+\varphi_z^2}{U^2}\right),\qquad F^2=\frac{U^2}{g\,h},\qquad(29.30)$$

where for any (x, z) we obtain φ_x and φ_z from (29.27). For example, on the surface of the two-dimensional profile $(z=z_0)$, (29.30) reduces to

$$\frac{\eta(x,z_0)}{h}=-\frac{U^2}{g\,h}\,\frac{1}{\pi\beta}\left[\mathrm{PV}\int\limits_0^L\frac{z_0'(\xi)\,d\xi}{x-\xi}+O(z_0^2)\right]\qquad(29.31)$$

where φ_x and φ_z are both of $O(z_0')$.

These relations are, of course, completely restricted to flows that are everywhere subcritical since (29.23) shows that the Froude number $(F=U/\sqrt{g h})$ must be everywhere less than unity to keep $\beta>0$. The effect of increasing Froude number is to increase φ_x, and therefore decrease η, since β decreases. It is seen that this effect increases as z increases, the greatest effect being on $\varphi_x\sim 1/\beta^3$ in the limiting case of very large values of z as shown in (29.29). This relation, or preferably (29.27), could be used to predict the additional change in $\eta(x,z)$ due to a finite stream width by using the increment of φ_x from one mirror image to represent the first approximation to the channel boundary wall as indicated in Fig. 37. For slender cylinders in a narrow channel the "one-dimensional" approximation of Sect. 30γ is generally used, this allows an approximation for frictional head loss which becomes relatively more important as the channel width decreases.

For supercritical flow $(F=U/\sqrt{g h}>1)$, (29.23) must be written as

$$\left.\begin{aligned}B^2\varphi_{xx}-\varphi_{zz}=0\quad\text{or}\quad\frac{\partial^2\varphi}{\partial x^2}=\frac{\partial^2\varphi}{\partial(B z)^2},\\ B^2=F^2-1=\frac{U^2}{g\,h}-1=\text{const}>0.\end{aligned}\right\}\qquad(29.32)$$

Now, however, (29.32) cannot provide a satisfactory approximation of the change in water depth at some distance from the two-dimensional profile since its general solution is

$$\varphi(x,z)=G(x-B z)+g(x+B z),\qquad(29.33)$$

which predicts no change, even upon approaching infinity, along the lines of constant slope $dz/dx=\pm 1/B=\pm[F^2-1]^{-\frac{1}{2}}$. Consequently the nonlinear method of characteristics, which will be described in Sect. 30, must be used in predicting the depth variation at any finite distance from the profile. Although the method of characteristics will directly and easily give the velocity distribution or depth variation on the profile itself, we will also derive the variation on the profile surface according to the linearized theory. The result will be of crucial importance in evaluating the validity of the nonlinear first-order shallow-water theory (28.1), since any great discrepancy between the linearized result and the nonlinear results from (28.1) would indicate that the perturbations involved are sufficiently large that the second-order shallow-water theory of Sect. 31 must be introduced.

The linearized solution of (29.32) for any sharp-nosed slender two-dimensional profile, as in Fig. 37, is obtained from the general solution (29.33) and the following linearized boundary condition:

$$z_0'(x) = \frac{\varphi_z(x, z_0)}{U} = -\frac{B}{U} G'(x - B z_0); \qquad z = z_0 > 0.$$

Therefore $G'(x - B z_0) = -U z_0'(x)/B$, so that on the profile surface, $z = z_0(x)$,

$$\left.\begin{aligned} u(x, z_0) &= U + \varphi_x = U + G'(x - B z_0) = U\left[1 - \frac{z_0'(x)}{B}\right], \\ w(x, z_0) &= U z_0'(x). \end{aligned}\right\} \tag{29.34}$$

Then the variation in water depth on the profile surface is given by (29.30) as

$$\frac{\eta(x, z_0)}{h} = F^2\left[\frac{z_0'(x)}{B} + O(z_0'^2)\right] \tag{29.35}$$

for flow that is everywhere supercritical, i.e. $B^2 = F^2 - 1 > 0$.

It should be noted that (29.23) and (29.32) are identical to the linearized potential equations for two-dimensional steady subsonic flow and supersonic flow, respectively, if we simply replace the Froude number ($F = U/\sqrt{gh}$) by the Mach number ($M = U/c$) [see (28.3)]. This is in complete accord with the statement that the hydraulic analogy is valid for the flow over a flat horizontal bottom (i.e., the flow is equivalent to the two-dimensional isentropic flow of a fictitous perfect gas having a specific heat ratio $\gamma = 2$). Consequently, Eqs. (29.24) through (29.29) are identical to these for subsonic flow about slender two-dimensional profiles in free air or in a wind tunnel of rectangular cross-section as derived by LAITONE (1946). These equations confirm the known result that the linearized equations are independent of the value of the specific heat ratio γ. Similarly, Eq. (29.34) is identical to the well-known linearized two-dimensional supersonic-flow solution if we let $F^2 - 1 \equiv B^2 = M^2 - 1 > 0$.

Although these linearized results are very satisfactory for slender sharp-nosed profiles, they only apply for Froude numbers that are not too near unity, that is they are not applicable to flows near the critical velocity $U = \sqrt{gh} = c$, equivalent to sonic flow. For these cases we must return to the nonlinear equation (28.1), as discussed in Sect. 30.

30. Nonlinear shallow-water theory. This section will primarily discuss methods for obtaining solutions of the nonlinear equations (28.1) which provide the first-order approximation of the shallow-water theory. The special cases to be considered are the one-dimensional unsteady flow and the two-dimensional steady flow in open channels. This will provide a basis for discussing the one-dimensional assumption of open-channel flow. Finally the hydraulic jumps, and their relation to the first-order shallow-water theory, will be discussed.

α) One-dimensional non-steady, first-order, shallow-water theory. By assuming one-dimensional flow in the x direction only, the nonlinear equations (28.1) reduce to

$$\left.\begin{aligned} u_t + u\,u_x + g(\eta + h)_x &= g\,h_x,\\ (\eta + h)_t + [u(\eta + h)]_x &= h_t = 0. \end{aligned}\right\} \tag{30.1}$$

Again it should be noted that these are equivalent to the gas dynamic equations, upon introducing (28.3), only if the bottom is flat and horizontal, i.e. $h_x = 0$.

Now, if we let

$$\left.\begin{aligned} c^2(x,t) &= g[\eta(x,t) + h(x)],\\ 2c\,c_x = g(\eta + h)_x, &\quad 2c\,c_t = g(\eta + h)_t, \end{aligned}\right\} \tag{30.2}$$

and give the initial conditions as $du/d\alpha$ and $dc/d\alpha$ along a curve in the (x, t)-plane defined by $x(\alpha)$, $t(\alpha)$, then we may write (30.1) as

$$\left.\begin{aligned} u\,u_x + u_t + 2c\,c_x + 0 &= g\,h_x,\\ c\,u_x + 0 + 2u\,c_x + 2c_t &= 0,\\ x_\alpha u_x + t_\alpha u_t + 0 + 0 &= \frac{du}{d\alpha},\\ 0 + 0 + x_\alpha c_x + t_\alpha c_t &= \frac{dc}{d\alpha}, \end{aligned}\right\} \tag{30.3}$$

This set of four equations can be solved uniquely for u_x, u_t, c_x, c_t in terms of u, c, h_x and the initial conditions as long as the determinant of the coefficients in (30.3) does not vanish. This condition is violated along the characteristic curves $x(\alpha)$, $t(\alpha)$ defined by

$$\begin{vmatrix} u & 1 & 2c & 0\\ c & 0 & 2u & 2\\ x_\alpha & t_\alpha & 0 & 0\\ 0 & 0 & x_\alpha & t_\alpha \end{vmatrix} = 0, \tag{30.4}$$

which may be easily expanded by the minors of the bottom row to give

$$x_\alpha^2 - 2u\,x_\alpha t_\alpha + (u^2 - c^2)\,t_\alpha^2 = [x_\alpha - (u - c)\,t_\alpha]\,[x_\alpha - (u + c)\,t_\alpha] = 0.$$

Therefore the characteristic curves, C_+ and C_-, are defined by

$$\frac{x_\alpha}{t_\alpha} = \left(\frac{dx}{dt}\right)_{C_\pm} = u(x,t) \pm c(x,t). \tag{30.5}$$

Since h_x is given, and appears only on the right-hand side of the first equation in (30.3), therefore the characteristic curves as defined in (30.5) are identical to those in the gas-dynamics case [see, e.g., COURANT and FRIEDRICHS (1948)]. However, the Riemann invariants, or quantities that can be constant along a characteristic curve, now depend upon the bottom slope, as may be seen by adding the two equations in (30.1) after introducing (30.2) so as to obtain

$$(u + 2c)_t + (u + c)\,(u + 2c)_x = \left[\frac{\partial}{\partial t} + (u + c)\,\frac{\partial}{\partial x}\right][u(x,t) + 2c(x,t)] = g\,h_x. \tag{30.6}$$

These give the same Riemann invariants as in the isentropic one-dimensional unsteady gas flow with a specific heat ratio $\gamma = 2$ only if $h_x = 0$ [see, e.g., COURANT and FRIEDRICHS (1948, p. 87)]. No simple Riemann invariant involving only u and c is possible if h_x varies with x; however, if h_x is constant, so that $g h_x = m = \text{const}$, then (30.6) may be written

$$\left[\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right][u + 2c - m\,t] = 0. \tag{30.7}$$

Similarly, by subtracting the two equations in (30.1), we obtain

$$\left[\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right][u - 2c - m\,t] = 0. \tag{30.8}$$

Consequently, the basic statements relating the characteristic curves and Riemann invariants of Eq. (30.1) with $g h_x = m = \text{const}$ may be summarized as follows:

$$\left.\begin{aligned} u + 2c - m\,t = R(x,t) &= \text{const along a curve } C_+ \\ &\quad\text{defined by } \frac{dx}{dt} = u + c; \\ u - 2c - m\,t = -\,S(x,t) &= \text{const along a curve } C_- \\ &\quad\text{defined by } \frac{dx}{dt} = u - c. \end{aligned}\right\} \tag{30.9}$$

Fig. 38 shows typical sets of curves in the (x, t)-plane. The above equations show that in any given region in the (x, t)-plane there are three basic types of solutions, namely:

(1) the constant steady state in which u and c remain constant everywhere in the region, so that all characteristics form straight lines;

(2) the general flow in which neither R nor S is constant in a finite region;

(3) the special case of a simple wave over a flat horizontal bottom ($m=0$) wherein a constant steady-state region is separated from a varying region by a straight characteristic line along which either R or S is constant.

The first type of solution obviously has R and S constant throughout the region only if the bottom is flat and horizontal ($m=0$). The second type of solution is complicated and can best be obtained by the method of finite differences [see, e.g., STOKER (1957, pp. 293—300)]. The third type of solution will now be discussed since it has considerable physical significance for many problems concerning the propagation of a disturbance into water that is originally at constant depth and constant velocity, and extends an unlimited distance for $x > 0$.

When a disturbance moves into still water at constant depth over a flat horizontal bottom ($m=0$), then it is obvious that $(dx/dt)_0 = c(\infty)$ is the characteristic, now a straight line, which must continually separate the steady-state region from the disturbance region in the (x, t)-plane, as indicated in Fig. 38. This characteristic curve must be a straight line since there is a constant steady state always ahead of it so that $(dx/dt)_0 = \text{const}$ and therefore $x_0 = c(\infty)\,t$. Also, either R or S must be constant along the characteristic, and since R_0 corresponds to C_+^0 or $(dx/dt)_0 = c(\infty) > 0$, as in Fig. 38, therefore $R_0 = 2c(\infty) = \text{const}$. This type of simple wave, having $(dx/dt)_0 = c(\infty) > 0$ and $R_0 = 2c(\infty) = \text{const.}$, is called a forward-facing wave since the particle paths enter from the side with greater values of x, as in Fig. 38. The value of R varies as one passes from one to another C_+ characteristic inside the region of the disturbance since u and c both vary due to the disturbance and none of the C_+ characteristic lines can ever intersect C_+^0. However, every C_- characteristic intersects C_+^0, as shown in Fig. 38, and since S remains constant on any given C_- characteristic curve, therefore S is everywhere constant since every C_- characteristic must have the same value $S(x,t)=R_0=2c(\infty)=\text{const}$ on C_+^0.

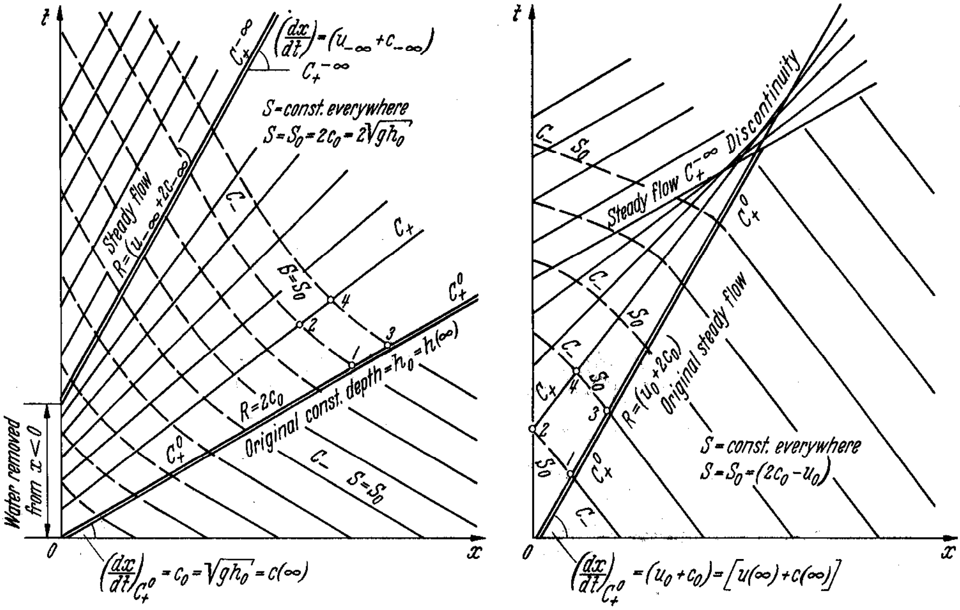

Fig. 38. Simple forward-facing waves, $S=\text{const}$.

The same considerations are true even if the water of constant depth into which the disturbance is being propagated is flowing with a constant velocity $u(\infty)<c(\infty)$. The only change is that now the following are constant:

$$\left(\frac{dx}{dt}\right)_0 = u(\infty)+c(\infty)>0, \qquad R_0 = 2c(\infty)+u(\infty)$$

on C_+^0 only, while on all C_-,

$$2c(x,t)-u(x,t)=S(x,t)=2c(\infty)-u(\infty)=\text{const}.$$

Similarly all R in the disturbance region vary as

$$R(x,t)=2c(x,t)+u(x,t),$$

as indicated in Fig. 38 for the simple forward-facing (C_+^0) wave. As shown in Fig. 39 a simple backward-facing (C_-^0) occurs if $R=\text{const}$ and $S=2c-u$. These waves are called simple waves because all the characteristics of the family for which the Riemann invariant takes on a different constant for each line form straight lines. For example, referring to Fig. 38, the forward-facing waves ($dx/dt>0$) have $S(x,t)$ constant everywhere and $R(x,t)$ varying so that the C_+ characteristics form straight lines. On the other hand, in Fig. 39 the backward-facing wave ($dx/dt<0$) has $R(x,t)$ constant everywhere and $S(x,t)$ varying, so that now only the C_- characteristics form straight lines. The characteristics of one family only must form straight lines in a simple wave because only one of the Riemann invariants (S or R) is constant in the entire region of the disturbance. For example, in the case of the forward-facing simple wave in Fig. 38, we have S constant in the region of the disturbance. Therefore from (30.9) and Fig. 38 we may write,

$$-S_1 = -S_2 = -S_3 = -S_4 = u_1 - 2c_1 = u_2 - 2c_2 = u_3 - 2c_3 = u_4 - 2c_4 = \text{const},$$
$$R_1 = R_3 = u_1 + 2c_1 = u_3 + 2c_3 \neq R_2 = R_4 = u_2 + 2c_2 = u_4 + 2c_4.$$

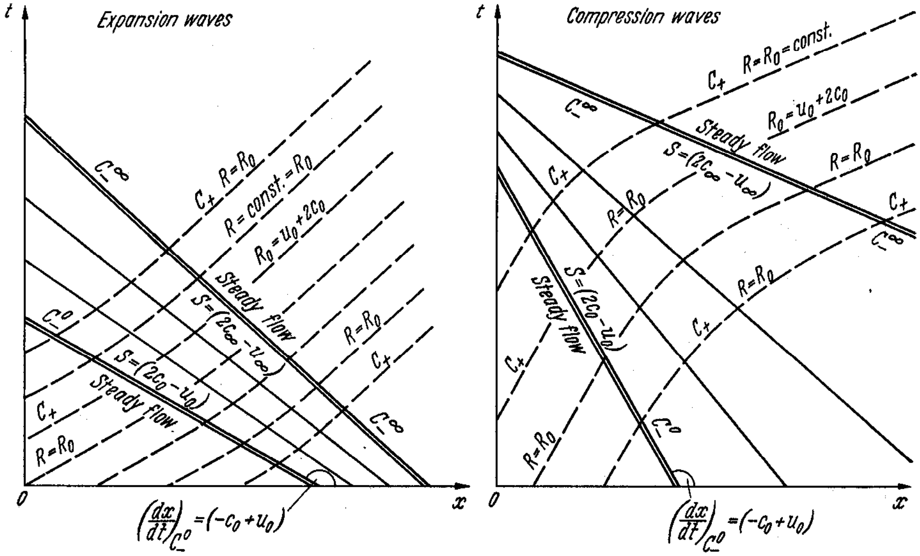

Fig. 39. Simple backward-facing waves, $R = \text{const.}$

Consequently, $u_1 = u_3$, $c_1 = c_3$, $u_2 = u_4$, $c_2 = c_4$, and $u_1 \neq u_2$, $c_1 \neq c_2$, $u_3 \neq u_4$, $c_3 \neq c_4$, so that

$$\left(\frac{dx}{dt}\right)_{C_-} = u_1 - c_1 \neq u_2 - c_2 \neq \text{const}, \qquad (C_- \text{ curved}),$$

$$\left(\frac{dx}{dt}\right)_{C_+^0} = u_1 + c_1 = u_3 + c_3 = \text{const}, \qquad (C_+^0 \text{ straight}),$$

$$\left(\frac{dx}{dt}\right)_{C_+} = u_2 + c_2 = u_4 + c_4 = \text{const}, \qquad (C_+ \text{ straight}).$$

It is important to note that these simple waves can exist only over a flat horizontal bottom, i.e. when $m = 0$.

We have now shown how the method of characteristics for one-dimensional unsteady flow has resulted in the concept of the simple wave which quickly gives a numerical evaluation of the propagation of a one-dimensional disturbance into water of constant depth moving at constant speed. The solution of this problem in the (x, t)-plane can be obtained by direct application of (30.9). For example, the usual case of a forward-facing wave having S everywhere constant, and straight C_+ characteristic lines, as shown in Fig. 38, has the slope of the C_+ straight lines determined directly by the time history of the disturbance at $x = 0$, and Eqs. (30.2) and (30.9) which show that

$$\left(\frac{dx}{dt}\right)_{C_+} = u(0, t) + c(0, t) = \text{const} = u(0, t) + \sqrt{g[h + \eta(0, t)]}. \qquad (30.10)$$

Along any given C_+ straight line having this constant slope

$$R(x, t)_{C_+} = u(0, t) + 2c(0, t) = \text{const} = u(0, t) + 2\sqrt{g[h + \eta(0, t)]}. \qquad (30.11)$$

Consequently, the values of u and c are determined in the entire region shown in Fig. 38 by the given values on the t-axis. The curved C_- characteristics need not be calculated, since the desired numerical solution is independent of them. Their existence, however, can lead to a simplification in the numerical calculation of (30.10) since, in the case shown in Fig. 38, each curved C_- characteristic extends from the C_+^0 characteristic to the t-axis, and on each and every C_- characteristic, $-S = u(\infty) - 2c(\infty) = \text{const}$. Therefore, at every point on the t-axis that can be reached by a C_- characteristic we must have

$$S = 2c(0, t) - u(0, t) = 2c(\infty) - u(\infty) = 2\sqrt{gh} - u(\infty) = \text{const}. \qquad (30.12)$$

Of course the C_- characteristics can continue from C_+^0 to the t-axis only if $(dx/dt)_{C_-} = u - c < 0$, or $u < c$, so that in this case (30.10) may be simplified to

$$\left.\begin{aligned}\left(\frac{dx}{dt}\right)_{C_+} &= u(0, t) + c(0, t) = \text{const}\\ &= \tfrac{3}{2}u(0, t) - \tfrac{1}{2}[u(\infty) - 2c(\infty)] = \tfrac{3}{2}u(0, t) - \tfrac{1}{2}u(\infty) + \sqrt{gh}\\ &= 3c(0, t) + [u(\infty) - 2c(\infty)] = 3\sqrt{g[h + \eta(0, t)]} + u(\infty) - 2\sqrt{gh}.\end{aligned}\right\} \qquad (30.13)$$

Consequently, the problem is solved in the region so defined if either $u(0, t)$ or $c(0, t)$ is alone given. The surface elevation is given by (30.2) as

$$h + \eta(x, t) = \frac{c^2(x, t)}{g}, \qquad h = \frac{c^2(\infty)}{g} = \text{const} \qquad (30.14)$$

in every case of disturbance propagations into a constant water depth over a flat horizontal bottom ($m = 0$).

Many other physical problems can be simulated by giving the data along a prescribed curve in the (x, t)-plane for $x \leqq 0$; e.g., see STOKER (1957) where the disturbance created by the breaking of a dam, and the effect of moving a vertical end plate in a tank of still water of rectangular cross-section, $u(\infty) = 0$, are considered. Since the bottom is flat and horizontal ($m = 0$), all of the equations following (30.9) are equivalent to the gas-dynamics equations with a specific heat ratio $\gamma = 2$. Consequently, the problems solved in COURANT and FRIEDRICHS (1948) for channels of finite length which produce wave reflections at either end are also applicable. In this hydraulic analogy to compressible flow it is important to remember that (30.13) is only applicable to subcritical flow, which is equivalent to subsonic gas flow, since we must have $(dx/dt)_{C_-} = u - c < 0$, or $u(\infty) < c(\infty) = \sqrt{gh}$. When the flow is supercritical, so that $u(\infty) > c(\infty) = \sqrt{gh}$, corresponding to supersonic gas flow, then the slopes of both the C_+ and C_- characteristics are positive. Consequently the two families can meet in a cusp, and the C_- characteristics cannot intersect both the t-axis and the undisturbed steady supercritical state that lies at, and to the right of, C_+^0. Therefore, in order to apply (30.13) for supercritical flow, the region of the constant value of S, as given by (30.12), must be very carefully defined.

Another limitation on all the preceding equations is indicated for the compression wave depicted in Figs. 38 and 39. This limitation is defined by the envelope of the straight characteristic lines that *must* always form for a compression wave in this first-order theory, as will be proven later. This envelope of the straight characteristic lines corresponds to a discontinuity that can be interpreted as a discontinuity in η, or the breaking of the wave crest. This leads to the hydraulic jump or surge that will be discussed later. The gas dynamic case has the envelope of the straight characteristic lines interpreted as a steady-state shock wave [see, e.g., COURANT and FRIEDRICHS (1948, pp. 110—181)].

β) Two-dimensional, steady, supercritical flow by the first-order shallow-water theory. We will now investigate the characteristic curves of the nonlinear equations of the first-order shallow-water theory for the case of steady two-dimensional flow. We will find that real characteristic curves, which are a great aid to numerical calculations, exist only in the regions wherein the flow is everywhere supercritical.

If we consider the steady two-dimensional flow over a flat horizontal bottom, then we may write (28.2) as

$$\left.\begin{aligned} &u\,u_x + w\,u_z = -g(\eta+h_0)_x = -(c^2)_x,\\ &u\,w_x + w\,w_z = -g(\eta+h_0)_z = -(c^2)_z,\\ &[u(\eta+h_0)]_x + [w(\eta+h_0)]_z = 0 \quad\text{or}\quad (u\,c^2)_x + (w\,c^2)_z = 0,\\ &u=\varphi_x,\quad w=\varphi_z,\quad u_z = w_x = \varphi_{xz}. \end{aligned}\right\} \tag{30.15}$$

By multiplying the first equation by $u=\varphi_x$ and the second by $w=\varphi_z$, and adding, we obtain

$$\varphi_x^2\,\varphi_{xx} + 2\varphi_x\,\varphi_z\,\varphi_{xz} + \varphi_z^2\,\varphi_{zz} = -[\varphi_x(c^2)_x + \varphi_z(c^2)_z] = (\varphi_{xx}+\varphi_{zz})\,c^2. \tag{30.16}$$

Therefore

$$\left(\frac{\varphi_x^2}{c^2}-1\right)\varphi_{xx} + 2\,\frac{\varphi_x\,\varphi_z}{c^2}\,\varphi_{xz} + \left(\frac{\varphi_z^2}{c^2}-1\right)\varphi_{zz} = 0 \tag{30.17}$$

or

$$\left(1-\frac{u^2}{c^2}\right)\varphi_{xx} - 2\,\frac{u\,w}{c^2}\,\varphi_{xz} + \left(1-\frac{w^2}{c^2}\right)\varphi_{zz} = 0 \tag{30.18}$$

where $c^2(x,z) = g[h_0+\eta(x,z)]$ and h_0 now is the still water depth found whenever $(u^2+w^2)=0=\eta$. Note that (30.18) immediately linearizes to (29.3), so that the numerical differences between the solutions of (29.3) and (30.18) will provide an estimate of whether or not the second-order shallow-water theory, as discussed in Sect. 31, must be introduced.

The characteristic curves of (30.18) may be found in a manner similar to that used for (30.3) by finding the curve $[x(\alpha), z(\alpha)]$ in the (x,z)-plane along which prescribed values of φ_x and φ_z cannot determine φ_{xx}, φ_{xz} and φ_{zz}. Therefore we write

$$\left.\begin{aligned} &\left(1-\frac{u^2}{c^2}\right)\varphi_{xx} + \left(-2\,\frac{u\,w}{c^2}\right)\varphi_{xz} + \left(1-\frac{w^2}{c^2}\right)\varphi_{zz} = 0,\\ &x_\alpha\,\varphi_{xx} + z_\alpha\,\varphi_{xz} + 0 = \frac{d\varphi_x}{d\alpha},\\ &0 + x_\alpha\,\varphi_{xz} + z_\alpha\,\varphi_{zz} = \frac{d\varphi_z}{d\alpha}, \end{aligned}\right\} \tag{30.19}$$

which may not have a solution if the determinant of the coefficient is zero, that is if

$$\begin{vmatrix} 1-\dfrac{u^2}{c^2} & -\dfrac{2u\,w}{c^2} & 1-\dfrac{w^2}{c^2}\\ x_\alpha & z_\alpha & 0\\ 0 & x_\alpha & z_\alpha \end{vmatrix} = 0, \tag{30.20}$$

or

$$\left.\begin{aligned} &\left(1-\frac{u^2}{c^2}\right)z_\alpha^2 + 2\,\frac{u\,w}{c^2}\,z_\alpha\,x_\alpha + \left(1-\frac{w^2}{c^2}\right)x_\alpha^2 = 0,\\ &\frac{z_\alpha}{x_\alpha} = \left(\frac{dz}{dx}\right)_{C_\pm} = \frac{\dfrac{u\,w}{c^2} \pm \sqrt{\dfrac{u^2+w^2}{c^2}-1}}{\dfrac{u^2}{c^2}-1} \end{aligned}\right\} \tag{30.21}$$

which therefore gives the slopes of the two families (C_+ and C_-) of characteristic curves. Now, however, entirely unlike the previous one-dimensional unsteady flow solution, the characteristic curves exist only for supercritical flow, i.e., for $u^2+w^2>c^2=g(h_0+\eta)$. The fact that the characteristic curves are real for supercritical flow means that in this case the nonlinear equation (30.17) is hyperbolic. However, for subcritical flow, since the characteristic curves are then complex functions, it is of elliptic type [see, e.g., COURANT and FRIEDRICHS (1948, pp. 40—55) or PREISWERK (1938)].

We can obtain a solution for the behavior of the quantity

$$F(x, z) = \sqrt{\frac{u^2+w^2}{c^2}} \geqq 1 \tag{30.22}$$

(which defines the Froude number of the supercritical flow) along a characteristic curve by transforming (30.18) into the (u, w)-plane, called the hodograph plane, through the use of the Legendre contact transformation which is given by [see, e.g., COURANT and FRIEDRICHS (1948, p. 249) or PREISWERK (1938)]

$$\chi = (x\,\varphi_x + z\,\varphi_z - \varphi) = (x\,u + z\,w - \varphi),$$
$$d\chi = (x\,du + u\,dx + z\,dw + w\,dz - d\varphi) = (x\,du + z\,dw).$$

Hence

see errata

$$x = \chi_n, \quad z = \chi_w,$$
$$dx = x_u\,du + x_w\,d_w = \chi_{uu}\,du + \chi_{uw}\,dw,$$
$$dy = z_u\,du + z_w\,dw = \chi_{uw}\,du + \chi_{ww}\,dw.$$

Solving for du and dw, we obtain

$$du = N^{-1}(\chi_{ww}\,dx - \chi_{uw}\,dz) = d\varphi_x = \varphi_{xx}\,dx + \varphi_{xz}\,dz,$$
$$dw = N^{-1}(-\chi_{uw}\,dx + \chi_{uu}\,dz) = d\varphi_z = \varphi_{xz}\,dx + \varphi_{zz}\,dz,$$

where

$$N = \begin{vmatrix} \chi_{uu} & \chi_{uw} \\ \chi_{uw} & \chi_{ww} \end{vmatrix} \neq 0,$$

so that

$$\varphi_{xx} = \frac{\chi_{ww}}{N}, \qquad \varphi_{xz} = -\frac{\chi_{uw}}{N}, \qquad \varphi_{zz} = \frac{\chi_{uu}}{N}.$$

The nonlinear equation (30.17) in the physical (x, z)-plane is transformed into the following linear equation in the hodograph (u, w)-plane:

$$\left(\frac{w^2}{c^2} - 1\right)\chi_{uu} - 2\frac{u\,w}{c^2}\chi_{uw} + \left(\frac{u^2}{c^2} - 1\right)\chi_{ww} = 0. \tag{30.23}$$

The same procedure used in (30.19) through (30.21), or a simple comparison of (30.17), (30.21), and (30.23), shows that the characteristic curves of (30.23) in the hodograph (u, w)-plane are defined by

$$\frac{w_\alpha}{u_\alpha} = \left(\frac{dw}{du}\right)_{\Gamma_\pm} = \frac{-\dfrac{u\,w}{c^2} \pm \sqrt{\dfrac{u^2+w^2}{c^2} - 1}}{\dfrac{w^2}{c^2} - 1}. \tag{30.24}$$

The characteristic curve Γ_- in the hodograph (u, w)-plane is orthogonal to the characteristic curve C_+ in the physical (x, z)-plane if we superimpose the two planes so that the velocity vectors coincide. This may be easily shown by rotating the axes for (30.21) and (30.24) so that $w=0$ (see Fig. 40); then the equations for the slopes of the characteristic curves C_+ and Γ_- simplify to

$$\left(\frac{dz}{dx}\right)_{C_+} = \frac{1}{\sqrt{\frac{u^2}{c^2}-1}} = \frac{1}{\sqrt{F^2-1}} = -\frac{1}{\left(\frac{dw}{du}\right)_{\Gamma_-}}. \tag{30.25}$$

Similarly, Γ_+ is orthogonal to C_- when the planes are superimposed so that the velocity vectors are coincident (see Fig. 40).

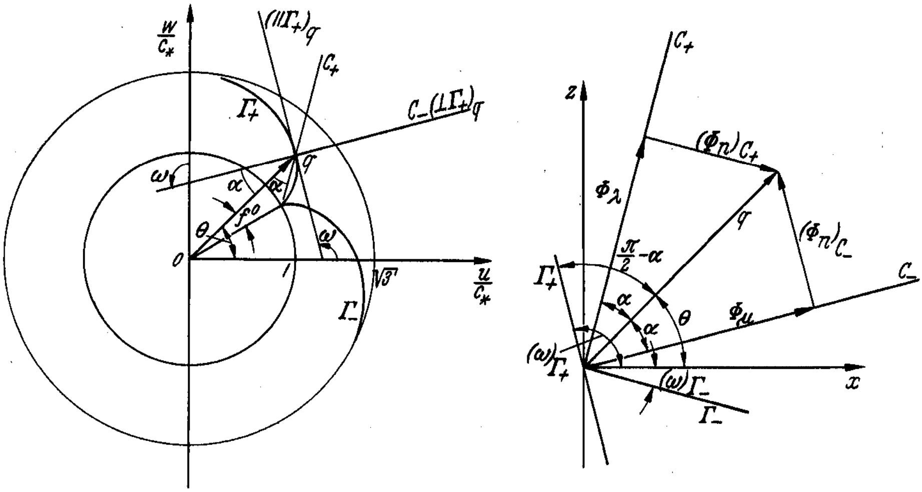

Fig. 40. Characteristic directions in the hodograph (u, w)-plane and the physical (x, z)-plane.

Eqs. (30.24) and (30.25) show that along any characteristic curve there exists a simple solution which is independent of the boundary conditions of a particular problem, for we can directly integrate (30.25) with axes rotated so that $w=0$ and hence $dw = u\, d\vartheta$:

$$\left(\frac{dw}{du}\right)_{\Gamma_-} = \left(\frac{u\, d\vartheta}{du}\right)_{\Gamma_-} = -\sqrt{\frac{u^2}{c^2}-1} = -\sqrt{F^2-1},$$

We integrate[1] (30.25) as follows:

$$\left.\begin{aligned}\int_{\Gamma_-} d\vartheta &= -\int_{\Gamma_-}\sqrt{\frac{u^2}{c^2}-1}\,\frac{du}{u} = -\int_{\Gamma_-}\frac{\sqrt{F^2-1}}{1+\frac{1}{2}F^2}\,\frac{dF}{F},\\ |\Delta\vartheta| &= \sqrt{3}\tan^{-1}\sqrt{\tfrac{1}{3}(F^2-1)} - \tan^{-1}\sqrt{F^2-1} = f(F).\end{aligned}\right\} \tag{30.26}$$

Consequently (30.26) provides a general solution, independent of the boundary conditions in the physical plane, for any two-dimensional potential flow that possesses the property of having simple waves in the given region, so that the end of the velocity vector follows Γ_- in the hodograph plane. The numerical values from (30.26) are indicated in Fig. 40 and are tabulated in Table 1 on page 688 [taken from PREISWERK (1938)].

The useful relation between c and (u, w) that was used to integrate (30.26) and calculate Table 1 is obtained by multiplying the first equation in (30.15)

[1] See (30.27) and (30.29) which show that with $w=0$

$$\frac{du}{u} = -\frac{d(c^2)}{u^2} = +\frac{c_0^2}{u^2}\left(1+\frac{1}{2}F^2\right)^{-2} F\,dF = +\frac{dF}{F(1+\frac{1}{2}F^2)}.$$

by dx and the second equation by dz, and then adding them. One obtains successively

$$\left.\begin{aligned} &u(u_x\,dx + u_z\,dz) + w(w_x\,dx + w_z\,dz) = -\left[(c^2)_x\,dx + (c^2)_z\,dz\right],\\ &u\,du + w\,dw = -d(c^2) = -g\,d\eta,\\ &\tfrac{1}{2}(u^2+w^2) + c^2 = \text{const} = \tfrac{1}{2}(u^2+w^2) + g(h_0+\eta).\end{aligned}\right\}\quad(30.27)$$

Therefore

$$\left.\begin{aligned} \tfrac{1}{2}(u^2+w^2) + c^2 &= \tfrac{1}{2}(u^2+w^2) + g(h_0+\eta)\\ &= g\,h_0 = \tfrac{1}{2}(u^2+w^2)_{\max} = \tfrac{3}{2}c_*^2 = \tfrac{1}{2}q^2 + c^2 = \tfrac{1}{2}q_{\max}^2,\end{aligned}\right\}\quad(30.28)$$

where (see Fig. 41) h_0 is the still water depth (or stagnation total head depth) that corresponds to $u_0^2 + w_0^2 = 0 = \eta_0$, $(u^2+w^2)_{\max}$ is the limiting resultant velocity

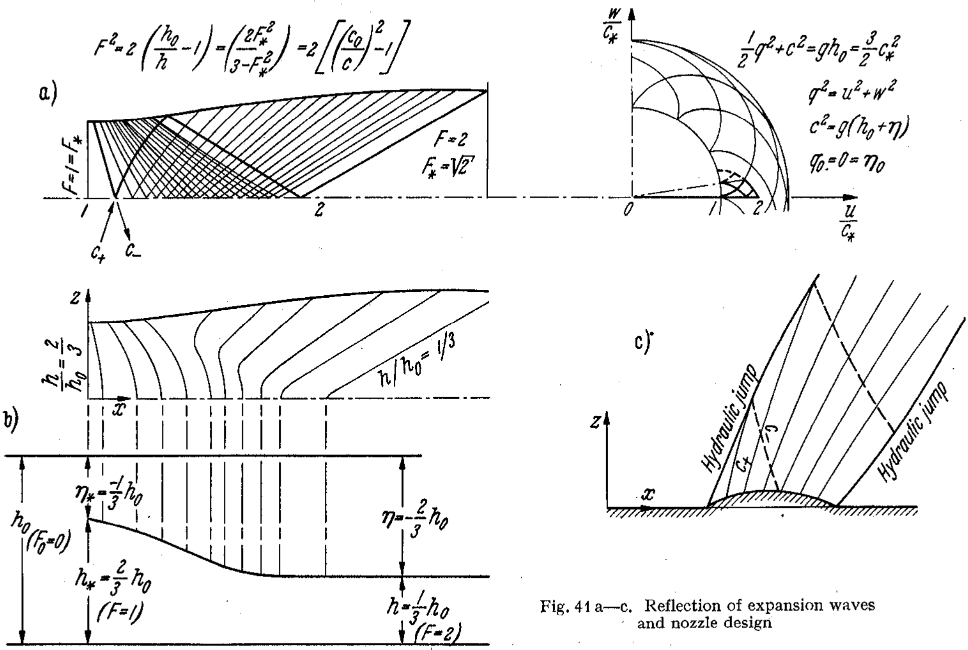

Fig. 41 a—c. Reflection of expansion waves and nozzle design

squared which is approached when the depth of flowing water approaches zero ($\eta\to -h_0$), and c^* is the speed when the resultant velocity $\sqrt{u^2+w^2} = c^*$ is critical ($F=1$), so that

$$\left.\begin{aligned} &c_* = \sqrt{g(h_0+\eta^*)} = \sqrt{\frac{2}{3}g\,h_0} = \sqrt{\frac{2}{3}\left[\frac{1}{2}c_*^2 + g(h_0+\eta_*)\right]} = \sqrt{\frac{1}{3}(u^2+w^2)_{\max}},\\ &\left(\frac{c_*}{c_0}\right)^2 = \frac{c_*^2}{g\,h_0} = \frac{2}{3} = -2\frac{\eta_*}{h_0},\\ &\frac{\eta_*}{h_0} = -\frac{1}{3},\qquad \frac{h_0+\eta_*}{h_0} = \frac{2}{3},\qquad \frac{(u^2+w^2)_{\max}}{c_*^2} = 3,\\ &\left(\frac{c_0}{c}\right)^2 = \frac{g\,h_0}{c^2} = 1 + \frac{u^2+w^2}{2c^2} = 1 + \frac{1}{2}F^2 = \left(\frac{c_0}{c_*}\right)^2\left(\frac{c_*}{c}\right)^2 = \frac{3}{2}\left(\frac{c_*}{c}\right)^2 = \frac{h_0}{h},\\ &F^2 = \frac{u^2+w^2}{c^2} = \left(\frac{c_*}{c}\right)^2 F_*^2 = \frac{2F_*^2}{3-F_*^2},\\ &F_*^2 = \frac{u^2+w^2}{c_*^2} = \left(\frac{c}{c_*}\right)^2 F^2 = \frac{3F^2}{2+F^2}.\end{aligned}\right\}\quad(30.29)$$

It is very useful to note that Eqs. (30.26) through (30.29) may all be obtained directly from the two-dimensional isentropic gas-flow equations by simply letting the specific heat ratio $\gamma = 2$ and $F \equiv M$, $F_* \equiv M_*$, as had been previously shown by PREISWERK (1938) [see also COURANT and FRIEDRICHS (1948)].

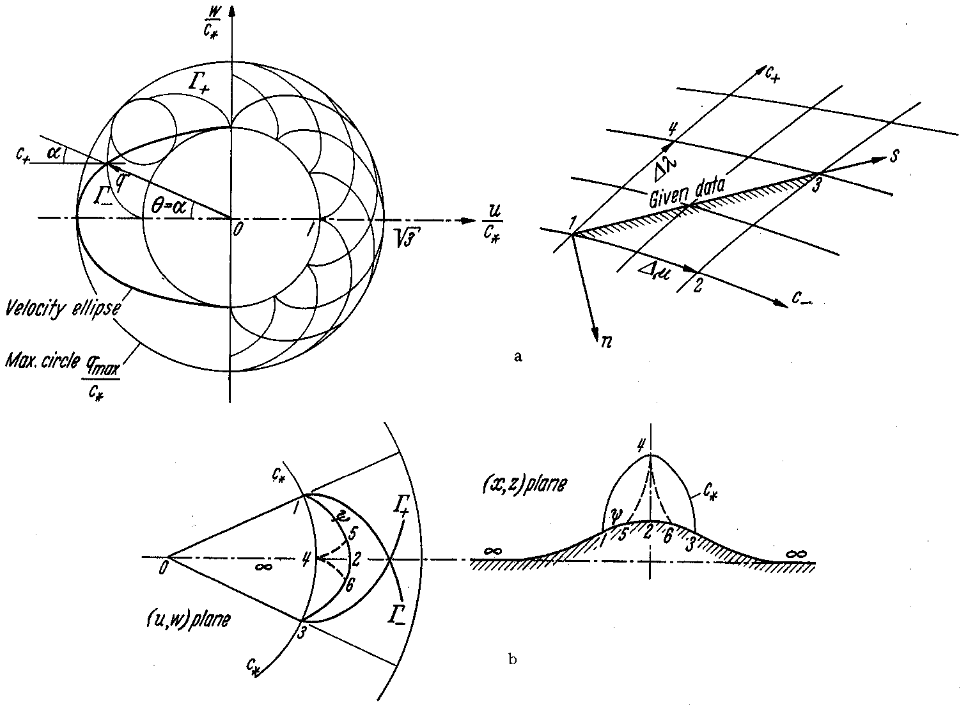

Fig. 42 a and b. Hodograph (u, w)-plane characteristic epicycloids.

The Riemann invariants for the characteristic curves (C_+, C_-) will now be determined. First we can show that the velocity component normal to the characteristic curves is always the local velocity of the shallow-water wave propagation, $c(x, z)$. We do this by writing (30.21) as

$$\left.\begin{aligned} (u\,dz - w\,dx)^2_{C_+} &= c^2\left[(dx)^2 + (dz)^2\right]_{C_+} = c^2 (d\lambda)^2_{C_+}, \\ c^2 &= \left(u\frac{dz}{d\lambda} - w\frac{dx}{d\lambda}\right)^2_{C_+} = (\varphi_x x_n + \varphi_z z_n)^2_{C_+} = (\varphi_n)^2_{C_+}, \end{aligned}\right\} \tag{30.30}$$

since the relation between the normal direction (n), and the tangential direction (λ) along the characteristic curve (C_+) is given by (see Fig. 40)

$$\left(\frac{dx}{dn}\right)_{C_+} = \left(\frac{dz}{d\lambda}\right)_{C_+}, \quad \left(\frac{dz}{dn}\right)_{C_+} = -\left(\frac{dx}{d\lambda}\right)_{C_+}.$$

Similarly, if μ is the tangential direction along C_-,

$$c^2 = \left(u\frac{dz}{d\mu} - w\frac{dx}{d\mu}\right)^2_{C_-} = (\varphi_x x_n + \varphi_z z_n)^2_{C_-} = (\varphi_n)^2_{C_-}.$$

Also, from Fig. 40 and (30.21)

$$\tan\alpha = \left(\frac{dz}{dx}\right)_{C_\pm,\,\vartheta=0} = \pm\frac{1}{\sqrt{F^2-1}}, \qquad \sin\alpha = \frac{\pm 1}{F} = \pm\frac{c}{q} \tag{30.31}$$

where q is the resultant velocity magnitude. Hence

$$\left.\begin{aligned} &q^2 = (u^2 + w^2) = (\varphi_\lambda^2 + c^2) = (\varphi_\mu^2 + c^2), \qquad \vartheta = \tan^{-1}\frac{w}{u},\\ &u = q\cos\vartheta, \qquad w = q\sin\vartheta,\\ &\varphi_n = c = q\sin\alpha, \qquad \varphi_\lambda = q\cos\alpha = \varphi_\mu. \end{aligned}\right\} \tag{30.32}$$

Substituting (30.31) and (30.32) into (30.21) and (30.24) we obtain

$$\left(\frac{dz}{dx}\right)_{C_\pm} = \frac{\dfrac{\cos\vartheta\sin\vartheta}{\sin^2\alpha} \pm \dfrac{1}{\tan\alpha}}{\dfrac{\cos^2\vartheta}{\sin^2\alpha} - 1} = \tan(\vartheta \pm \alpha), \tag{30.33}$$

$$\left(\frac{dw}{du}\right)_{\Gamma_\mp} = \frac{-\dfrac{\cos\vartheta\sin\vartheta}{\sin^2\alpha} \pm \dfrac{1}{\tan\alpha}}{\dfrac{\sin^2\vartheta}{\sin^2\alpha} - 1} = -\cot(\vartheta \pm \alpha). \tag{30.34}$$

Therefore, as proven before in (30.25),

$$\left(\frac{dz}{dx}\right)_{C_+}\left(\frac{dw}{du}\right)_{\Gamma_-} = -1 = \left(\frac{dz}{dx}\right)_{C_-}\left(\frac{dw}{du}\right)_{\Gamma_+}, \tag{30.35}$$

that is, as shown in Fig. 40, the C_+ characteristic curves in the physical (x, z) plane are at every corresponding point orthogonal to the Γ_- characteristic curves in the hodograph (u, w) plane. All these results are the same as in the gas-dynamics case where the $C_\pm$ characteristic curves are referred to as the Mach lines since, as shown by (30.30), the normal velocity component is always the local speed of sound.

Now, as shown in Fig. 40,

$$\left(\frac{dw}{du}\right)_{\Gamma_-} = \tan\omega_{\Gamma_-}; \qquad \omega_{\Gamma_-} = \vartheta + \alpha - \frac{1}{2}\pi,$$

$$\left(\frac{dw}{du}\right)_{\Gamma_+} = \tan\omega_{\Gamma_+}; \qquad \omega_{\Gamma_+} = \vartheta - \alpha + \frac{1}{2}\pi;$$

therefore (30.33) may be written as

$$\left(\frac{dz}{dx}\right)_{C_+} = \tan(\vartheta + \alpha) = \tan\left(\omega_{\Gamma_-} + \frac{1}{2}\pi\right),$$

$$\left(\frac{dz}{dx}\right)_{C_-} = \tan(\vartheta - \alpha) = \tan\left(\omega_{\Gamma_+} - \frac{1}{2}\pi\right).$$

Consequently the Riemann invariants are given by

$$R = \vartheta + \alpha - \omega_{\Gamma_-} - \tfrac{1}{2}\pi, \qquad S = \vartheta - \alpha - \omega_{\Gamma_+} + \tfrac{1}{2}\pi.$$

These may be simplified by calculating

$$\omega_{\Gamma_-} = \operatorname{arc\,tan}\left(\frac{dw}{du}\right)_{\Gamma_-} = -\sqrt{3}\operatorname{arc\,cot}\left[\sqrt{\frac{3}{F^2-1}}\right] = -\sqrt{3}\operatorname{arc\,tan}\left[\sqrt{\frac{F^2-1}{3}}\right]$$

from (30.24) and Fig. 40 since

$$\frac{1}{\sqrt{F^2-1}} = |\tan\alpha| = \sqrt{\frac{1}{3}}\left|\cot\left(\frac{\omega}{\sqrt{3}}\right)\right|$$

[or see COURANT and FRIEDRICHS (1948, p. 266)].

Table 1.

f (deg.)	$1+\frac{\eta}{h_0}$	F_*	F	K	f (deg.)	$1+\frac{\eta}{h_0}$	F_*	F	K
0	2/3	1.000	1.000	∞	26	0.234	1.516	2.56	−0.160
1	0.624	1.062	1.098	2.68	27	0.223	1.527	2.64	−0.177
2	0.598	1.101	1.160	2.07	28	0.212	1.538	2.73	−0.196
3	0.576	1.129	1.214	1.40	29	0.201	1.549	2.82	−0.216
4	0.555	1.156	1.267	1.014	30	0.190	1.559	2.92	−0.234
5	0.535	1.182	1.319	0.758	31	0.180	1.569	3.02	−0.252
6	0.516	1.207	1.371	0.590	32	0.170	1.579	3.13	−0.271
7	0.498	1.229	1.422	0.476	33	0.160	1.588	3.24	−0.291
8	0.481	1.249	1.470	0.394	34	0.151	1.597	3.36	−0.313
9	0.464	1.269	1.520	0.318	35	0.141	1.605	3.49	−0.336
10	0.448	1.288	1.570	0.263	36	0.132	1.613	3.63	−0.36
11	0.432	1.306	1.622	0.215	37	0.123	1.621	3.78	−0.38
12	0.417	1.323	1.674	0.170	38	0.115	1.629	3.93	−0.40
13	0.402	1.340	1.727	0.133	39	0.107	1.637	4.01	−0.43
14	0.387	1.356	1.781	0.103	40	0.099	1.644	4.26	−0.46
15	0.373	1.372	1.835	0.072	41	0.092	1.651	4.44	−0.49
16	0.359	1.387	1.89	0.046	42	0.085	1.657	4.63	−0.52
17	0.345	1.402	1.95	0.020	43	0.078	1.663	4.85	−0.54
18	0.331	1.416	2.01	−0.004	44	0.072	1.669	5.08	−0.58
19	0.318	1.430	2.07	−0.028	45	0.066	1.675	5.33	−0.62
20	0.305	1.444	2.13	−0.050	46	0.060	1.681	5.62	−0.66
21	0.292	1.457	2.20	−0.071	47	0.054	1.686	5.95	−0.70
22	0.280	1.470	2.27	−0.089	48	0.048	1.691	6.30	−0.75
23	0.268	1.482	2.34	−0.108	49	0.043	1.696	6.68	−0.81
24	0.256	1.494	2.41	−0.126	50	0.038	1.700	7.11	−0.86
25	0.245	1.505	2.48	−0.143	65°53′	0	$\sqrt{3}$	∞	$-\infty$

Therefore

$$\begin{aligned} R &= \vartheta + \arctan\frac{1}{\sqrt{F^2-1}} - \frac{\pi}{2} + \sqrt{3}\arctan\sqrt{\frac{F^2-1}{3}} \\ &= \vartheta + \sqrt{3}\arctan\sqrt{\frac{F^2-1}{3}} - \arctan\sqrt{F^2-1} = \vartheta + f(F), \\ S &= \vartheta - \arctan\frac{1}{\sqrt{F^2-1}} + \frac{\pi}{2} - \sqrt{3}\arctan\sqrt{\frac{F^2-1}{3}} \\ &= \vartheta - \sqrt{3}\arctan\sqrt{\frac{F^2-1}{3}} + \arctan\sqrt{F^2-1} = \vartheta - f(F), \end{aligned}$$

where $f(F)$ is given by (30.26) and Table 1. Consequently the Riemann invariants are very simply expressed for the characteristic curves in the physical plane as

$$\vartheta - f(F) = \text{const on } C_+, \qquad \vartheta + f(F) = \text{const on } C_-. \tag{30.36}$$

The function $f(F)$, which was derived from the fact that the end point of the velocity vector follows a characteristic in the hodograph plane in (30.26), is seen to have important physical significance, and directly provides the Riemann invariants for the steady two-dimensional potential flow. In gas dynamics $f(F) \equiv f(M)$ is referred to as the Prandtl-Meyer expansion function and, in the form in which it is given in Table 1, it corresponds to the supersonic free expansion about a sharp corner as shown in Fig. 43 for the centered simple wave with a specific heat ratio $\gamma = 2$. Since this Prandtl-Meyer function is so important, let us re-derive it on another basis that will further illustrate its physical significance. From the fact that f forms the Riemann invariant it is evident that u and w cannot be independent of one another on any such simple characteristic.

Consequently, if we write the original potential equation (30.18) in the physical plane as

$$\left(\frac{u^2}{c^2}-1\right)u_x+2\frac{u\,w}{c^2}u_y+\left(\frac{w^2}{c^2}-1\right)w_y=0$$

and introduce $w=w(u)$ so that

$$w_y=u_y\,w'(u)\,,\quad u_x\,w'(u)=w_x=\varphi_{xz}=u_y\,,$$

we obtain

$$\left(\frac{u^2}{c^2}-1\right)\frac{u_y}{w'}+2\frac{u\,w}{c^2}u_y+\left(\frac{w^2}{c^2}-1\right)u_y\,w'=0$$

or

$$\frac{d\,w}{d\,u}=w'(u)=\frac{-\frac{u\,w}{c^2}\pm\sqrt{\frac{u^2+w^2}{c^2}-1}}{\frac{w^2}{c^2}-1}. \tag{30.24'}$$

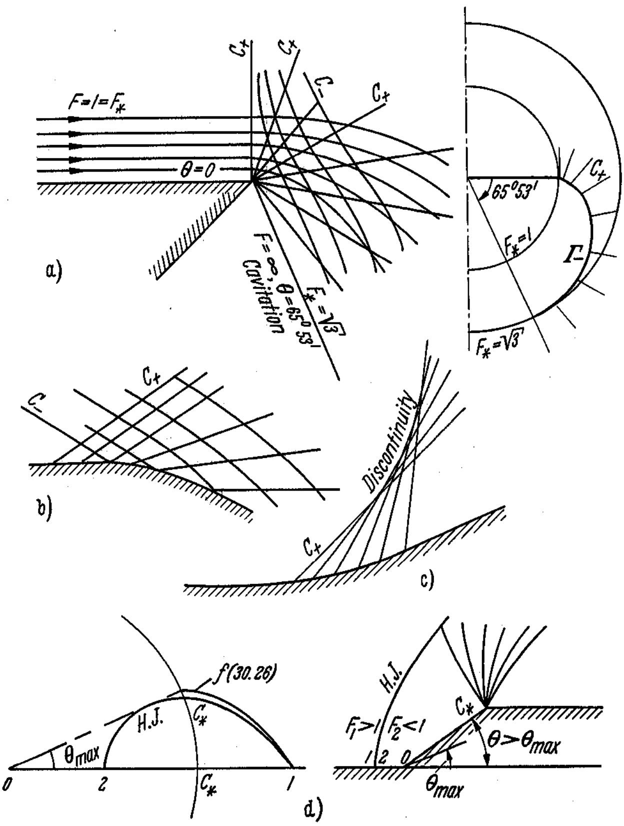

Fig. 43a–d. Simple waves and the formation of hydraulic jumps. (a) Complete centered simple expansion wave. (b) Simple(C_+) expansion waves. (c) Simple (C_+) compression waves forming a discontinuous (hydraulic jump) increase in water depth. (d) Detached hydraulic jump.

This derivation gives exactly the same result as in (30.24) and verifies the fact that discontinuities can occur in the first derivatives normal to a characteristic curve. If we introduce (30.32) into (30.24) we obtain the equivalent of (30.26)

$$\frac{1}{q}\,\frac{dq}{d\vartheta}=\tan\alpha=\frac{1}{\sqrt{F^2-1}} \tag{30.37}$$

which again has $f(F)$ as the general integral because (30.28) shows that

$$\frac{d\vartheta}{\sqrt{F^2-1}}=\frac{dq}{q}=\frac{d(q/c_*)}{(q/c_*)}=\frac{dF_*}{F_*}=\frac{1}{1+\frac{1}{2}F^2}\,\frac{dF}{F}\,. \tag{30.38}$$

However, neither of these methods gives the direct proof that $f(F)$ provides the Riemann invariant. This fact may be proved directly by the following derivation which utilizes the velocity component φ_λ along the C_+ characteristic, and $\varphi_n=c$, from (30.30), normal to C_+ as shown in Fig. 40. Hence

$$\left.\begin{aligned}\varphi_\lambda&=q\cos\alpha,\qquad \varphi_n=c=q\sin\alpha,\\ d\varphi_\lambda&=\cos\alpha\,dq-q\sin\alpha\,d\alpha=c(d\vartheta-d\alpha),\\ \lambda_z&=1/\sin(\vartheta+\alpha),\qquad \mu_z=1/\sin(\vartheta-\alpha)\end{aligned}\right\} \tag{30.39}$$

since, from (30.37),

$$dq=\frac{\sin\alpha}{\cos\alpha}\,q\,d\vartheta.$$

Then from (30.28) and (30.32) we have

$$\left.\begin{aligned}\tfrac{1}{2}q^2+c^2&=\tfrac{1}{2}(\varphi_\lambda^2+\varphi_n^2)+c^2=\tfrac{1}{2}\varphi_\lambda^2+\tfrac{3}{2}c^2=\tfrac{1}{2}q_{\max}^2=\tfrac{3}{2}c_*^2,\\ c^2&=\tfrac{1}{3}(q_{\max}^2-\varphi_\lambda^2)=c_*^2-\tfrac{1}{3}\varphi_\lambda^2,\end{aligned}\right\} \tag{30.40}$$

so that (30.39) may be written

$$\left.\begin{aligned}\vartheta-\alpha+\text{const}&=\int\frac{d(\varphi_\lambda)}{c}=\int\frac{d(\varphi_\lambda)}{\sqrt{\frac{1}{3}(q_{\max}^2-\varphi_\lambda^2)}}=\sqrt{3}\int\frac{d(\varphi_\lambda/q_{\max})}{\sqrt{1-(\varphi_\lambda/q_{\max})^2}}\\ &=\sqrt{3}\arcsin\left(\frac{\varphi_\lambda}{q_{\max}}\right)=\sqrt{3}\arcsin\left(\frac{\varphi_\lambda}{\sqrt{3c^2+\varphi_\lambda^2}}\right)=\sqrt{3}\arctan\left(\frac{\varphi_\lambda}{c\sqrt{3}}\right)\end{aligned}\right\} \tag{30.41}$$

This may be finally written in terms of (F) alone by noting from (30.39) that

$$\frac{\varphi_\lambda}{c}=\frac{\varphi_\lambda}{\varphi_n}=\frac{q\cos\alpha}{q\sin\alpha}=\frac{1}{\tan\alpha}=\sqrt{F^2-1}\,.$$

Consequently (30.41) reduces to

$$\left.\begin{aligned}&\vartheta(F)-\sqrt{3}\arctan\left(\sqrt{\frac{1}{3}(F^2-1)}\right)-\arctan\frac{1}{\sqrt{F^2-1}}+\text{const}\\ &=\vartheta(F)-\sqrt{3}\arctan\left(\sqrt{\frac{1}{3}(F^2-1)}\right)+\arctan\sqrt{F^2-1}=\vartheta-f(F)=\text{const},\end{aligned}\right\} \tag{30.42}$$

where $f(F)$ is the same Prandtl-Meyer function as given in (30.26) and Table 1. Therefore we have proven that the Riemann invariants are given by (30.36) and (30.26). In addition to the relation between f and F in (30.26) it is sometimes convenient to use one of the following:

$$\left.\begin{aligned}f(\alpha)&=\sqrt{3}\,\text{arc cot}(\sqrt{3}\tan\alpha)+\alpha-\frac{1}{2}\pi\\ &=f(F_*)=\sqrt{3}\arctan\sqrt{\frac{F_*^2-1}{3-F_*^2}}-\arctan\sqrt{\frac{F_*^2-1}{1-\frac{1}{3}F_*^2}}=\frac{\lambda+\mu}{2}\end{aligned}\right\} \tag{30.26'}$$

It now follows that a numerical solution can be obtained for the general problem in which both families of characteristics represent curved non-simple waves by carrying on a simultaneous finite-difference solution in the physical (x, z)-plane with (30.33), and in the hodograph (u, w)-plane by (30.26), (30.34), and (30.36). Almost any initial- or boundary-value data can be handled in this manner as long as the curve on which the data are given is not coincident with a characteristic curve. The solution cannot be obtained in the neighborhood of any portion of the boundary-value curve that happens to be tangent to any characteristic curve, because, as proven by (30.20), the solution is indeterminate for boundary-value data given on a characteristic. It is easily seen by this finite-difference method that the data along a smooth non-characteristic curve can only determine the solution inside the quadrilateral formed by the characteristic curves passing through its end points (Fig. 42) [see, e.g., PREISWERK (1938) or COURANT and FRIEDRICHS (1948)]. This well-known behavior of hyperbolic-type partial differential equations is most directly demonstrated by writing them in their normal or canonical form by transforming the coordinates to curvilinear axes which are the characteristic curves themselves. For example, PREISWERK (1938) transforms the equivalent of (30.23) onto the curvilinear characteristic-coordinate (λ, μ) system to obtain

$$\left.\begin{aligned} f(F_*) &= \tfrac{1}{2}(\lambda+\mu), \qquad \vartheta = \tfrac{1}{2}(\lambda-\mu), \\ \chi_{\lambda\mu} &= -K(\lambda,\mu)\,(\chi_\lambda+\chi_\mu), \\ K(\lambda,\mu) &= \frac{F_*^2(1-\frac{1}{2}F_*^2)}{\sqrt{3}\,(3-F_*^2)^{\frac{1}{2}}\,(F_*^2-1)^{\frac{3}{2}}}. \end{aligned}\right\} \tag{30.43}$$

This normal or canonical form is so useful in carrying out the finite-difference method of solution that the values of K have also been included in Table 1. It can be used in the following type of approximation, as indicated in Fig. 42 where (1, 3) are known values and (2, 4) are to be calculated,

$$\left.\begin{aligned} &\chi_\lambda = \frac{\chi_4-\chi_1}{\lambda_4-\lambda_1}, \qquad \chi_\mu = \frac{\chi_2-\chi_1}{\mu_2-\mu_1}, \\ &\left(-\frac{K_1+K_3}{2}\right)(\chi_\lambda+\chi_\mu) = \chi_{\lambda\mu} = \frac{(\chi_3+\chi_1)-(\chi_4+\chi_2)}{(\lambda_4-\lambda_1)(\mu_2-\mu_1)}. \end{aligned}\right\} \tag{30.43'}$$

Consequently, if the data were given on only one characteristic curve the method would fail since the values must be known on *both* characteristics, or on the non-characteristic curve s in Fig. 42, so that one can also write

$$\begin{aligned} \chi_s &= \chi_\lambda\lambda_s + \chi_\mu\mu_s = g(s), \\ \chi_n &= \chi_\lambda\lambda_n + \chi_\mu\mu_n = G(s). \end{aligned}$$

The numerical method of solution by finite differences following (30.43′) is known as the "lattice-point method" and replaces the original partial differential equation (30.43) by a set of linear algebraic equations. The other commonly used semi-graphical method of solving hyperbolic partial differential equations is called the network or "mesh method" and can be illustrated by writing (30.43) in the form

$$\begin{aligned} \Delta\chi_\lambda &= -K(\chi_\lambda+\chi_\mu)\,\Delta\mu, \\ \Delta\chi_\mu &= -K(\chi_\lambda+\chi_\mu)\,\Delta\lambda. \end{aligned} \tag{30.43''}$$

The average value at the center of each mesh formed by the characteristic network is used for the trial and error numerical calculation of each Δ increment. The increments are drawn tangent to the characteristic curves as indicated in Fig. 42. The simultaneous semi-graphical solution must be carried out in the physical plane as shown in Fig. 42 by using (30.33) and writing (30.39) and (30.41) in finite difference form.

As a further aid to numerical and graphical solutions it is useful to plot $f(F_*)$ from (30.26′) or Table 1 on the hodograph $(u/c_*, w/c_*)$-plane as shown in Fig. 42. The single curve defined by Table 1 may be drawn and then rotated by equal increments of $\Delta\vartheta$, or the construction may be accomplished entirely by graphical means as indicated in Fig. 42 by rotating the small circle upon the inner unit circle representing critical flow, while the outer maximum circle has a radius of $\sqrt{3}$ representing $q_{\max}/c_*$ from (30.29). This geometrical construction yields $f(F_*)$ since it is an epicycloid, as proven by PREISWERK (1938), or COURANT and FRIEDRICHS (1948, p. 262). All simple waves must follow the characteristic epicycloid in the hodograph plane because simple waves are defined by (30.37) which has been proven to have $f(F_*)$ as its integral. It can be shown that all streamlines corresponding to non-simple waves must lie within the corresponding characteristic epicycloids as indicated in Fig. 42, since the streamline must have

$$\left|\frac{1}{F_*}\left(\frac{dF_*}{d\vartheta}\right)_{\psi\,\text{const}}\right| \leq \tan\alpha \tag{30.44}$$

unless a finite discontinuity corresponding to a hydraulic jump (or shock wave in a gas) is formed.

Another useful aid in the hodograph graphical construction is the velocity ellipse, which is also drawn in Fig. 42. Wherever the velocity vector q touches the curve of the ellipse, it will be found that the major axis of the ellipse is in the direction of the tangent to the corresponding characteristic (either C_+ or C_-) in the physical plane because, as a consequence of (30.29) and (30.32), if we assume that $\vartheta = \alpha$ then

$$(w/c_*)^2 = F_*^2 \sin^2\alpha = F_*^2/F^2 = \tfrac{1}{2}(3 - F_*^2),$$

$$(u/c_*)^2 = F_*^2(1 - \sin^2\alpha) = \tfrac{3}{2}(F_*^2 - 1),$$

$$\tfrac{2}{3}(u/c_*)^2 + 1 = F_*^2 = 3 - 2(w/c_*)^2,$$

or

$$\tfrac{1}{3}(u/c_*)^2 + (w/c_*)^2 = 1 = [\tfrac{1}{3}(\varphi_\mu/c_*)^2 + (c/c_*)^2]_{C_-,\vartheta=0} = [\tfrac{1}{3}(\varphi_\lambda/c_*)^2 + (c/c^*)^2]_{C_+,\vartheta=0}. \tag{30.45}$$

This gives the velocity ellipse shown in Fig. 42 with a major axis of $\sqrt{3}$ and a minor axis of unity. The major axis is always at the Mach angle α with respect to the velocity vector q because we find from (30.28) and (30.32) that when $\alpha = \vartheta$

$$(w/c_*)^2 = (\varphi_n/c_*)^2 = (c/c_*)^2 = 1 - \tfrac{1}{3}(u/c_*)^2.$$

As in the previous case of unsteady one-dimensional flow over a flat bottom we can obtain very simple solutions for the case of simple waves. In this case there is an analogy between the (t, x) diagram and the (x, z) diagram [see, e.g., COURANT and FRIEDRICHS (1948)]. As before, the simple wave corresponds to having the characteristics in the (x, z)-plane of one family become straight lines, as in the examples shown in Fig. 43, so that $(q, \vartheta, \alpha, \eta)$ are all constant on the straight line $dz/dx = \text{const}$ in the physical plane. Therefore any given straight characteristic line has all of its properties determined by $f(F_*)$ from (30.26) and each of the straight lines in the physical plane maps onto a single point of the same single characteristic epicycloid in the hodograph plane. The characteristics of the other family remain curved in the physical plane and map in a unique continuous manner upon the corresponding characteristic epicycloid arcs in the hodograph plane. As before, in a simple wave these curved characteristics are not required for a numerical solution.

Common examples of simple-wave problems are shown in Fig. 43, and they always occur whenever a region of constant uniform properties adjoins a region having any variation in its properties, the two regions always being joined by a straight-line physical characteristic ($dz/dx = \text{const}$) as long as no finite discontinuities, corresponding to hydraulic jumps or shock waves, have been formed. These finite discontinuities correspond to an envelope of the straight characteristic lines that must form whenever the boundary-surface curves towards the oncoming flow, resulting in a flow compression or decrease of velocity and increase in water depth as indicated in Fig. 43. The solution is no longer single valued at, or downstream of the envelope so this region must be replaced by a hydraulic jump having a finite discontinuity.

If the local flow velocity and water depth are required only on the curved boundary itself, then neither family of characteristics has to be determined (except as a precaution to verify that no finite discontinuities have formed near the boundary due to flow compression). The solution on the curved boundary itself is given directly from Table 1 by simply measuring $f(F_*)$ as the value corresponding to (see Fig. 43)

$$f[F_*(\vartheta)] = f[F_{*\infty}] \pm \vartheta. \tag{30.46}$$

If this expression becomes zero it signifies that the supercritical flow has been compressed to critical speed and a detached hydraulic jump can occur as in Fig. 43.

Whenever disturbance waves enter along both families, either due to another boundary or by reflection from a hydraulic jump, as in Fig. 41, then the mixed region contains non-simple waves, and only a numerical solution, similar to the ones discussed in conjunction with (30.43), can yield the exact solution. However, an approximate solution for the particular cases shown in Fig. 41 can be obtained by approximating the curved characteristics in the non-simple region by means of simple-wave straight characteristic lines. The geometrical construction assumes that the curved boundary wall of the nozzle can be replaced by a series of straight chord lines which each have the same magnitude of $\Delta\vartheta$ at every corner, as depicted in Fig. 44. At each expansion corner it is assumed that the centered simple wave (corresponding to a portion of the complete Prandtl-Meyer expansion, f) can be approximated by a single physical characteristic that is the average of the actual expansion fan of characteristics. This is the (dz/dx) straight line that is normal to the midpoint of the $\Delta\vartheta$ epicycloid arc representing the expansion-angle change at this corner, as shown in Fig. 44. Similarly, the compression corner that turns into the flow is represented by the single compression simple wave that is normal to the midpoint of the $\Delta\vartheta$ epicycloid arc representing the compression angle change at this corner. It will be shown that the angle of this single average compression wave is actually the correct limiting value for a weak hydraulic jump. The geometrical construction is carried out in the manner indicated in Fig. 44. Whenever a streamline crosses one of these finite amplitude construction characteristics the flow is assumed to bend through the $\Delta\vartheta$ associated with the finite corner bend which supposedly produced this single finite wave. The corresponding construction in the hodograph plane transfers to the epicycloid arc that is normal to the single finite wave in the physical plane as shown in Fig. 44.

Also shown in Fig. 44 are the geometric constructions required for the reflection of these simple finite waves in the physical plane from either solid boundaries, or from free boundaries with constant water depth. In the reflection from a solid boundary the original boundary slope is again attained by the velocity vector after passing through the reflected wave which has the same strength for flow deflection as the original oncoming finite simple wave. In the hodograph plane the streamline has gone from one family of epicycloids to the other, ending at the same value of ϑ. The completed solution for the flow inside a channel

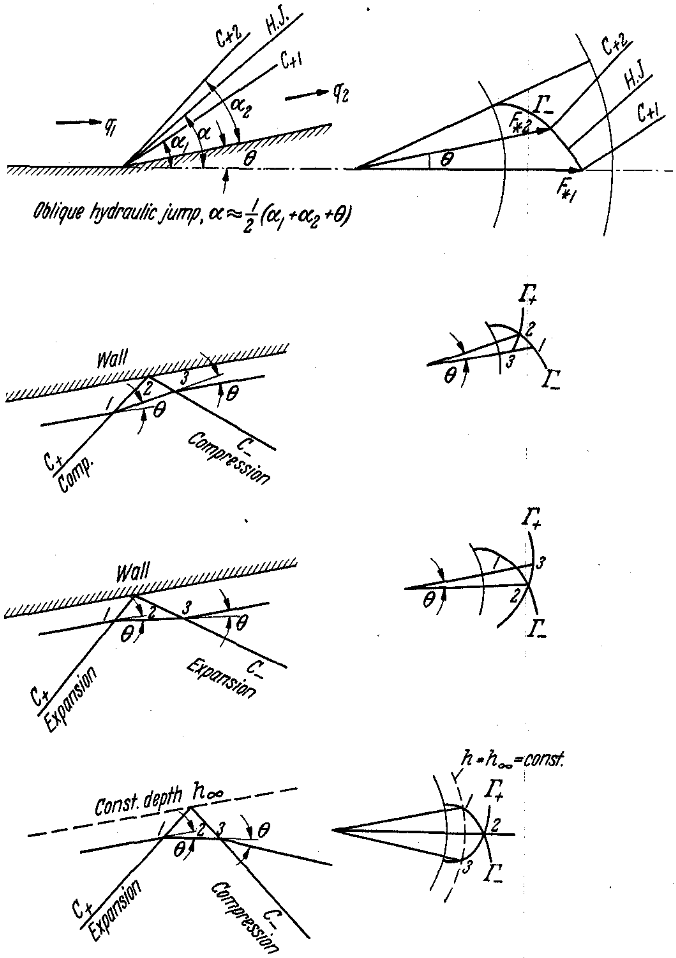

Fig. 44. Reflection of compression and expansion waves.

of varying width having supercritical flow ($F > 1$) is presented in Fig. 41. For additional details and aids on the graphical constructions see PREISWERK (1938). As another example in Fig. 44, consider the reflection from a free jet, hydraulic jump, or any constant-water-depth free boundary, which must occur in such a manner that the same water depth is maintained after passing through the reflected wave which is not only on the opposite family of epicycloid arcs, but now must have the negative algebraic strength of the flow deflection of the original oncoming wave; consequently, the value of $\Delta\vartheta$ is exactly doubled after passing through the reflected wave. That is to say, unlike the ordinary reflection from a solid boundary, the reflection from a constant-depth free boundary results in the opposite type of wave, an expansion wave becoming a compression wave and vice versa.

In conclusion it must be noted that this two-dimensional steady-flow analysis is only valid for a flat horizontal bottom, as was already shown by (29.3) for the linearized equations. If the bottom slope varies, then the Riemann invariants do not exist, simple waves do not occur, and the numerical solution is much more complicated. However, there is an even more important criterion that must be satisfied before *any* of the solutions given so far can be applicable. This is the necessary requirement that all the perturbation quantities involved $(u-U, w, \eta)$ must be sufficiently small so that it is not necessary to introduce the second-order terms from Sect. 31. A satisfactory evaluation of this criterion, at least for F not too near unity, can be obtained by comparing the solutions of the non-linear equation (30.17) with the linearized equation (29.23) or (29.32). As is well-known in gas dynamics, and is apparent by inspection, (29.23) and (29.32) are not satisfactory for F approaching unity since additional terms must then be retained. For example, on the boundary profile itself, (29.3) for a flat horizontal bottom must include the additional term $3F^2(\varphi_x/U)\varphi_{xx}$, which corresponds to the "transonic approximation" of the gas dynamic equation (with a specific heat ratio $\gamma=2$) in the limit as F approaches unity. However, for the solution of the steady flow everywhere about a two-dimensional profile it may be necessary to use

$$(1-F^2)\,\varphi_{xx}+\varphi_{zz}=F^2\left[3\,\frac{\varphi_x}{U}\,\varphi_{xx}+2\,\frac{\varphi_z}{U}\,\varphi_{xz}\right] \tag{30.47}$$

since (29.29) indicates that far from the profile $w/U=\varphi_z/U\sim 1/z$, whereas $(u-U)/U=\varphi_x/U\sim 1/z^2$. In any case any radical increase in the order of magnitude of any perturbation term immediately indicates that the second-order terms discussed in Sect. 31 must be introduced, since the non-linear equation (28.1) and all the preceding results are based only on the first-order terms of the shallow-water theory.

γ) One-dimensional, steady, open-channel hydraulics and the hydraulic jump. see errata The reltations given in Eqs. (30.27) (30.28) and (39.29), and shown in Fig. 41, can be used in what is commonly known as the steady "one-dimensional" hydraulics of open-channel flow. Here we assume that even though the channel width $b(x)$ is varying, still the values of $q(x)$ and $\eta(x)$ do not depend upon z and therefore do not vary on any given cross-section. In conjunction with the steady "one-dimensional" concept it is necessary that $w\approx 0\approx\vartheta$. Consequently the basic equations to be used for a flat horizontal bottom are given by $q(x)=u(x)$ in (30.27), (30.28) and (30.29), and, in addition, by the "one-dimensional" continuity equation

$$b(x)\,h(x)\,u(x)=A(x)\,u(x)=Q=\left(\frac{\text{meters}^3}{\text{sec}}\right)=\text{const}, \tag{30.48}$$

where, from Fig. 41, $h(x)=h_0+\eta(x)=A(x)/b(x)$.

The validity of the "one-dimensional" assumption can be considerably in error if $b'(x)$ is large since it is obvious that in this case w or ϑ cannot be small. However, the "one-dimensional" approximation gives surprisingly good numerical values, even in supercritical flow *if* the channel is well designed as in Fig. 41 so as to maintain the flow as uniform as possible. However in supercritical flow the velocity over any cross-section remains uniform only near the design Froude number (F). Preiswerk (1938) gives the calculated and measured water depths in a Laval-type nozzle (the same one duplicated in Fig. 41) at various supercritical Froude numbers $(F>1)$. His results indicate that "one-dimensional" hydraulics gives a satisfactory approximation, having an error probably less than 10 %, even for critical or supercritical flow. This method should be especially useful for subcritical flow since the more exact numerical solution is now very difficult to obtain because the simple method of characteristics is no longer applicable.

The most useful, and obviously the most accurate, application of "one-dimensional" hydraulics is to the constant-width rectangular-cross-section open-channel flow. In this application the friction effect of the vertical channel walls generally has a greater effect on the variation of $q(x, y, z)$ than would any of the more exact terms of the complete first-order shallow-water equations (28.1) which have been derived on the assumption of negligible viscosity effects. Consequently the

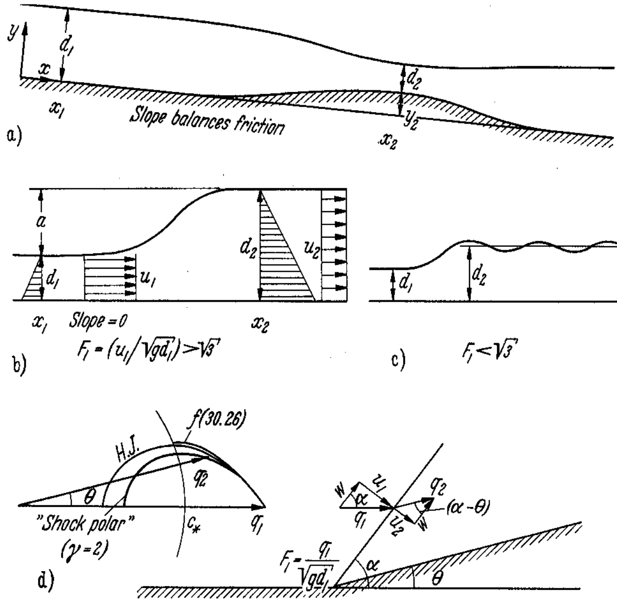

Fig. 45a—d. (a) One-dimensional flow over a sloping bottom, $|dy/dx| \ll 1$ so centrifugal force negligible. (b) Normal hydraulic jump. (c) Undulating hydraulic jump. (d) Oblique hydraulic jump with $w = w_1 = w_2$.

"one-dimensional" assumption that $q = u(x)$ provides a satisfactory approximation for the constant width (b), rectangular-cross-section, vertical-wall channel having $A(x) = b\,d(x)$. Even more important, this open-channel flow analysis may be further generalized, with but little additional difficulty, to apply to a bottom slope varying also with x. The "one-dimensional" continuity equation (30.48) then becomes

$$u(x)\,d(x) = \frac{Q}{b} = \text{const}, \tag{30.49}$$

where $d(x)$ is measured vertically from the varying bottom as shown in Fig. 45. The generalization of the Bernoulli equation (30.27) to include extraneous head losses (h_L), other than those due to friction, and local variations in the bottom contour $y(x)$, as shown in Fig. 45, may be written as

$$\left.\begin{aligned}(\text{specific energy}) = \left(\frac{\text{kg meters}}{\text{kg}}\right) = (\text{meters}) &= d(x) + \frac{u^2(x)}{2g} + y(x) + h_L(x)\\ &= d(x) + \frac{(Q/b)^2}{2g\,d^2(x)} + y(x) + h_L(x) = \text{const}.\end{aligned}\right\} \tag{30.50}$$

This assumes that in steady flow the work of gravity, through the known average slope of the flow, is wholly spent in overcoming the frictional resistance.

Another relation, that is necessary for calculating the sudden additional head loss h_L in hydraulic jumps or other discontinuous flow phenomenon, is given by the impulse-momentum relation [see KELVIN (1886), RAYLEIGH (1914), or BAKHMETEFF (1932)],

$$\left.\begin{aligned}(\text{specific momentum}) = \left(\frac{\text{meters}^3}{\sec^2}\right) &= \frac{1}{2}\, g\, d^2(x) + d(x)\, u^2(x)\\ &= \frac{1}{2}\, g\, d^2(x) + \frac{(Q/b)^2}{d(x)}\end{aligned}\right\} \tag{30.51}$$

which is constant across the hydraulic jump over a flat horizontal bottom as shown in Fig. 45.

Eq. (30.50) with zero additional head loss ($h_L = 0$), gives the "one-dimensional" solution for the open-channel flow that has no finite discontinuities in the flow itself, and either has the hydraulic frictional resistance exactly balanced by the given average slope for steady flow (so that if $y = 0$ the surface slope is parallel to the bottom), or the hydraulic frictional resistance can be approximated by the Chézy formula for the case of varying open-channel flow [see BAKHMETEFF (1932) or STOKER (1957)]. A useful concept for nearly all solutions is the definition of the critical depth d_*, which corresponds to our previous definition of critical flow velocity in (30.28), that is, with $w \approx 0 \approx \vartheta$ we assume that

$$\left.\begin{aligned}u_* = c_* = \sqrt{\frac{2}{3}\, g\, h_0} = \frac{u_{\max}}{\sqrt{3}}, \qquad F_* = 1 = F,\\ d_* = \frac{2}{3}\, h_0 = \frac{c_*^2}{g} = \left[\frac{(Q/b)^2}{g}\right]^{\frac{1}{3}}.\end{aligned}\right\} \tag{30.52}$$

The last relation for d_* can be obtained either directly from (30.27), or by substituting the expression $u_* = \sqrt{\frac{2}{3} g h_0}$ into (30.50) with y and h_L both zero. Also from (30.29) we have

$$\left.\begin{aligned}\frac{d}{h_0} = \frac{2}{2 + F^2}, \qquad F_*^2 = F^2\left(\frac{c}{c_*}\right)^2 = \frac{3F^2}{2 + F^2},\\ F^2 = \frac{u^2}{g\, d} = \frac{(Q/b)^2}{g\, d^3} = \left(\frac{d_*}{d}\right)^3.\end{aligned}\right\} \tag{30.53}$$

As an example, if we apply Eqs. (30.50) and (30.52) to determine the flow relations between stations (1) and (2) in Fig. 45a we obtain, since h_L is generally negligible for a smooth variation in y_2,

$$\frac{d_2}{d_*} + \frac{1}{2}\left(\frac{d_*}{d_2}\right)^2 = \frac{d_1}{d_*} + \frac{1}{2}\left(\frac{d_*}{d_1}\right)^2 - \frac{y_2}{d_*}$$

as a satisfactory approximation for "one-dimensional" hydraulics, at least as long as y_2/d_* is sufficiently small. It is interesting to note that here is another resemblance to gas-dynamics behaviour since

$$\frac{y_2 + d_2}{d_*} < \frac{d_1}{d_*} > 1 \qquad \text{for } F_1 < 1,$$

$$\frac{y_2 + d_2}{d_*} > \frac{d_1}{d_*} < 1 \qquad \text{for } F_1 > 1.$$

As another example, if we consider the hydraulic jump shown in Fig. 45b, now we find that a solution can only be obtained by using the impulse-momentum relation (30.51), thereby proving that the discontinuous change occurring in a hydraulic jump must result in a head loss. If the bottom slope is negligible, as indicated in Fig. 45b, then the impulse-momentum relation (30.51) may be written, with $Q/b = u_1 d_1 = u_2 d_2$, in the following manner, first given by RAYLEIGH (1914):

$$\left.\begin{aligned} \frac{1}{2} g\, d_1^2 + d_1 u_1^2 &= \frac{1}{2} g\, d_2^2 + \frac{(Q/b)^2}{d_2} = \frac{1}{2} g\, d_2^2 + \frac{(u_1 d_1)^2}{d_2}, \\ F_1^2 = \frac{(Q/b)^2}{g\, d_1^3} &= \frac{u_1^2}{g\, d_1} = \frac{1}{2} \frac{\left(\frac{d_2}{d_1}\right)^2 - 1}{1 - \frac{d_1}{d_2}} = \frac{1}{2} \frac{d_2}{d_1}\left(1 + \frac{d_2}{d_1}\right), \end{aligned}\right\} \tag{30.54}$$

or, if we let the actual rise in water level be $a = d_2 - d_1$,

$$F_1 = \frac{u_1}{\sqrt{g\, d_1}} = \left[1 + \frac{3}{2}\frac{a}{d_1} + \frac{1}{2}\left(\frac{a}{d_1}\right)^2\right]^{\frac{1}{2}}. \tag{30.55}$$

where

$$1 + \frac{a}{d_1} = \frac{d_2}{d_1} = \frac{1}{2}\left[\sqrt{1 + 8\frac{(Q/b)^2}{g\, d_1^3}} - 1\right] = \frac{1}{2}\left[\sqrt{1 + 8F_1^2} - 1\right]. \tag{30.56}$$

Similarly, (30.54) can also be solved for

$$F_2^2 = \frac{(Q/b)^2}{g\, d_2^3} = \frac{u_2^2}{g\, d_2} = \frac{1}{2}\frac{d_1}{d_2}\left(1 + \frac{d_1}{d_2}\right) = F_1^2\left(\frac{d_1}{d_2}\right)^3. \tag{30.57}$$

Eqs. (30.54) and (30.57) may be multiplied together to yield

$$u_1 u_2 = \frac{1}{2} g\,(d_1 + d_2) = c_{1*}^2\left[\frac{3}{2 + F_1^2}\right]\left[\frac{1 + \sqrt{1 + 8F_1^2}}{4}\right] < c_{1*}^2 > c_{2*}^2. \tag{30.58}$$

The last inequality in (30.58) is obtained from (30.50) and (30.52) by noting that in any finite hydraulic jump the head loss must also be finite, so that $h_L = h_{0_1} - h_{0_2} > 0$ and (30.50) must be written as

$$\left.\begin{aligned} d_1^2 + \frac{u_1^2}{2g} &= d_2^2 + \frac{u_1^2}{2g}\left(\frac{d_1}{d_2}\right)^2 + h_{0_1} - h_{0_2}, \\ h_L = h_{0_1} - h_{0_2} &= \frac{1}{2} F_1^2 \frac{(1 - d_1/d_2)^3}{1 + d_2/d_1} d_2 \\ &= \frac{1}{4} F_1^2 \left[\sqrt{1 + 8F_1^2} - 1\right] \frac{(1 - d_1/d_2)^3}{1 + d_2/d_1} d_1. \end{aligned}\right\} \tag{30.59}$$

Thus (30.52) and (30.59) give the total head ratio, and therefore the critical speed ratio, as

$$\left.\begin{aligned} \frac{h_{0_2}}{h_{0_1}} &= \left(\frac{c_{2*}}{c_{1*}}\right)^2 = 1 - \frac{h_L}{h_{0_1}} = \frac{d_{2*}}{d_{1*}} \\ &= \frac{(\sqrt{1 + 8F_1^2} - 1)^3 + 4F_1^2}{(\sqrt{1 + 8F_1^2} - 1)^2\,(2 + F_1^2)} < 1. \end{aligned}\right\} \tag{30.60}$$

Consequently there is no direct analogy between finite hydraulic jumps and gas-dynamic shock waves, as was pointed out by PREISWERK (1938), since in gas dynamics the well-known Prandtl relation for normal shock waves gives $u_1 u_2 = c_*^2$, and c_* is constant through the shock wave [see, e.g., COURANT and FRIEDRICHS (1948, p. 146)]. The equations are similar only for the limiting case as the hydraulic jump vanishes so that $F_1 = F_2 = 1$, $d_2 = d_1$, and $h_{0_2} = h_{0_1}$. However this limiting process corresponds to the isentropic potential-flow case where there is an analogy for small perturbations over a flat horizontal bottom, as previously discussed. Also, as indicated by (30.59) the head loss and variation in c_* could be neglected until the third-order terms become important, so that for F_1 near unity the first- and second-order terms of the hydraulic-jump relations correspond to the gas-dynamic shock-wave relations having a specific heat ratio $\gamma=2$. However, this is identical to the known fact that weak shock waves may be considered isentropic to the third order of approximation; consequently the hydraulic analogy to compressible gas dynamics exists only for small perturbations in potential flow.

There is no direct analogy between the finite hydraulic jump and the gas-dynamic shock wave because the hydraulic jump has a head loss that must be included in the specific-energy equation (30.50). This head loss results in a loss of kinetic energy that is no longer available as flow energy since it is converted into an insignificant temperature rise in the water itself. In the gas dynamics energy equation the entropy increase through a shock wave of course corresponds to a loss of kinetic energy, but this is converted, through the increase of the temperature of the gas, into an adiabatic enthalpy increase that maintains constant flow energy through the shock wave [see, e.g., COURANT and FRIEDRICHS (1948, p. 125)]. The most unusual effect of this loss in flow energy (or h_L) in the hydraulic jump is revealed in (30.58) which shows that the flow velocity downstream of a hydraulic jump is always less than in the corresponding gas dynamics case, which maintains c_* constant so that $u_1 u_2 = c_*^2$. For example, in the gas-dynamic case when $F_1 \to \infty$, then $u_1/c_* \to \sqrt{3}$ (for $\gamma=2$), and therefore $u_2/c_* \to 1/\sqrt{3}$. However, in a hydraulic jump (30.58) shows that $u_2/c_* \to 0$ when $F_1 \to \infty$ (or $u_1/c_* \to \sqrt{3}$).

The experimental investigations by BAKHMETEFF (1932) have shown that the hydraulic jumps in a horizontal rectangular channel are in excellent agreement with the predictions of the "one-dimensional" hydraulic equations (30.54) through (30.60). BAKHMETEFF found that depth increases as high as $10 d_1$ were in excellent agreement with (30.56). However, he found that for Froude numbers of the oncoming flow less than $\sqrt{3}$ (i.e., $F_1 < \sqrt{3}$) the profile of the normal hydraulic jump developed undulations, and the relative length of transition became indeterminate because the undulating surface made the region of parallel flow increasingly remote from the start of the wave front, as indicated in Fig. 45c. It is interesting to note that $F_1 = \sqrt{3}$ corresponds to the maximum absolute elevation that a hydraulic jump can reach with a given h_0 (although there is no limit to d_2/d_1), since (30.53) and (30.56) may be combined to give

$$\frac{d_2}{h_{0_1}} = \frac{1}{2+F_1^2}\left(\sqrt{1+8F_1^2}-1\right) \leqq \frac{4}{5}$$

which attains its maximum elevation of $\frac{4}{5} h_{0_1}$ above the channel bottom only for $F_1 = \sqrt{3}$; at this condition we have

$$\frac{d_2}{d_1} = 2, \quad F_2^2 = \frac{3}{8}, \quad \frac{h_L}{d_1} = \frac{1}{8}, \quad \frac{h_{0_2}}{h_{0_1}} = \frac{19}{20}, \quad u_1 u_2 = \frac{9}{10} c_{1*}^2 .$$

This shows that for all the undulating hydraulic jumps ($F_1 < \sqrt{3}$) the change in total head is less than 5 %; consequently these jumps can be approximated by the isentropic, potential-flow relations. This is of great aid in calculating the slant or oblique hydraulic jumps, as shown in Fig. 45d, since the characteristic epicycloid values of $f(F_*)$, as given in Table 1, may be used in the manner indicated in (30.46) to approximate the change in F_*, F, or η upon turning through an angle ϑ by means of an oblique hydraulic jump whenever $d_2/d_1 \to 1$. The comparison between the value given by $f(F_1)$ in Table 1 for compression to $F_2 = 1$, is compared with the exact values for the corresponding oblique hydraulic jumps in Fig. 46. It is seen that, although the gas-dynamic shock wave is not a satisfactory approximation for $F_1 > \sqrt{3}$, still the isentropic potential relation $f(F_1)$ provides an excellent approximation for much greater values of F_1 since the criterion for oblique hydraulic jumps is that the flow component normal to the discontinuity satisfy $F_1 \sin\alpha < \sqrt{3}$.

The exact relations for the oblique hydraulic jump are given by PREISWERK (1938) and can be obtained by simply adding the same velocity component ($w = w_1 = w_2$) tangent to both faces of the hydraulic jump as shown in Fig. 45d.

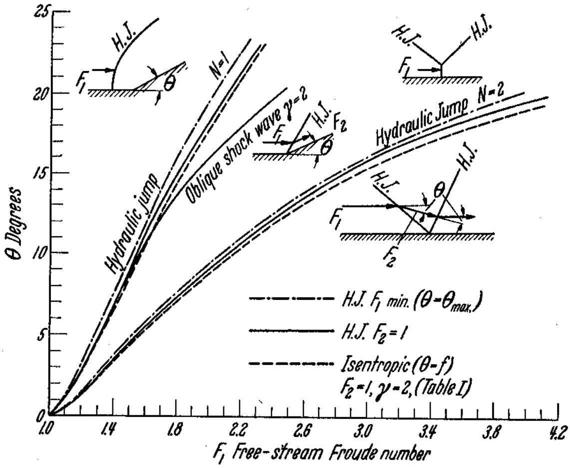

Fig. 46. Maximum flow deflection ($n=1$), and reflection ($n=2$).

This results in the following equations (which reduce to the preceding ones for a normal hydraulic jump by simply letting $\vartheta \to 0$ and $\alpha \to \pi/2$):

$$\left.\begin{aligned}
F_1^2 &= \frac{1}{2\sin^2\alpha}\,\frac{\tan\alpha}{\tan(\alpha-\vartheta)}\left[1+\frac{\tan\alpha}{\tan(\alpha-\vartheta)}\right] = \left[\frac{u_1/\sqrt{g\,d_1}}{\sin\alpha}\right]^2,\\
F_2^2 &= F_1^2\left(\frac{d_1}{d_2}\right)^3\left[\frac{\sin\alpha}{\sin(\alpha-\vartheta)}\right]^2 = \left[\frac{u_2/\sqrt{g\,d_2}}{\sin(\alpha-\vartheta)}\right]^2,\\
\frac{d_2}{d_1} &= \frac{\tan\alpha}{\tan(\alpha-\vartheta)} = \frac{1}{2}\left[\sqrt{1+8F_1^2\sin^2\alpha}-1\right] = \frac{u_1}{u_2},\\
\tan\vartheta &= \frac{\tan\alpha\left[\sqrt{1+8F_1^2\sin^2\alpha}-3\right]}{2\tan^2\alpha-1+\sqrt{1+8F_1^2\sin^2\alpha}},\\
\sin\alpha &= \frac{1}{F_1}\sqrt{\frac{1}{2}\left(\frac{d_2}{d_1}\right)\left(1+\frac{d_2}{d_1}\right)} = \frac{u_1/\sqrt{g\,d_1}}{F_1},\\
\sin(\alpha-\vartheta) &= \frac{1}{F_2}\sqrt{\frac{1}{2}\left(\frac{d_1}{d_2}\right)\left(1+\frac{d_1}{d_2}\right)} = \frac{u_2/\sqrt{g\,d_2}}{F_2}.
\end{aligned}\right\} \qquad (30.61)$$

The last two equations in (30.61) clearly show how the oblique hydraulic jump approaches the same value as given by the isentropic, potential relation $f(F_1)$ at *any* value of F_1 as long as $d_2/d_1 \to 1$, since they reduce to the isentropic, potential characteristic curve given by (30.31) whenever $\vartheta \to 0$ and $d_2/d_1 \to 1$. As a matter of fact, as previously mentioned, (30.61) shows that the oblique hydraulic jump angle (α) can be approximated as in Fig. 44 by

$$\alpha = \frac{1}{2}\left(\arcsin F_1^{-1} + \arcsin F_2^{-1} + \vartheta\right) + O\left(\frac{d_2}{d_1}-1\right)^2 + O(\vartheta^3), \qquad (30.62)$$

that is, by taking α as defined by the average line between the two characteristics in the physical plane, or the line normal to the midpoint of the corresponding characteristic epicycloid arc in the hodograph plane (see Fig. 44). The close approximation of the characteristic epicycloid as given by f from (30.26) and Table 1, to the oblique hydraulic-jump relations (30.61) is strikingly illustrated when they are both plotted in the hodograph plane as in Fig. 45d [see also PREISWERK (1938)]. The trace of the endpoint of the velocity vector for the oblique shock wave of gas dynamics, generally called the "shock polar", is also shown in Fig. 45d for a specific heat ratio $\gamma = 2$.

As long as f from (30.26) is in close agreement with the oblique hydraulic jump relations (30.61), then the problems involving the interaction and reflection of hydraulic jumps can be closely approximated by the same procedure as detailed previously for the characteristic epicycloids involving compression waves (see Figs. 43 and 44). Whenever the required flow deflection ϑ is greater than that provided by the epicycloid passing through F_1, as shown in Fig. 43, then subcritical flow follows the curved or normal hydraulic jump as indicated by $N = 1$ in Fig. 46. Similarly, $N = 2$ defines the maximum flow reflection angle (ϑ) that can occur without ending in subcritical flow with a curved or normal hydraulic jump. In both cases two curves are shown for the oblique hydraulic jump: one shows the turning angle ϑ that will make the flow critical $(F_2 = 1)$, and the other one is the maximum possible turning angle $\vartheta_{\max}$ for any oblique hydraulic jump at the given value of F_1. The latter always produces subcritical flow $(F_2 < 1)$ as indicated in Fig. 43.

All of the preceding results primarily hold for hydraulic jumps in rectangular cross-section channels with a nearly horizontal bottom. BAKHMETEFF (1932) shows experimentally the various effects of steepening bottom slopes. He also generalizes (30.51) so that it will apply to any constant cross-section shape. However, it must be noted that our Eq. (29.3) shows conclusively that (30.51) which completely neglects the w velocity component, cannot be applicable to channel walls that are not nearly vertical. Sloping sides on a channel would increase the vertical velocity gradients, make a normal hydraulic jump impossible, and induce unsteady vortex motions.

It must also be noted that all of the preceding results are valid only for relatively small bottom slopes, as indicated by the direct comparison of (30.50) and (30.51) with (28.1) and (29.3). When the flow is rapidly varying because of large changes in the bottom slope, then the change in surface profile curvature is so pronounced that the pressure variation can no longer be considered as hydrostatic. For example, over the spillway of a dam the centrifugal force due to the streamline curvature can actually exceed the hydrostatic pressure, thereby leading to a pressure less than atmospheric resulting in flow separation or violent oscillations. At present spillway design is based on semi-empirical methods or model tests since no satisfactory mathematical analysis is available.

31. Higher-order theories and the solitary and cnoidal waves. It will now be shown that many of the preceding methods and results based on the shallow-water approximation are valid only if the local variations in water depth are not too large, and the average or undisturbed water depth is sufficiently small. The first requirement implies that the solutions of the first-order nonlinear shallow-water equations (28.1) do not greatly differ (at least for Froude numbers not near unity) from the linearized solutions given by (29.3) or (29.7). The second requirement essentially demands that the depth h be much less than the effective wavelength λ in any application, say $\frac{h}{\lambda} < \frac{1}{10}$, in order to reduce the effects associated with the infinitesimal-wave approximation.

As already discussed, the infinitesimal-wave approximation predicts that the fluid particle motion varies with the distance below the free surface, and also that the propagation velocity depends upon the wavelength, as shown in Sect. 15. There it was proved that the velocity defined by

$$c = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi h}{\lambda}\right)} = \sqrt{gh}\left[1 - \frac{1}{6}\left(\frac{2\pi h}{\lambda}\right)^2 + \cdots\right] \tag{31.1}$$

can only be considered a phase velocity while the actual rate of propagation of energy is associated with the group velocity defined by

$$c - \lambda\frac{dc}{d\lambda} = \frac{1}{2}c\left(1 + \frac{4\pi h/\lambda}{\sinh 4\pi h/\lambda}\right) = c\left[1 - \frac{1}{3}\left(\frac{2\pi h}{\lambda}\right)^2 + \cdots\right] \tag{31.2}$$

[see also LAMB (1932, p. 381)]. Any such variation will directly interfere with the applicability of the shallow-water results. However, as long as $\frac{h}{\lambda} < \frac{1}{10}$ it is seen that the phase velocity and the group velocity are both satisfactory approximations to the shallow-water first-order result that $c = \sqrt{gh}$ and is independent of the effective wave length.

This means that if small-scale model tests are used to simulate results appropriate to the shallow-water theory, then the undisturbed water depth should be less than $\frac{1}{10}$ the principal model dimensions. Consequently, if models less than 10 cm in effective dimensions are used, the depth of test water should be less than 1 cm, so the capillary ripples produced by surface tension must be considered. As shown in Sects. 15 and 24 the effect of the surface tension T is to increase the phase velocity for the short wavelength capillary ripples so that (31.1) is replaced by

$$c = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\varrho}\right)\tanh\frac{2\pi h}{\lambda}}. \tag{31.1'}$$

For ordinary water (at 20° C, $T = 72.8$ dynes/cm, $\varrho = 0.998$ gm/cm³) this gives the interesting result that both the phase velocity and the group velocity are closely approximated by $\sqrt{gh}$ for all $\lambda > 2$ cm if $h \approx \frac{1}{2}$ cm. However, in any small-model tests the surface wave patterns formed by the capillary ripples must be ignored since they are short-wavelength surface waves that are never in accord with the long-wavelength shallow-water theory.

Except for the section on hydraulic jumps the preceding shallow-water results have all been based entirely on (28.1), the first approximation to shallow-water theory, and this will now be shown to be limited to relatively small wave amplitudes even though the complete nonlinear equation (28.1) be used, and even though the bottom surface be flat and horizontal. The second approximation to shallow-water theory will be shown to immediately yield particular solutions corresponding to continuous permanent wave profiles of finite amplitude that can be propagated without a change in form or shape if viscosity effects are neglected. These permanent, finite-amplitude wave forms are the cnoidal waves discovered by KORTEWEG and DE VRIES (1895) which reduce, in the limiting case of essentially infinite wavelength, to the solitary wave of RUSSELL (1837, 1844) which was first analyzed theoretically by BOUSSINESQ (1871, 1872) and RAYLEIGH (1876).

The second approximation to shallow-water theory will show that the limitation of the nonlinear first approximation to relatively small amplitudes is primarily due to the fact that the variation in the vertical velocity cannot be neglected as the wave amplitude is increased. This of course invalidates even the rectangular channel hydraulic analogy to compressible gas flow, since, as previously discussed, the principal assumption of the hydraulic analogy is that the vertical acceleration be negligible.

The third approximation to shallow-water theory will then be presented to obtain new relations which will predict the limiting heights of the continuous finite-amplitude steady-state wave forms and give, for the first time, the complete second approximation to the cnoidal and solitary waves. It will be found that the pressure is no longer hydrostatic, thereby violating the remaining principal assumption of the hydraulic analogy and the ordinary classical shallow-water theory.

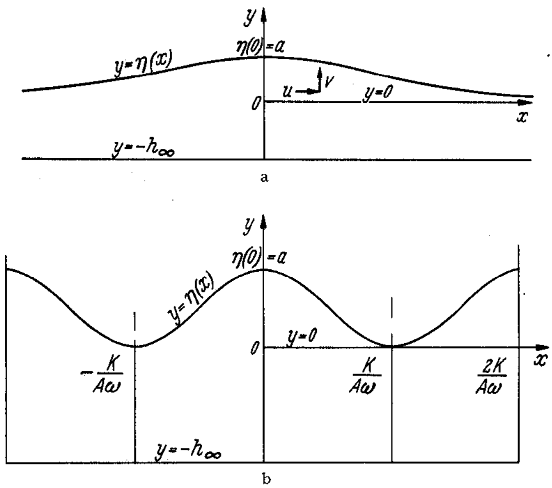

Fig. 47a and b. (a) Solitary wave over a flat horizontal bottom. (b) Cnoidal wave $\eta(x) = a\,\mathrm{cn}^2(A\omega x, k)$.

α) The first and second approximations to the cnoidal and solitary waves. We will now extend the perturbation method of FRIEDRICHS (1948), which was used to derive the nonlinear first-order approximation (28.1) to shallow-water theory, to obtain the second and higher orders of approximations for the special case of the steady-state propagation of a wave independent of z and t over a flat horizontal bottom described by $y = -h_\infty = \text{const}$ as in Fig. 47.

First we will show that the only steady-state finite-amplitude solution of the first-order equation (28.2) is $y^{(0)} = \eta_0 = \text{const}$ and $u^{(0)} = u_0 = \text{const}$. This is most easily proved by substituting the solution of the zeroth-order terms in (10.24) for steady water flow over a flat horizontal bottom, namely

$$u^{(0)} = u^{(0)}(x), \quad v^{(0)} = 0, \quad p^{(0)} = 0, \quad \eta^{(0)} = \eta^{(0)}(x), \tag{31.3}$$

into the first-order terms in (10.27) to obtain

$$u^{(0)} = u_0 = \text{const}, \quad v^{(1)} = 0, \quad p_y^{(0)} = -\varrho g, \quad \eta^{(0)} = \eta_0 = \text{const} \tag{31.4}$$

since $\eta_x^{(0)} = 0 = p_x^{(0)}$. Consequently the only finite-amplitude first-order steady-state solution must have $\eta_x^{(0)} = 0$, which would permit only the hydraulic jump as a solution since $\eta_x^{(0)} = 0$ and $u^{(0)} = \text{const}$ on each side of the finite discontinuity defining the hydraulic jump. This is in agreement with the well-known fact that the gas-dynamics equation or (28.2), predicts that any finite amplitude disturbance must form a finite discontinuity which is a shock wave, or hydraulic jump [see, e.g., LAMB (1932, pp. 278, 481)]. However, the second-order approximation of shallow-water theory (10.33) does yield a permanent finite-amplitude, steady-state wave profile that does not form a discontinuity. These are called the cnoidal waves, discovered by KORTEWEG and DE VRIES (1895), and the solitary waves of RUSSELL (1837, 1844), BOUSSINESQ (1871, 1872), and RAYLEIGH (1876). In order to obtain the higher-order approximations and limiting heights of these waves, it is more convenient to use exactly the same non-dimensional variables introduced by FRIEDRICHS (1948), and also used by KELLER (1948), namely,

$$\left.\begin{aligned}
&\varepsilon=\omega^2 h^2, \quad \alpha=\omega x, \quad \beta=y/h, \quad H=h_\infty/h,\\
&u(\alpha,\beta)=u(x,y)/\sqrt{g h}, \qquad v(\alpha,\beta)=\frac{v(x,y)}{\sqrt{g h}}\,\omega h,\\
&Y(\alpha)=\eta(x)/h, \quad \pi(\alpha,\beta)=p(x,y)/\varrho g h,\\
&Y(\alpha)=Y^{(0)}+\varepsilon Y^{(1)}+\varepsilon^2 Y^{(2)}+\cdots,\\
&\eta(x)=h Y^{(0)}+\omega^2 h^3 Y^{(1)}+\omega^4 h^5 Y^{(2)}+\cdots,
\end{aligned}\right\} \tag{31.5}$$

the only difference in notation being that x and y are now defined as in Fig. 47; consequently, the flat horizontal bottom is given by $y=-h_\infty$ or $\beta=-h_\infty/h=-H$, and the expansion parameter $\varepsilon=\omega^2 h^2$ is used as defined in (10.23), with (31.5) replacing (10.21).

Introducing the transformation defined by (31.5) into (31.4) and into the corresponding equivalent of (10.33), we obtain

$$\left.\begin{aligned}
&v^{(0)}=0=v^{(1)}\\
&u^{(0)}(\alpha,\beta)=u_0=\text{const}, \qquad Y^{(0)}(\alpha)=Y_0=\text{const}=\eta_0/h,\\
&\pi_\beta^{(0)}=-1, \quad \pi_\beta^{(1)}=0, \quad u_\beta^{(1)}=v_\alpha^{(1)}=0, \quad u_\alpha^{(1)}=-v_\beta^{(2)},\\
&u_0 u_\alpha^{(1)}+\pi_\alpha^{(1)}=0, \quad v^{(2)}(Y_0)=u_0 Y_\alpha^{(1)},\\
&v^{(2)}(-H)=0, \quad \pi^{(1)}(Y_0)=-Y^{(1)}\pi_\beta^{(0)}=Y^{(1)}(\alpha).
\end{aligned}\right\} \tag{31.6}$$

These expressions may be integrated to obtain

$$\left.\begin{aligned}
&u^{(1)}(\alpha,\beta)=f(\alpha)=u^{(1)}(\alpha),\\
&v^{(2)}(\alpha,\beta)=v^{(2)}(\alpha,\beta)-v^{(2)}(\alpha,-H)\\
&\qquad=-\int_{-H}^{\beta} f_\alpha\, d\beta=-(\beta+H)\, f_\alpha,\\
&\pi^{(1)}(\alpha,\beta)=-\int u_0 f_\alpha\, d\alpha=-(u_0 f+C)=\pi^{(1)}(\alpha)=Y^{(1)}(\alpha),\\
&Y_\alpha^{(1)}=\pi_\alpha^{(1)}=-u_0 f_\alpha=\frac{v_2(Y_0)}{u_0}=-\left(\frac{Y_0+H}{u_0}\right) f_\alpha.
\end{aligned}\right\} \tag{31.7}$$

The identities in the last equation show that the solution for constant u_0 is restricted to the unique value defined by

$$u(x,y)=u_0\sqrt{g h}=\sqrt{g h\,(Y_0+H)}=\sqrt{g(\eta_0+h_\infty)}=\text{const} \tag{31.8}$$

which corresponds to the infinitesimal-wave propagation velocity (28.3) and shows that the steady-state solution will be in the neighborhood of the critical speed defined by a Froude number of unity. However, $u^{(1)}=f(\alpha)$ now provides a finite-amplitude steady-state solution that does not form a discontinuity; consequently, the behavior of the second-order shallow-water theory is mathematically completely different from the first-order (28.2) shallow-water theory or the gas-dynamics equations. The pressure variation is still hydrostatic, since $\pi^{(1)}$ does not depend upon β, and only $v^{(2)}$ has a direct dependence upon $\beta\,(=y/h)$.

Now in order to continue the solution and determine $f(\alpha)$ we must introduce some ε^3 terms. By following the same procedure as used in collecting the ε^2 terms for (10.33) we obtain for the particular case of steady flow over a flat horizontal bottom the following additional terms that are required for completing the second order solution:

$$\left.\begin{aligned} &u_\beta^{(2)} = v_\alpha^{(2)}, \quad u_\alpha^{(2)} = -v_\beta^{(3)},\\ &u_0 u_\alpha^{(2)} + u^{(1)} u_\alpha^{(1)} + \pi_\alpha^{(2)} = 0, \quad u_0 v_\alpha^{(2)} + \pi_\beta^{(2)} = 0,\\ &v^{(3)}(Y_0) = u_0 Y_\alpha^{(2)} + u^{(1)} Y_\alpha^{(1)} - v_\beta^{(2)} Y^{(1)}, \quad v^{(3)}(-H) = 0,\\ &\pi^{(2)}(Y_0) = -Y^{(2)} \pi_\beta^{(0)} = Y^{(2)}. \end{aligned}\right\} \tag{31.9}$$

These expressions were first given by KELLER (1948) and they may be directly integrated to give the following:

$$\left.\begin{aligned} u^{(2)} &= \int v_\alpha^{(2)} d\beta = -f_{\alpha\alpha}\int(\beta+H)\,d\beta\\ &= -\tfrac{1}{2}(\beta^2+2H\beta) f_{\alpha\alpha} + R(\alpha) = u^{(2)}(\alpha,\beta)\\ \pi^{(2)}(\alpha,\beta) &= Y^{(2)}(\alpha) - [\tfrac{1}{2}(Y_0^2-\beta^2) + H(Y_0-\beta)]\, u_0 f_{\alpha\alpha},\\ v^{(3)}(\alpha,\beta) &= v^{(3)}(\alpha,\beta) - v^{(3)}(\alpha,-H) = \int_{-H}^{\beta} - u_\alpha^{(2)}\, d\beta\\ &= [\tfrac{1}{6}(\beta^3+H^3) + \tfrac{1}{2}H(\beta^2-H^2)]\, f_{\alpha\alpha\alpha} - (\beta+H)\, R_\alpha,\\ v^{(3)}(Y_0) &= u_0 Y_\alpha^{(2)} - u_0 f f_\alpha - (u_0 f + C) f_\alpha\\ &= \tfrac{1}{6}(Y_0^3 + 3H Y_0^2 - 2H^3) f_{\alpha\alpha\alpha} - (Y_0+H) R_\alpha. \end{aligned}\right\} \tag{31.10}$$

The last equation for $v^{(3)}$ gives the following expression for the ε^2 term in the surface profile

$$u_0 Y^{(2)}(\alpha) = u_0 f^2 + C f + \tfrac{1}{6}(Y_0^3 + 3H Y_0^2 - 2H^3) f_{\alpha\alpha} - (Y_0+H) R + \text{const}, \tag{31.11}$$

while a similar expression may be obtained directly from $\pi^{(2)}(Y_0)$ by equating its relation in (31.9) and (31.10) so as to obtain

$$\left.\begin{aligned} u_0 Y^{(2)}(\alpha) &= u_0 \pi^{(2)}(Y_0) = -u_0 [u_0 u^{(2)} + \tfrac{1}{2} u^{(1)2}]_{\beta=Y_0}\\ &= \tfrac{1}{2} u_0^2 (Y_0^2 + 2H Y_0) f_{\alpha\alpha} - u_0^2 R(\alpha) - \tfrac{1}{2} u_0 f^2 + \text{const}. \end{aligned}\right\} \tag{31.12}$$

Since (31.11) and (31.12) must be identical, we may equate them and find that $f(\alpha)$ must satisfy the ordinary differential equation

$$f_{\alpha\alpha} - \frac{9}{2u_0^5} f^2 - \frac{3C}{u_0^6} f + C_0 = 0 \tag{31.13}$$

after having introduced (31.8) to eliminate Y_0. Eq. (31.13) may be integrated to

$$\tfrac{1}{3} u_0^6 f_\alpha^2 - u_0 f^3 - C f^2 + \tfrac{2}{3} u_0^6 C_0 f = \text{const}. \tag{31.14}$$

Upon noting from (31.7) that $f(\alpha) = u^{(1)}(\alpha)$ and

$$\frac{u_0}{C} f = -\left[1 + \frac{Y^{(1)}(\alpha)}{C}\right],$$

it is evident that (31.13) and (31.14) are the same equations as obtained by BOUSSINESQ (1871, 1872), RAYLEIGH (1876), KORTEWEG and DE VRIES (1895), LAVRENT'EV (1943), and KELLER (1948). The physical significance of each term in (31.14) was first pointed out by BENJAMIN and LIGHTHILL (1954), who derived (31.14) in an entirely different manner, starting with the same series expansion of the stream function as was introduced by RAYLEIGH (1876). BENJAMIN and LIGHTHILL (1954) use the continuity equation (30.49), the specific-energy equation (30.50), and the specific-momentum equation (30.51) to derive the equivalent of (31.14), and then they give a very useful discussion of the mathematical and physical behavior of its solutions.

The appropriate solution of (31.13) for the boundary conditions shown in Fig. 47 is given by the square of the Jacobian elliptic function "cn" having the modulus $0<k\leqq 1$ and the real period

$$4K(k) = 4\int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} > 2\pi.$$

Substituting

$$f(\alpha) = -B\,\mathrm{cn}^2(A\,\alpha, k) \tag{31.15}$$

into (31.13) we find that (31.15) is a solution if, and *only* if, $0<k\leqq 1$ and

$$B = \frac{4}{3}u_0^5 A^2 k^2 = \frac{C}{u_0}\,\frac{k^2}{2k^2-1} = \frac{C_0}{2A^2(1-k^2)}. \tag{31.16}$$

Substituting (31.15) into (31.7) and (31.5) we obtain

$$\eta(x) = \eta_0 + \omega^2 h^3 B u_0\left[\mathrm{cn}^2(A\,\omega\,x, k) - \frac{2k^2-1}{k^2}\right] + O(\varepsilon^2).$$

The boundary conditions in Fig. 47 then yield

$$\left.\begin{aligned}
\eta(0) &= \eta_0 + \omega^2 h^3 B u_0\,\frac{1-k^2}{k^2} = a,\\
\eta\left(\frac{K}{A\,\omega}\right) &= \eta_0 - \omega^2 h^3 B u_0\,\frac{2k^2-1}{k^2} = 0,\\
a &= \omega^2 h^3 B u_0, \quad \eta_0 = a\,\frac{2k^2-1}{k^2},\\
\eta(x) &= a\,\mathrm{cn}^2(A\,\omega\,x, k).
\end{aligned}\right\} \tag{31.17}$$

Then upon introducing (31.8) and (31.16) into (31.17) we obtain

$$\left.\begin{aligned}
A\,\omega\,x &= \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\,\frac{a/h_\infty}{(1+\eta_0/h_\infty)^3}} = \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\,\frac{a/h_\infty}{[1+(a/h_\infty)(2k^2-1)/k^2]^3}}\\
&= \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\,\frac{a}{h_\infty}}\left\{1-\frac{3}{2}\,\frac{a}{h_\infty}\,\frac{2k^2-1}{k^2}+\cdots\right\}\\
&= \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\,\frac{a}{h_\infty}} + O\left[\frac{a}{h_\infty}\,\frac{2k^2-1}{k^2}\right]^{\frac{3}{2}}
\end{aligned}\right\} \tag{31.18}$$

as the exact second-order shallow-water theory solution for the first approximation to the cnoidal waves of KORTEWEG and DE VRIES (1895). The remaining terms in (31.6) and (31.7) may similarly be solved to give

$$\left.\begin{aligned}
\frac{p(x,y)}{\varrho\, g\, h_\infty} &= \frac{\eta(x)-y}{h_\infty} + O\left[\frac{a}{h_\infty}\,\frac{2k^2-1}{k^2}\right]^2,\\
\frac{u(x)}{\sqrt{g\,h_\infty}} &= 1 + \left(1-\frac{1}{2k^2}\right)\frac{a}{h_\infty} - \frac{\eta(x)}{h_\infty} + O\left[\frac{a}{h_\infty}\,\frac{2k^2-1}{k^2}\right]^2,\\
\frac{v(x,y)}{\sqrt{g\,h_\infty}} &= -\left(1+\frac{y}{h_\infty}\right)\sqrt{\frac{3}{k^2}\left(\frac{a}{h_\infty}\right)^3}\,\mathrm{cn}(A\,\omega\,x,k)\,\mathrm{sn}(A\,\omega\,x,k)\,\mathrm{dn}(A\,\omega\,x,k) + {}\\
&\quad + O\left[\frac{a}{h_\infty}\,\frac{2k^2-1}{k^2}\right]^{\frac{5}{2}}\\
&= +\left(1+\frac{y}{h_\infty}\right)\frac{d\eta(x)}{dx} + O\left[\frac{a}{h_\infty}\,\frac{2k^2-1}{k^2}\right]^{\frac{5}{2}},
\end{aligned}\right\} \tag{31.19}$$

where cn, sn, dn are the Jacobian elliptic functions with the argument $A\omega x$ defined by (31.18). It must be noted that $0<k\leqq 1$; k can never become identically zero for two reasons. First, because for $k=0$

$$\mathrm{cn}^2(A\,\omega\,x, 0) = \cos^2(A\,\omega\,x)$$

is not a solution of (31.13) or (31.14), and second, because the asymptotic expansions given in (31.18) and (31.19) are only valid as $a^2/k^2 \to 0$.

The limiting case of $k=1$ corresponds to an essentially infinite wavelength since $K(k^2)\to\infty$ as $k^2\to 1$, and the cnoidal-wave solutions reduce to

$$\left.\begin{aligned}
\frac{\eta(x)}{h_\infty} &= \frac{a}{h_\infty}\,\mathrm{sech}^2\left(\frac{x}{h_\infty}\sqrt{\frac{3}{4}\frac{a}{h_\infty}}\right) + O\left(\frac{a}{h_\infty}\right)^2,\\
\frac{p(x,y)}{\varrho\, g\, h_\infty} &= \frac{\eta(x)-y}{h_\infty} + O\left(\frac{a}{h_\infty}\right)^2,\\
\frac{u(x)}{\sqrt{g\,h_\infty}} &= 1 + \frac{1}{2}\frac{a}{h_\infty} - \frac{\eta(x)}{h_\infty} + O\left(\frac{a}{h_\infty}\right)^2,\\
\frac{v(x,y)}{\sqrt{g\,h_\infty}} &= \left(1+\frac{y}{h_\infty}\right)\frac{d\eta(x)}{dx} + O\left(\frac{a}{h_\infty}\right)^{\frac{5}{2}},
\end{aligned}\right\} \tag{31.20}$$

which provides the exact first approximation to the solitary wave.

All of these solutions for the cnoidal wave and the solitary wave are in exact agreement with the expressions first given by KORTEWEG and DE VRIES (1895, pp. 430—431) if one neglects the terms of $O(a/h_\infty)^2$. It will now be proved that the terms of $O(a/h_\infty)^2$ must be neglected in these first approximations because the second approximations introduce additional terms having this order of magnitude.

We can continue to the next order of approximation by collecting the remaining terms corresponding to ε^3, and adding some of the ε^4 terms that are necessary in order to complete the solution

$$\left.\begin{aligned}
&\pi^{(3)}(Y_0) = Y^{(3)} - Y^{(1)}\pi^{(2)}_\beta(Y_0),\\
&u^{(3)}_\beta = v^{(3)}_\alpha, \qquad u^{(3)}_\alpha = -v^{(4)}_\beta,\\
&u_0\, u^{(3)}_\alpha + u^{(1)} u^{(2)}_\alpha + u^{(2)} u^{(1)}_\alpha + \pi^{(3)}_\alpha + v^{(2)} u^{(2)}_\beta = 0,\\
&u_0\, v^{(3)}_\alpha + u^{(1)} v^{(2)}_\alpha + \pi^{(3)}_\beta + v^{(2)} v^{(2)}_\beta = 0,\\
&v^{(4)}(Y_0) = u_0\, Y^{(3)}_\alpha + u^{(1)}\, Y^{(2)}_\alpha + u^{(2)}\, Y^{(1)}_\alpha - v^{(2)}_\beta\, Y^{(2)} - v^{(3)}_\beta\, Y^{(1)},\\
&v^{(4)}(-H) = 0.
\end{aligned}\right\} \tag{31.21}$$

Now we can combine the expression for $v^{(3)}$ in (31.10) with that in (31.21) to write

$$\left.\begin{aligned}
u^{(3)}(\alpha,\beta) = \int v^{(3)}_\alpha\, d\beta = \tfrac{1}{24}(\beta^4 + 4H\beta^3 - 8H^3\beta)\, f_{\alpha\alpha\alpha\alpha} -\\
- \tfrac{1}{2}(\beta^2 + 2H\beta)\, R_{\alpha\alpha} + S(\alpha).
\end{aligned}\right\} \tag{31.22}$$

Then the expression for $v^{(4)}$ in (31.21) yields

$$\left.\begin{aligned}
v^{(4)}(\alpha,\beta) = -\int_{-H}^{\beta} u^{(3)}_\alpha(\alpha,\beta)\, d\beta = -\,[(\beta+H)\, S_\alpha +\\
+ \tfrac{1}{120}(\beta^5 + 5H\beta^4 - 20H^3\beta^2 + 16H^5)\, f_{\alpha\alpha\alpha\alpha\alpha} - \tfrac{1}{6}(\beta^3 + 3H\beta^2 - 2H^3)\, R_{\alpha\alpha\alpha}].
\end{aligned}\right\} \tag{31.23}$$

The boundary condition defined by the expression for $v^{(4)}$ in (31.21) thereby gives one relation for $Y^{(3)}$ that may be written as

$$\left.\begin{aligned} u_0 Y_\alpha^{(3)}(\alpha) &= v^{(4)}(Y_0) - [u^{(1)} Y_\alpha^{(2)} - v_\beta^{(2)} Y^{(2)}] - [u^{(2)} Y_\alpha^{(1)} - v_\beta^{(3)} Y^{(1)}] \\ &= v^4(Y_0) - [u^{(1)} Y^{(2)}]_\alpha - [u^{(2)} Y^{(1)}]_\alpha \end{aligned}\right\} \quad (31.24)$$

which may be directly integrated, upon substituting (31.23) for $v^{(4)}$, as

$$\left.\begin{aligned} u_0 Y_\alpha^{(3)} &= \int v^{(4)}(Y_0)\, d\alpha - u^{(1)}(Y_0)\, Y^{(2)} - u^{(2)}(Y_0)\, Y^{(1)} \\ &= \text{const} - \left\{\frac{1}{120}(Y_0^5 + 5H Y_0^4 - 20H^3 Y_0^2 + 16H^5) f_{\alpha\alpha\alpha\alpha} + \right. \\ &\quad - \frac{1}{6}(Y_0^3 + 3H Y_0^2 - 2H^3) R_{\alpha\alpha} + (Y_0 + H)\, S(\alpha) + \\ &\quad + \frac{f}{u_0}\left[\frac{1}{2} u_0^2 (Y_0^2 + 2H Y_0) f_{\alpha\alpha} - u_0^2 R - \frac{1}{2} u_0 f^2\right] + \\ &\quad \left. + \left[\frac{1}{2}(Y_0^2 + 2H Y_0) f_{\alpha\alpha} - R(\alpha)\right][u_0 f + C]\right\}. \end{aligned}\right\} \quad (31.25)$$

Another relation for $Y^{(3)}$ may also be obtained from the other boundary condition defined by the expression for $\pi^{(3)}$ in (31.21), namely

$$Y^{(3)}(\alpha) = \pi^{(3)}(Y_0) + Y^{(1)} \pi_\beta^{(2)}(Y_0), \quad (31.26)$$

where $\pi^{(3)}$ itself may be obtained by integrating the expressions for $\pi_\alpha^{(3)}$ and $\pi_\beta^{(3)}$ in (31.21) to obtain

$$\pi^{(3)}(\alpha, \beta) = - u_0 u^{(3)} - u^{(1)} u^{(2)} - \tfrac{1}{2}[v^{(2)}]^2 + \text{const}. \quad (31.27)$$

Then substituting $\pi^{(3)}$ from (31.27), $\pi_\beta^{(2)}$ from (31.10), $u^{(1)} = f$, $u^{(2)}$ from (31.10), $u^{(3)}$ from (31.22), $y^{(1)} = -(u_0 f + C)$ and $Y^{(2)}$ from (31.12) into (31.26) we obtain another relation for $Y^{(3)}$, namely,

$$\left.\begin{aligned} u_0 Y^{(3)}(\alpha) = &-\{\tfrac{1}{24} u_0^2 (Y_0^4 + 4H Y_0^3 - 8H^3 Y_0) f_{\alpha\alpha\alpha\alpha} - \tfrac{1}{2} u_0^2 (Y_0^2 + 2H Y_0) R_{\alpha\alpha} + \\ &+ u_0^2 S - \tfrac{1}{2} u_0 (Y_0^2 + 2H Y_0) f f_\alpha + u_0 f R + \tfrac{1}{2} u_0^5 (f_\alpha)^2 + \\ &+ (u_0 f + C) u_0^4 f_{\alpha\alpha} + \text{const}\}. \end{aligned}\right\} \quad (31.28)$$

These two expressions for $Y^{(3)}$, (31.25) and (31.28), must be identically equal; therefore, since u_0 is defined by (31.8), we find that the unknown function R must satisfy the ordinary differential equation

$$\left.\begin{aligned} \frac{1}{3} u_0^5 R_{\alpha\alpha} - &\left(\frac{C}{u_0} + 3f\right) R + \text{const} \\ &= \frac{1}{30} u_0^5 (u_0^4 - 5H^2) f_{\alpha\alpha\alpha\alpha} - \frac{1}{2}(u_0^4 - 3H^2) f f_{\alpha\alpha} + \\ &\quad + \frac{C}{2u_0}(u_0^4 + H^2) f_{\alpha\alpha} + \frac{1}{2} u_0^4 (f_\alpha)^2 + \frac{1}{2u_0} f^3, \end{aligned}\right\} \quad (31.29)$$

the other unknown function $S(\alpha)$ having been eliminated since $u_0^2 = Y_0 + H$.

When $f(\alpha)$ is given by (31.15), then the solution of (31.29) is

$$\left.\begin{aligned} R(\alpha) = \frac{C^2}{u_0^3}\Big\{&\left(\frac{k^2}{2k^2-1}\right)^2\left(1-\frac{9}{4}\frac{H^2}{u_0^4}\right)\mathrm{cn}^4(A\,\alpha,k)+\\ &+\frac{k^2}{2k^2-1}\left(1+\frac{3}{2}\frac{H^2}{u_0^4}\right)\mathrm{cn}^2(A\,\alpha,k)-\frac{3}{10}\frac{k^2(1-k^2)}{(2k^2-1)^2}\left(1-\frac{5}{2}\frac{H^2}{u_0^4}\right)-\frac{3}{5}\Big\}\end{aligned}\right\}\quad(31.30)$$

and (31.11) or (31.12) give the ε^2 term of the wave profile as

$$\left.\begin{aligned} Y^{(2)}(\alpha) = \left(\frac{C}{u_0}\right)^2\Big\{&\frac{3}{4}\left(\frac{k^2}{2k^2-1}\right)^2\mathrm{cn}^4(A\,\alpha,k)-\\ &-\frac{5}{2}\frac{k^2}{2k^2-1}\mathrm{cn}^2(A\,\alpha,k)+\frac{12-57k^2+57k^4}{20(2k^2-1)^2}\Big\}.\end{aligned}\right\}\quad(31.31)$$

Consequently, the second approximation to the cnoidal-wave profile is obtained from the preceding and (31.5) as

$$\left.\begin{aligned} \eta(x) &= \eta_0+\omega^2h^3\,Y^{(1)}+\omega^4h^5\,Y^{(2)}+O(\varepsilon^3)\\ &=\eta_0-\eta_1\left[1-\frac{k^2}{2k^2-1}\mathrm{cn}^2(A\,\omega\,x,k)\right]+\\ &\quad+\frac{\eta_1^2}{\eta_0+h_\infty}\left[\frac{3}{4}\left(\frac{k^2}{2k^2-1}\right)^2\mathrm{cn}^4(A\,\omega\,x,k)-\frac{5}{2}\frac{k^2}{2k^2-1}\mathrm{cn}^2(A\,\omega\,x,k)+\right.\\ &\quad\left.+\frac{12-57k^2+57k^4}{20(2k^2-1)^2}\right],\end{aligned}\right\}\quad(31.32)$$

where

$$\begin{aligned}\eta_1 &= C\,\omega^2h^3=(A\,\omega)^2\tfrac{4}{3}(2k^2-1)\,(h\,u_0^2)^3\\ &=(A\,\omega)^2\tfrac{4}{3}(2k^2-1)\,(\eta_0+h_\infty)^3.\end{aligned}$$

Then the boundary conditions shown in Fig. 37c yield the relations:

$$\left.\begin{aligned} \eta(0) &= a=\eta_0-\eta_1\frac{k^2-1}{2k^2-1}+\frac{\eta_1^2}{\eta_0+h_\infty}\frac{12-7k^2-28k^4}{20(2k^2-1)^2},\\ \eta\left(\frac{K}{A\,\omega}\right) &= 0=\eta_0-\eta_1+\frac{\eta_1^2}{\eta_0+h_\infty}\left[\frac{12-57k^2+57k^4}{20(2k^2-1)^2}\right],\end{aligned}\right\}\quad(31.33)$$

which may be solved to give the second approximation

$$\left.\begin{aligned} \frac{\eta_0}{h_\infty} &= \left(\frac{a}{h_\infty}\right)\frac{2k^2-1}{k^2}+\left(\frac{a}{h_\infty}\right)^2\frac{38-128k^2+113k^4}{20k^4}+O\left(\frac{a}{h_\infty}\right)^3,\\ \frac{\eta_1}{h_\infty} &= \frac{2k^2-1}{k^2}\left(\frac{a}{h_\infty}\right)\left[1+\left(\frac{a}{h_\infty}\right)\left(\frac{85k^2-50}{20k^2}\right)\right]+O\left(\frac{a}{h_\infty}\right)^3,\\ \frac{\eta(x)}{h_\infty} &= \left(\frac{a}{h_\infty}\right)\mathrm{cn}^2(A\,\omega\,x,k)-\frac{3}{4}\left(\frac{a}{h_\infty}\right)^2\mathrm{cn}^2(A\,\omega\,x,k)\times\\ &\quad\times[1-\mathrm{cn}^2(A\,\omega\,x,k)]+O\left(\frac{a}{h_\infty}\right)^3,\end{aligned}\right\}\quad(31.34)$$

where now

$$\left.\begin{aligned} A\,\omega\,x &= \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\left(\frac{a}{h_\infty}\right)}\left(1+\frac{\eta_0}{h_\infty}\right)^{-\frac{3}{2}}\left[1+\left(\frac{a}{h_\infty}\right)\frac{85k^2-50}{40k^2}\right]\\ &=\frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\left(\frac{a}{h_\infty}\right)}\left[1-\left(\frac{a}{h_\infty}\right)\frac{7k^2-2}{8k^2}\right]+O\left[\left(\frac{a}{h_\infty}\right)\frac{2k^2-1}{k^2}\right]^{\frac{5}{2}}.\end{aligned}\right\}\quad(31.35)$$

The remaining ε^2 terms from (31.10) may then be combined with the ε terms from (31.7), by means of (31.5), to give

$$\left.\begin{aligned}
\frac{p(x,y)}{\varrho g h_\infty} &= \frac{\eta(x)-y}{h_\infty} - \left(\frac{a}{h_\infty}\right)^2 \frac{3}{4k^2}\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right)[1-k^2+ \\
&\quad +2(2k^2-1)\,\mathrm{cn}^2(A\,\omega\,x,k) - 3k^2\,\mathrm{cn}^4(A\,\omega\,x,k)] + O\left[\left(\frac{a}{h_\infty}\right)\frac{2k^2-1}{k^2}\right]^3, \\
\frac{u(x,y)}{\sqrt{g h_\infty}} &= 1+\left(\frac{a}{h_\infty}\right)\left(1-\frac{1}{2k^2}\right) - \left(\frac{a}{h_\infty}\right)^2 \frac{21k^4-6k^2-9}{40k^4} + \\
&\quad -\left(\frac{a}{h_\infty}\right)\left[1-\left(\frac{a}{h_\infty}\right)\frac{7k^2-2}{4k^2} - \left(\frac{a}{h_\infty}\right)\frac{3}{2}\left(2-\frac{1}{k^2}\right)\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right)\right]\mathrm{cn}^2(A\,\omega\,x,k) - \\
&\quad -\left(\frac{a}{h_\infty}\right)^2\left[\frac{5}{4}+\frac{9}{4}\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right)\right]\mathrm{cn}^4(A\,\omega\,x,k) + \\
&\quad +\left(\frac{a}{h_\infty}\right)^2\frac{3}{4}\left(\frac{1}{k^2}-1\right)\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right) + O\left[\left(\frac{a}{h_\infty}\right)\frac{2k^2-1}{k^2}\right]^3, \\
\frac{v(x,y)}{\sqrt{g h_\infty}} &= -\sqrt{\frac{3}{k^2}\left(\frac{a}{h_\infty}\right)^3}\left(1+\frac{y}{h_\infty}\right)\mathrm{cn}(A\,\omega\,x,k)\,\mathrm{sn}(A\,\omega\,x,k)\,\mathrm{dn}(A\,\omega\,x,k)\times \\
&\quad \times\left\{1-\left(\frac{a}{h_\infty}\right)\left(\frac{5k^2+2}{8k^2}\right) - \left(\frac{a}{h_\infty}\right)\left(1-\frac{1}{2k^2}\right)\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right) - \right. \\
&\quad \left. -\frac{1}{2}\left(\frac{a}{h_\infty}\right)\left(1-6\frac{y}{h_\infty}-3\frac{y^2}{h_\infty^2}\right)\mathrm{cn}^2(A\,\omega\,x,k)\right\} + O\left[\left(\frac{a}{h_\infty}\right)\left(\frac{2k^2-1}{k^2}\right)\right]^{\frac{7}{2}}.
\end{aligned}\right\}\quad (31.36)$$

For the solitary wave we have $k=1$ and essentially infinite wavelength, so that (31.36) reduces to

$$\left.\begin{aligned}
\frac{\eta(x)}{h_\infty} &= \frac{a}{h_\infty}\,\mathrm{sech}^2(A\,\omega\,x) - \frac{3}{4}\left(\frac{a}{h_\infty}\right)^2\mathrm{sech}^2(A\,\omega\,x)\times \\
&\quad \times[1-\mathrm{sech}^2(A\,\omega\,x)] + O\left(\frac{a}{h_\infty}\right)^3, \\
A\,\omega\,x &= \frac{x}{h_\infty}\sqrt{\frac{3}{4}\left(\frac{a}{h_\infty}\right)}\left\{1-\frac{5}{8}\left(\frac{a}{h_\infty}\right)\right\} + O\left(\frac{a}{h_\infty}\right)^{\frac{5}{2}}, \\
\frac{p}{\varrho g h_\infty} &= \frac{\eta(x)-y}{h_\infty} - \left(\frac{a}{h_\infty}\right)^2\frac{3}{4}\left(2\frac{y}{h_\infty}+\frac{y^2}{h_\infty^2}\right)\times \\
&\quad \times[2\,\mathrm{sech}^2(A\,\omega\,x) - 3\,\mathrm{sech}^4(A\,\omega\,x)] + O\left(\frac{a}{h_\infty}\right)^3, \\
\frac{u(x,y)}{\sqrt{g h_\infty}} &= 1+\frac{1}{2}\left(\frac{a}{h_\infty}\right) - \frac{3}{20}\left(\frac{a}{h_\infty}\right)^2 - \frac{\eta(x)}{h_\infty} + \\
&\quad +\frac{1}{2}\left(\frac{a}{h_\infty}\right)^2\left[1+6\left(\frac{y}{h_\infty}\right)+3\left(\frac{y}{h_\infty}\right)^2\right]\mathrm{sech}^2(A\,\omega\,x) + \\
&\quad -\frac{1}{2}\left(\frac{a}{h_\infty}\right)^2\left[1+9\left(\frac{y}{h_\infty}\right)+\frac{9}{2}\left(\frac{y}{h_\infty}\right)^2\right]\mathrm{sech}^4(A\,\omega\,x) + O\left(\frac{a}{h_\infty}\right)^3, \\
\frac{v(x,y)}{\sqrt{g h_\infty}} &= -\sqrt{3}\left(\frac{a}{h_\infty}\right)^{\frac{3}{2}}\left(1+\frac{y}{h_\infty}\right)\mathrm{sech}^2(A\,\omega\,x)\tanh(A\,\omega\,x)\times \\
&\quad \times\left\{1-\frac{7}{8}\left(\frac{a}{h_\infty}\right) - \left(\frac{a}{h_\infty}\right)\left(\frac{y}{h_\infty}+\frac{1}{2}\frac{y^2}{h_\infty^2}\right) - \frac{1}{2}\left(\frac{a}{h_\infty}\right)\times\right. \\
&\quad \left.\times\left(1-6\frac{y}{h_\infty}-3\frac{y^2}{h_\infty^2}\right)\mathrm{sech}^2(A\,\omega\,x)\right\} + O\left(\frac{a}{h_\infty}\right)^{\frac{7}{2}}.
\end{aligned}\right\}\quad (31.37)$$

The celerity or propagation velocity c of a solitary wave is defined by (31.37) as the constant uniform motion attained as $x\to\infty$,

$$\frac{c}{\sqrt{g h_\infty}} = \frac{u(\infty)}{\sqrt{g h_\infty}} = 1+\frac{1}{2}\left(\frac{a}{h_\infty}\right) - \frac{3}{20}\left(\frac{a}{h_\infty}\right)^2 + O\left(\frac{a}{h_\infty}\right)^3. \qquad (31.38)$$

In Fig. 48, Eq. (31.38) is shown to be in better agreement with recent experimental data than is the commonly used Boussinesq (1871)-Rayleigh (1876) propagation velocity given by

$$\frac{c}{\sqrt{g h_\infty}} \approx \sqrt{1+\left(\frac{a}{h_\infty}\right)} \approx 1+\frac{1}{2}\left(\frac{a}{h_\infty}\right)-\frac{1}{8}\left(\frac{a}{h_\infty}\right)^2+\cdots.$$

The past success of the Boussinesq-Rayleigh equation, as opposed to the propagation velocities derived by McCowan (1891), as indicated in Fig. 48, is easily explained when one notices the close numerical agreement of the coefficients of the Boussinesq-Rayleigh equation with the exact second approximation given by (31.38).

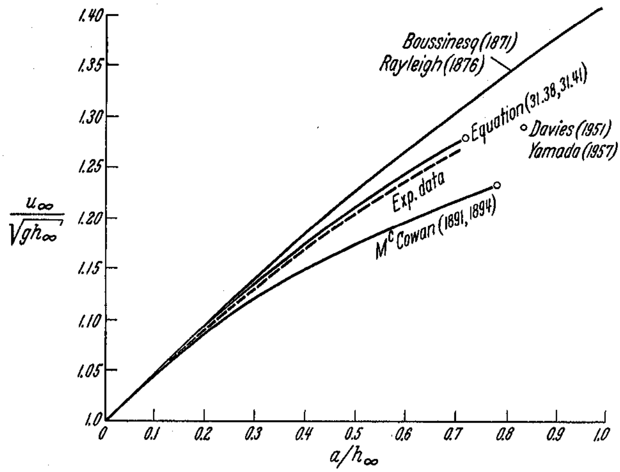

Fig. 48. Propagation velocity of solitary waves.

A comparison of the second approximations with the first approximations to the cnoidal waves proves conclusively that only the proper order of a/h_∞ must be retained for each order of approximation. For example, a comparison of (31.18) with (31.35) shows that a completely erroneous second approximation would be obtained by trying to extend the first approximation to include an additional a/h_∞ term. The reason for this is evident upon comparing the first and second approximations for η_0 in (31.17) and (31.34). Each successive approximation directly affects all the coefficients of the corresponding a/h_∞ terms. Fig. 49 shows the effect of the second approximation on a solitary wave.

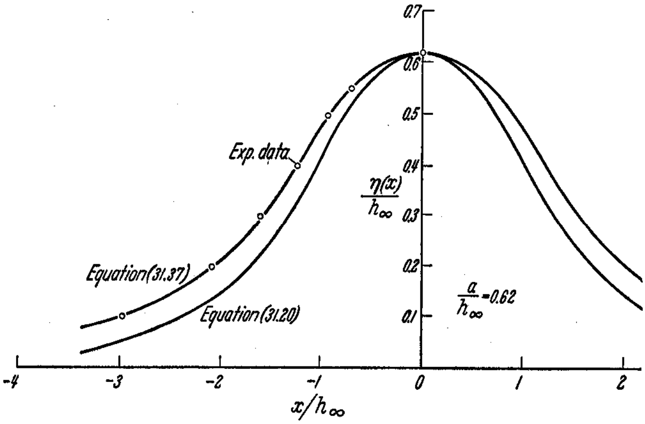

Fig. 49. Comparison of first (31.17) and second (31.37) approximation to the solitary-wave profile, $\eta(x)$.

Of course it must be remembered that the expansion method of Friedrichs (1948), which was used to obtain all the preceding results, is applicable only to shallow water, or long-wavelength wave propagations. However, this is precisely the nature of the solitary wave, especially if the amplitude a/h_∞ is relatively small, since its wavelength, as a member of the family of cnoidal waves, is essentially infinite since $K(k^2)\to\infty$ when $k\to 1$. Also, Friedrichs and Hyers (1954) proved that this expansion method does yield an existence proof for the solitary wave, and thereby demonstrate that it will at least provide asymptotic descriptions of the exact solution for the solitary-wave problem. The corresponding existence proof for cnoidal waves (in the neighborhood of the critical speed defined by a Froude number of unity) was given by LITTMAN (1957). Again this justified the Friedrichs expansion method, at least as an asymptotic type of series development. An additional discussion of these existence proofs is given in Sect. 35.

β) The limiting height and velocity of propagation of cnoidal and solitary waves. It is interesting to note that with the second approximation the pressure is still hydrostatic for $y \approx 0$, but is no longer hydrostatic as the bottom ($y = -h_\infty$) is approached. Similarly, the variation of the horizontal velocity component with depth below the surface becomes important in the second approximation only upon approaching the flat horizontal bottom. However, the finite vertical velocity component is now seen to be the principal variation from the basic assumptions of first-order shallow-water theory. The first approximation given in (31.19) gives a monotonic variation in $v(y)$ that is obviously necessary from physical considerations in order to satisfy the continuity equation. However, this monotonic variation in $v(y)$ is of the higher order $(a/h_\infty)^{\frac{3}{2}}$ so that it can be neglected in the first-order equations (28.2) as long as the resulting local variations in η are sufficiently small.

The second approximation to the vertical velocity component, as given in (31.36), now shows that the variation of $v(y)$ will no longer be monotonic as a/h_∞ increases. This leads one to suspect that there is a limiting value to a/h_∞ for cnoidal and solitary waves. For example, (31.36) shows that in the neighborhood of the wave crest, where $x \approx 0$ and

$$\mathrm{cn}^2(A\,\omega\,x) \approx 1 - (A\,\omega\,x)^2 = 1 - \frac{3}{4k^2}\left(\frac{a}{h_\infty}\right)\left(\frac{x}{h_\infty}\right)^2 + O\left(\frac{a}{h_\infty}\right)^2,$$

$v(y)$ actually has a reversal in its direction if a/h_∞ exceeds the value given by

$$\left(\frac{a}{h_\infty}\right)_{\max} = \frac{8k^2}{9k^2+2} \tag{31.39}$$

for any value of $y \geqq 0$.

This limiting value can be substantiated, at least in the limit as $k \to 1$, by noting that (31.33) has a real solution for η_1 only if

$$\frac{2k^2-1}{k^2}\,\frac{a}{h_\infty} < \frac{\eta_0}{h_\infty} \leqq \frac{5(2k^2-1)^2}{7-37k^2(1-k^2)},$$

leading to a limiting value of

$$\left(\frac{a}{h_\infty}\right)_{\max} < \frac{5k^2(2k^2-1)}{7-37k^2(1-k^2)}. \tag{31.40}$$

The most interesting application of these results is to the solitary wave, defined by $k=1$, in which case we find from (31.38), (31.39) and (31.40) that the limiting heights and the corresponding total velocity at infinity are given by

$$\left(\frac{a}{h_\infty}\right)_{\max} = \frac{8}{11} = 0.7273 > \frac{5}{7} = 0.7143, \qquad \left[\frac{u(\infty)}{\sqrt{g\,h}}\right]_{\max} = 1.284 > 1.281. \tag{31.41}$$

Either of these limiting heights would be satisfactory for a solitary wave since recent experimental investigations by IPPEN and KULIN (1955), DAILY and STEPHAN (1952), and PERROUD (1957) have shown that under properly controlled conditions most solitary waves have $a/h_\infty < 0.7$, the maximum recorded value being 0.72. Not only are the limiting values given by (31.39) or (31.40) in excellent agreement with recent experimental data, but they are consistent with the order of approximation involved. The value $\frac{8}{11}$ is derived from the vertical velocity variation given to the order $(a/h_\infty)^{\frac{5}{2}}$ by (31.36), while the value $\frac{5}{7}$ corresponds to the terms governed by ε^2 or $(a/h_\infty)^2$ in (31.32).

Many attempts have been made to determine the limiting height of a solitary wave. However, nearly all of the theoretical calculations have been based on STOKES' (1880, p. 227) relation which assumes that for the limiting heights of any wave the wave crest must form a sharp peak or corner having an enclosed angle of 120° in order to reduce the relative local velocity to zero at the crest itself [see, e.g., Sect. 33 or LAMB (1932, p. 418)]. This 120° enclosed angle at the wave crest was assumed by McCOWAN (1894), STOKES (1905), GWYTHER (1900), DAVIES (1952), PACKHAM (1952), GOODY and DAVIES (1957) and YAMADA (1957). Several of these values are compared in Fig. 48 with experimental data, and with the theoretical values given by (31.38) and (31.41). It is seen that none of these limiting heights for solitary waves are in as good an agreement with the experimental data as is (31.41). A reasonable explanation of the failure of the 120° sharp crest wave to provide a satisfactory limiting height for a solitary wave may be obtained by noting that KORTEWEG and DE VRIES (1895) proved that any finite-amplitude profile that did not correspond to (31.17) or (31.20) would not be steady with respect to time. Consequently, (31.37) defines the only possible steady-state solitary wave, and when $a/h_\infty > \frac{8}{11}$ the vertical velocity variation reverses its direction near the crest. This probably leads to an unsteady wave crest that breaks unsymmetrically.

Eqs. (31.38) or (32.52) show that the solitary wave occurs only in supercritical flow since the Froude number corresponding to the propagation velocity is always greater than unity. Its velocity of propagation is always less than that of the corresponding hydraulic jump of the same height as may be seen by comparing (30.55) with (31.38), after expanding it in powers of $a/d_1 = a/h_\infty$:

$$F_1 = \frac{u_1}{\sqrt{g\,h_\infty}} = 1 + \frac{3}{4}\left(\frac{a}{h_\infty}\right) - \frac{1}{32}\left(\frac{a}{h_\infty}\right)^2 + O\left(\frac{a}{h_\infty}\right)^3. \qquad (30.55')$$

However, the cnoidal wave can occur in subcritical as well as in supercritical flow, and as shown by BENJAMIN and LIGHTHILL (1954), the undulating flow in the subcritical region behind a hydraulic jump produced at all Froude numbers less than $\sqrt{3}$ may well be represented by these cnoidal waves. The fact that cnoidal waves can form in subcritical flow is easily shown, even in the first approximation, by writing the horizontal velocity component from (31.19) as

$$\left.\begin{aligned} \frac{u(0)}{\sqrt{g\,h_\infty}} &= 1 - \frac{1}{2k^2}\left(\frac{a}{h_\infty}\right) < 1 \qquad &&\text{for all } k \leq 1,\\ \frac{u\left(\frac{K}{A\,\omega}\right)}{\sqrt{g\,h_\infty}} &= 1 + \left(1 - \frac{1}{2k^2}\right)\left(\frac{a}{h_\infty}\right) < 1 \qquad &&\text{for } k^2 < \frac{1}{2}. \end{aligned}\right\} \qquad (31.42)$$

Therefore (31.19) shows that any definition of the wave propagation velocity would be subcritical when $k^2 < \frac{1}{2}$. STOKES (1847) (see Sect. 7) has given two logical definitions of the celerity or propagation velocity of permanent periodic wave forms, and each one would define a critical celerity corresponding to a different value of k, varying as $\frac{1}{2} \leq k^2 < 1$, the solitary wave ($k = 1$) being always supercritical for a finite amplitude. However, the existence proof for cnoidal waves by LITTMAN (1957) is only valid for average velocities (defined as the velocity of the vertical plane that would have zero average flux across it) that are near critical. An interesting physical and mathematical explanation of these flow restrictions is given by BENJAMIN and LIGHTHILL (1954). The main consideration, as shown in Fig. 56 on page 754 and Sect. 35, is that the finite-amplitude periodic waves corresponding to $k^2 < 0.9$ may be better described by using infinitesimal wave theory. This becomes necessary because the wavelength of the cnoidal waves decreases rapidly with k^2 when k is near unity. Fig. 56 indicates that not only must the wavelength be large compared to the water depth, in order to satisfy the shallow-water expansion method, but also the amplitude of the cnoidal wave must become extremely small for values of $k^2 < 0.9$, or for subcritical flow.

KORTEWEG and DE VRIES (1895) have also shown how negative cnoidal or solitary waves can be formed when the water is very shallow and surface tension T is considered. Their correct first approximation may be written as

$$\left.\begin{aligned} \frac{\eta(x)}{h_\infty} &= \pm\left(\frac{a}{h_\infty}\right)\mathrm{cn}^2(A\,\omega\,x, k), \\ A\,\omega\,x &= \frac{x}{h_\infty}\sqrt{\frac{3}{4k^2}\,\frac{a/h_\infty}{|1-3\,T/\varrho\,g\,h_\infty^2|}}, \end{aligned}\right\} \tag{31.43}$$

where the negative algebraic value is assigned to the surface profile whenever

$$h_\infty < \sqrt{\frac{3\,T}{\varrho\,g}} \approx \frac{1}{2}\,\mathrm{cm} \qquad \text{for water.} \tag{31.44}$$

These negative waves have a very small amplitude and a very large wavelength, but can create a surprising particle motion. It is interesting to notice that the depth of $h_\infty = \frac{1}{2}$ cm, which, if it could be maintained, would eliminate both solitary and cnoidal waves, is the same depth found from (31.1′) and (31.2) to give nearly the same value of $\sqrt{g\,h_\infty}$ for both the propagation velocity and the group velocity of infinitesimal waves (also see Sect. 15). Consequently the depth of $\frac{1}{2}$ cm seems to be the optimum for ordinary water ($T = 72.8$ dynes/cm) whenever one uses small models to simulate results appropriate to the first-order shallow-water theory of (28.2), since this particular depth minimizes the effect of group velocity and variation with wavelength for the infinitesimal waves, and minimizes the second-order effect due to the existence of finite-amplitude cnoidal or solitary waves. However, the variation of η must remain sufficiently small since a finite increase in η above $h_\infty = \frac{1}{2}$ cm could still produce cnoidal or solitary waves. Also, the short-wavelength or capillary ripples that will form must be neglected in these model tests.

F. Exact solutions.

The word "exact" in this context is generally understood to mean solutions in which there has been no approximation in the equations or boundary conditions. However, this usage of the word does not exclude neglect of viscosity and, in fact, since positive results have been obtained only for perfect fluids, the treatment below will be restricted to them. Indeed, the present results in the theory of exact solutions are restricted, with few exceptions, to a very special class of motions, namely, those which can be represented as steady two-dimensional flows.

In Sect. 32 some general theorems will be established. In Sect. 33 waves of maximum amplitude-to-length ratio are discussed; because the methods are intimately related, we have also included in this section a discussion of HAVELOCK's method of approximating periodic waves. Sect. 34 treats methods of obtaining explicit exact solutions and of various ones which have been obtained. In Sect. 35, the last, existence theorems are discussed, but only in a descriptive way, for proofs are highly technical and lengthy.

see errata

32. Some general theorems. This section will be devoted to several theorems of a rather general nature concerning the motion of a fluid with free surface in a gravitational field. The theorems in subsection 32α are mostly of a kinematical nature and are associated with the phenomenon of mass transport already discussed in subsection 27α. The last part of this section is devoted to several theorems on energy and momentum. In subsection 32β some theorems concerning waves in heterogeneous fluids will be established. In subsection 32γ several different ways of formulating the problem of motion with a free surface will be described.

see errata

α) *Kinematical theorems—mass transport—energy integrals.* The first theorem, due to M. S. LONGUET-HIGGINS (1953), is independent of the presence of a free surface or of the nature of the time dependence. Let $f(z) = \Phi + i\Psi$ describe a space-periodic motion, i.e. $f(z + n\lambda) = f(z)$. The definition of φ will be normalized so that

$$\int_0^\lambda \Phi(x, y, t)\, dx = 0. \tag{32.1}$$

Note that if this condition holds for one value of y, it holds for all since

$$\frac{\partial}{\partial y}\int_0^\lambda \Phi\, dx = \int_0^\lambda \Phi_y\, dx = -\int_0^\lambda \Psi_x\, dx = -\Psi(\lambda, y, t) + \Psi(0, y, t) = 0.$$

In Eq. (2.10′) we shall write

$$\frac{p}{\varrho} = \frac{p_0}{\varrho} - g y + \frac{p_d}{\varrho}, \qquad \frac{p_d}{\varrho} = A(t) - \Phi_t - \frac{1}{2}(u^2 + v^2). \tag{32.2}$$

In the following we define an average by

$$\overline{F}(y, t) = \frac{1}{\lambda}\int_0^\lambda F(x, y, t)\, dx. \tag{32.3}$$

Theorem. In a non-uniform space-periodic motion $\overline{u^2}$, $\overline{v^2}$, $-\overline{p_d}$ each decrease with increasing depth, provided either $\Phi_y(x, -h, t) = 0$ or $\lim\limits_{y \to -\infty} \Phi_y = 0$.

This may be proved as follows. Consider first $q^2 = u^2 + v^2$. Then

$$\left.\begin{aligned}
\frac{\partial}{\partial y}\overline{q^2} &= \frac{\partial}{\partial y}\frac{1}{\lambda}\int_0^\lambda (\Phi_x^2 + \Phi_y^2)\, dx = \frac{2}{\lambda}\int_0^\lambda (\Phi_x \Phi_{xy} + \Phi_y \Phi_{yy})\, dx \\
&= \frac{2}{\lambda}\int_0^\lambda [(\Phi_y \Phi_x)_x - 2\Phi_y \Phi_{xx}]\, dx \\
&= \frac{2}{\lambda}[\Phi_y \Phi_x]_0^\lambda - \frac{4}{\lambda}\int_0^\lambda \Phi_y \Phi_{xx}\, dx \\
&= -\frac{4}{\lambda}\int_0^\lambda \Phi_y \Phi_{xx}\, dx.
\end{aligned}\right\} \tag{32.4}$$

By a similar computation it follows that

$$\frac{\partial^2}{\partial y^2}\overline{q^2} = \frac{4}{\lambda}\int_0^\lambda (\Phi_{xx}^2 + \Phi_{xy}^2)\,dx > 0, \tag{32.5}$$

since we have assumed that Φ_x is not constant. It is evident from (32.4) that, if the fluid is bounded below by $y=-h$, then

$$\frac{\partial}{\partial y}\overline{q^2}(-h,t) = 0; \tag{32.6}$$

if it is infinitely deep, it is an assumed boundary condition that $\Phi_y \to 0$ as $y\to-\infty$ and hence

$$\frac{\partial}{\partial y}\overline{q^2} \to 0 \quad \text{as} \quad y\to-\infty. \tag{32.7}$$

In either case it then follows from (32.5) that $\partial q^2/\partial y$ is an increasing function of y and hence

$$\frac{\partial}{\partial y}\overline{q^2} \geqq 0, \tag{32.8}$$

with equality occurring only for $y=-h$. In fact, even more can be concluded, for (32.5) is like $\overline{q^2}$ itself with Φ replaced by $2\Phi_x$. Hence, by repeating the above reasoning one may establish that

$$\frac{\partial^{2n}}{\partial y^{2n}}\overline{q^2} > 0, \qquad \frac{\partial^{2n-1}}{\partial y^{2n-1}}\overline{q^2} \geqq 0, \qquad n=1,2,\ldots. \tag{32.9}$$

Next consider $\overline{u^2}-\overline{v^2}$. A similar computation shows that

$$\frac{\partial}{\partial y}(\overline{u^2}-\overline{v^2}) = \frac{2}{\lambda}\int_0^\lambda (\Phi_x\Phi_y)_x\,dx = \frac{2}{\lambda}[\Phi_x\Phi_y]_0^\lambda = 0.$$

Hence

$$\overline{u^2}-\overline{v^2} = C(t) = \overline{u^2}|_{y=-h \text{ or } -\infty}. \tag{32.10}$$

It follows from (32.8) that

$$\overline{u^2} = \tfrac{1}{2}[\overline{q^2}+C], \qquad \overline{v^2} = \tfrac{1}{2}[\overline{q^2}-C], \qquad -\overline{p_d} = \tfrac{1}{2}\overline{q^2} - A(t) \tag{32.11}$$

are each increasing functions of y, i.e., they decrease with increasing depth. For infinite depth Longuet-Higgins shows further that, if axes are chosen such that $u=0$ at $y=-\infty$, then the quantites $|u|$, $|v|$ and $|p_d|$ all decrease exponentially to zero. He had shown earlier (1950) for exact waves (we shall not carry through the proof) that

$$\overline{p_d} = \frac{1}{2}\frac{\partial^2}{\partial t^2}\overline{\eta^2} - \overline{v^2}. \tag{32.12}$$

Hence it follows that

$$p_d|_{y=-h \text{ or } -\infty} = \frac{1}{2}\frac{\partial^2}{\partial t^2}\overline{\eta^2}. \tag{32.13}$$

For purely progressive waves this quantity vanishes, but we recall that for standing waves we found earlier a constant pressure fluctuation of double the wave frequency [see (27.62) and (27.65)] if second-order terms were retained.

Mass transport. In Sect. 27 [see (27.39) and (27.41)] it was shown that a forward drift, called "mass transport", occurred in progressive waves if second-order terms were taken into account. It was shown by Rayleigh (1876) in a proof valid only for infinitely deep fluid that mass transport must always occur. The proof is independent of the dynamical free-surface condition. LEVI-CIVITA (1912) and later URSELL (1953) developed methods of analysis to include both finite depth and nonperiodic waves; essentially URSELL'S analysis has also been given by NEKRASOV (1951) for infinite depth. The analysis given below is due to LONGUET-HIGGINS (1953) and is similar to that used in the preceding theorem. We note that STARR (1945) has also given an instructive and perspicuous derivation of RAYLEIGH'S theorem for infinite depth.

Take the wave as moving to the left with velocity c (in the sense of Sect. 7) and impose a uniform velocity c in the opposite direction, so that the motion is reduced to a steady one, generally in the positive x-direction in the sense that $u > \varepsilon > 0$. We may then write the complex potential in the form

$$f(z) = \Phi + i\,\Psi = c\,z + \varphi + i\,\psi, \tag{32.14}$$

where $\operatorname{Re} f' > \varepsilon > 0$ and

$$\Phi(x + n\,\lambda, y) = n\,c\,\lambda + \Phi(x, y), \qquad \Psi(x + n\,\lambda, y) = \Psi(x, y). \tag{32.15}$$

We take $\Phi = 0$ at a crest and assume $\Psi = 0$ as the free-surface streamline and $\Psi = -Q$ as the bottom streamline if the depth is finite. One may invert the relation $f = f(z)$ and obtain $z = z(f)$. Then, since $q^2 \neq 0$,

$$z'(f) = \frac{1}{f'(z)} = \frac{\Phi_x + i\,\Phi_y}{\Phi_x^2 + \Phi_y^2} = \frac{1}{q^2}(u + i\,v) = x_\Phi + i\,y_\Phi. \tag{32.16}$$

Denote by $T(\Psi)$ the time required for a given particle to progress one wavelength along a streamline $\Psi = \text{const}$. In the original wave motion, the time elapsed between the passage of two successive crests over a given point is λ/c. If $T > \lambda/c$, the particle is being transported with the wave and it will be reasonable to call

$$U(\Psi) = c - \frac{\lambda}{T(\Psi)} \tag{32.17}$$

see errata
the mass transport in the direction of wave motion. The following theorem is true.

Theorem. Both T and U decrease with increasing depth, and, with the assumed definition of c, $U > 0$.

The theorem may be proved by the following computation:

$$T(\Psi) = \int_0^{s(\lambda)} \frac{1}{q}\,d s = \int_0^{c\lambda} \frac{1}{q}\,\frac{\partial s}{\partial \Phi}\,d\Phi = \int_0^{c\lambda} (x_\Phi^2 + y_\Phi^2)\,d\Phi = \int_0^{c\lambda} (x_\Phi^2 + x_\Psi^2)\,d\Phi, \tag{32.18}$$

$$T'(\Psi) = 4\int_0^{c\lambda} x_\Phi\, x_{\Phi\Psi}\,d\Phi, \tag{32.19}$$

$$T''(\Psi) = 4\int_0^{c\lambda} (x_{\Psi\Phi}^2 + x_{\Psi\Psi}^2)\,d\Phi. \tag{32.20}$$

The details of the computation are almost identical with those used in deriving (32.4) and (32.5). Since

$$x_\Psi = -\,y_\Phi = -\frac{1}{q^2}\,\Phi_y = 0 \quad \text{on} \quad \Psi = -\,Q,$$

it follows from (32.19) that $T'(-Q) = 0$. Then, since $T''(\Psi) > 0$ unless the flow is uniform, it follows that

$$T'(\Psi) \geqq 0, \tag{32.21}$$

with equality holding only for $\Psi = -Q$. As in the earlier theorem, the computations can be extended to yield

$$T^{(2n)}(\Psi) > 0, \qquad T^{(2n-1)}(\Psi) \geqq 0. \tag{32.22}$$

It now follows immediately from (32.17) that

$$U'(\Psi) \geqq 0, \tag{32.23}$$

with the equality holding only for the bottom streamline. If the fluid is infinitely deep, then $U' > 0$ for all Ψ. To complete the proof we must show that $U > 0$. If the fluid is infinitely deep, it is evident that

$$\lim_{\psi \to -\infty} T(\Psi) = \frac{\lambda}{c}. \tag{32.24}$$

Hence $\lim U = 0$ as $\Psi \to -\infty$ and the conclusion follows from $U' > 0$. If the depth is finite, we compute

$$T(-Q) = \int_0^{c\lambda} x_\Phi^2 \, d\Phi > \frac{1}{c\lambda} \left[\int_0^{c\lambda} x_\Phi \, d\Phi \right]^2 = \frac{1}{c\lambda} \lambda^2 = \frac{\lambda}{c}; \tag{32.25}$$

here use has been made of the Schwarz-Bunyakovskii inequality. (We have written $>$ rather than $\geqq$, for the equal sign will hold only in the trivial case of a uniform flow.) It now follows that $U(-Q) > 0$ and hence that

$$U(\Psi) > 0 \tag{32.26}$$

since $U' \geqq 0$. This completes the proof of the theorem.

The method of analysis can be extended to prove an analogous theorem for nonperiodic steady motions which approach uniform flows as $x \to \pm\infty$, in particular, to the solitary wave.

Momentum and energy integrals. We close this section with several momentum and energy integrals, most of which have been found by LEVI-CIVITA (1912, 1921), STARR (1947a, b, 1948) and STARR and PLATZMAN (1948).

Let us again take the wave as moving to the left without change of form and impose an opposite velocity c which brings the profile to rest (or, equivalently, consider the motion relative to a coordinate system moving with the wave). Let the velocity potential be as in (32.14). Consider the area bounded by two streamlines $\Psi = \Psi_1$ and $\Psi = \Psi_2$, say $y = \eta_1(x)$ and $y = \eta_2(x)$ and two vertical lines a wavelength λ apart. To this area apply the theorem

$$\iint (\Phi_x^2 + \Phi_y^2) \, d\sigma = \oint \Phi \Phi_n \, ds. \tag{32.27}$$

This yields

$$\iint [(c+u)^2 + v^2] \, d\sigma = \int_{\eta_1(x_0+\lambda)}^{\eta_2(x_0+\lambda)} \Phi(x_0+\lambda, y) \, \Phi_x \, dy - \int_{\eta_1(x_0)}^{\eta_2(x_0)} \Phi(x_0, y) \, \Phi_x \, dy \tag{32.28}$$

since $\Phi_n = 0$ on the streamlines. Moreover, since $\Phi(x+\lambda, y) = c\lambda + \Phi(x, y)$, $\Phi_x(x+\lambda, y) = \Phi_x(x, y) = c + u$ and $\eta_i(x+\lambda) = \eta_i(x)$, the right-hand side of (32.28) may be written as

$$c^2 \lambda [\eta_2(x_0) - \eta_1(x_0)] + c\lambda \int_{\eta_1(x_0)}^{\eta_2(x_0)} \varphi_x(x_0, y) \, dy = c^2 \lambda [\eta_2(x) - \eta_1(x)] + c\lambda \int_{\eta_1(x)}^{\eta_2(x)} u \, dy. \tag{32.29}$$

Expanding $(c+u)^2$ and rearranging give

$$\iint (u^2 + v^2) \, d\sigma + 2c \iint u \, d\sigma + c^2 \iint d\sigma = c^2 \lambda [\eta_2(x) - \eta_1(x)] + c\lambda \int_{\eta_1(x)}^{\eta_2(x)} u \, dy. \tag{32.30}$$

If one now applies the operator $\lambda^{-1}\int\limits_0^\lambda \dots dx$ to (32.30), one obtains

$$\iint (u^2+v^2)\,d\sigma + c\iint u\,d\sigma = 0 \tag{32.31}$$

or, after multiplying by $\frac{1}{2}\varrho$ and rearranging,

$$\iint \tfrac{1}{2}\varrho\,(u^2+v^2)\,d\sigma = \tfrac{1}{2}c\iint -\varrho\,u\,d\sigma, \tag{32.32}$$

i.e., the kinetic energy per wavelength between two streamlines equals $\frac{1}{2}c$ times the momentum in the direction of the wave (here to the left).

Next let us write the integral (2.10′) in the form

$$\tfrac{1}{2}\varrho\,[(c+u)^2+v^2] + \varrho\,g\,y + p = \tfrac{1}{2}\varrho\,c_1^2, \tag{32.33}$$

the form of the constant having been chosen for later convenience. Write the terms $p+\varrho g y$ as follows:

$$\left.\begin{aligned} p+\varrho\,g\,y &= \frac{\partial}{\partial y}[y(p+\varrho\,g\,y)] - y\frac{\partial}{\partial y}(p+\varrho\,g\,y)\\ &= \frac{\partial}{\partial y}[y(p+\varrho\,g\,y)] + y\frac{D}{Dt}v\\ &= \frac{\partial}{\partial y}[y(p+\varrho\,g\,y)] - v^2 + \frac{D}{Dt}(y\,v)\,. \end{aligned}\right\} \tag{32.34}$$

Here we have used the second equation of (2.6). We may now write (32.33) as follows

$$\frac{1}{2}\varrho\,(u^2-v^2) + \varrho\,c\,u + \frac{\partial}{\partial y}[y(p+\varrho\,g\,y)] + \frac{D}{Dt}(y\,v) = \frac{1}{2}\varrho\,(c_1^2-c^2)\,. \tag{32.35}$$

Next let us integrate Eq. (32.35) over the same area as is described in the preceding paragraph. First consider $D(yv)/Dt$. Since the motion is steady in the selected coordinate system,

$$\frac{D}{Dt}(y\,v) = (u+c)\frac{\partial(yv)}{\partial x} + v\frac{\partial(y\,v)}{\partial y} = \frac{\partial}{\partial x}(u+c)\,y\,v + \frac{\partial}{\partial y}\,y\,v^2,$$

where the last equality follows from the continuity equation. Hence

$$\iint \frac{D}{Dt}(y\,v)\,d\sigma = \oint y\,v\,(u+c,\,v)\cdot\boldsymbol{n}\,ds = \oint y\,v\,\Phi_n\,ds = 0 \tag{32.36}$$

since $\Phi_n=0$ on the streamline boundaries and the integrals over the vertical boundaries cancel from periodicity. The integrated equation then becomes

$$\left.\begin{aligned} &\iint \tfrac{1}{2}\varrho\,(u^2-v^2)\,d\sigma + c\iint \varrho\,u\,d\sigma + \int\limits_0^\lambda \{\eta_2(x)\,[p(x,\eta_2)+\varrho\,g\,\eta_2] - \\ &\quad - \eta_1(x)\,[p(x,\eta_1)+\varrho\,g\,\eta_1]\}\,dx = \tfrac{1}{2}\varrho\,(c_1^2-c^2)\iint d\sigma\,. \end{aligned}\right\} \tag{32.37}$$

If one eliminates the second integral by means of (32.32), one obtains

$$\left.\begin{aligned} &\iint \tfrac{1}{2}\varrho\,u^2\,d\sigma + 3\iint \tfrac{1}{2}\varrho\,v^2\,d\sigma - \int\limits_0^\lambda \{\eta_2[p(x,\eta_2)+\varrho\,g\,\eta_2] - \\ &\quad - \eta_1[p(x,\eta_1)+\varrho\,g\,\eta_1]\}\,dx = \tfrac{1}{2}\varrho\,(c^2-c_1^2)\iint d\sigma\,. \end{aligned}\right\} \tag{32.38}$$

Eq. (32.38) has a simpler aspect if the two streamlines are taken as the free surface $\eta(x)$ and the bottom $y=-h$. Then $p(x,\eta(x))=0$ and the third integral becomes

$$\int\limits_0^\lambda \varrho\,g\,\eta_2^2(x)\,dx + h\int\limits_0^\lambda [p(x,-h)-\varrho\,g\,h]\,dx\,.$$

Moreover,

$$\int_0^\lambda [p(x, -h) - \varrho g h]\, dx = 0 \tag{32.39}$$

if the x-axis is taken at the mean water level. This follows from the following sequence of equations, similar to those used in (32.36):

$$\left.\begin{aligned}
&\int_0^\lambda [p(x, -h) - \varrho g h]\, dx = \iint \frac{\partial}{\partial y}[p(x, y) + \varrho g y]\, d\sigma - \\
&\quad - \iint \left[(u + c)\frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v\right] d\sigma = - \iint \left[\frac{\partial}{\partial x} v(u + c) + \frac{\partial}{\partial y} v^2\right] d\sigma \\
&\quad = \oint v(u + c, v) \cdot \boldsymbol{n}\, d\sigma = -\oint v\, \Phi_n\, d\sigma = 0.
\end{aligned}\right\} \tag{32.40}$$

Eq. (32.39) now allows us to give a simple physical interpretation of the constant c_1 in (32.33). For if (32.33) is integrated along $y = -h$, and account is taken of (32.39), one finds

$$\frac{1}{\lambda}\int_0^\lambda (u + c)^2\, dx = c_1^2 \geqq c^2, \tag{32.41}$$

i.e., c_1^2 is the mean square velocity of fluid along the bottom. The inequality follows easily from

$$\int_0^\lambda u(x, -h)\, dx = \int_0^\lambda \varphi_x(x, -h)\, dx = \varphi(\lambda, -h) - \varphi(0, -h) = 0. \tag{32.42}$$

If the fluid is infinitely deep, $u \to 0$ as $y \to -\infty$, and (32.41) reduces to

$$c^2 = c_1^2. \tag{32.43}$$

If, following (15.27), we let $\mathcal{T}_{\text{av}}, \mathcal{T}_{x\,\text{av}}, \mathcal{T}_{y\,\text{av}}, \mathcal{V}_{\text{av}}, \mathcal{M}_{\text{av}}$ denote the average kinetic energy, the contributions to this due to the x and y velocity components, the potential energy, and the momentum in the direction of wave motion, respectively, then (32.32) and (32.88) may be expressed as follows: see errata

$$2\mathcal{T}_{\text{av}} = c\,\mathcal{M}_{\text{av}}, \qquad \mathcal{T}_{x\,\text{av}} + 3\,\mathcal{T}_{y\,\text{av}} = 2\,\mathcal{V}_{\text{av}} - \tfrac{1}{2}\varrho\,(c_1^2 - c^2)\, h, \tag{32.44}$$

where the last term of the second equation is zero for $h = \infty$. The first equation is essentially due to Levi-Civita (1912, 1921), the second to Starr (1947b).

We note another simple consequence of (32.41), due to Levi-Civita (1924). Let us integrate (32.33) along the free surface for a wavelength and divide by $\frac{1}{2}\varrho\lambda$. Then, remembering our choice of x-axis as the mean water level, we find

$$\frac{1}{\lambda}\int_0^\lambda [(c + u)^2 + v^2]\, dx = c_1^2. \tag{32.45}$$

On the other hand, if we compute the velocity at the intersection of the mean water level and the profile, we also find

$$(c + u)^2 + v^2\big|_{y=0} = c_1^2. \tag{32.46}$$

Hence the absolute value of the velocity at the mean water level equals the root-mean-square velocity along the surface profile or along the bottom, or, indeed, along any streamline, for in the reasoning in (32.40) we could have substituted any streamline $y=\eta_1(x)$ for $y=-h$ and obtained

$$\int_0^\lambda [p(x,\eta_1(x)) - \varrho g \eta_1(x)]\,dx = 0. \tag{32.47}$$

STARR and PLATZMAN (1948) have used the relations above to derive some general relations concerning the flow of energy in a periodic wave. We recall that the average flux of energy in the direction of wave motion is given by [cf. Sect. 8 and Eqs. (15.23) and (15.27)]

see errata

$$\mathscr{F}_{\text{av}} = \int_0^\lambda dx \int_{-h}^{\eta(x)} \varrho\, c\, \varphi_x^2(x,y)\,dy = 2\mathscr{T}_{x\,\text{av}}. \tag{32.48}$$

It follows from the second formula in (32.44) that

$$2\mathscr{T}_{x\,\text{av}} = 3\mathscr{T}_{\text{av}} - 2\mathscr{V}_{\text{av}} + \tfrac{1}{2}\varrho(c_1^2 - c^2)\,h. \tag{32.49}$$

Hence, with $\mathscr{E}_{\text{av}} = \mathscr{T}_{\text{av}} + \mathscr{V}_{\text{av}}$, we obtain from (32.48)

see errata

$$\frac{\mathscr{F}_{\text{av}}}{\mathscr{E}_{\text{av}}} = \frac{1}{2} + \frac{5}{2}\,\frac{\mathscr{T}_{\text{av}} - \mathscr{V}_{\text{av}} + \frac{1}{5}\varrho(c_1^2 - c^2)\,h}{\mathscr{E}_{\text{av}}}. \tag{32.50}$$

This should be compared with the result derived in Sect. 15 for infinitesimal waves with neglect of surface tension [cf. (15.25) and (15.26)], namely, $\mathscr{F}_{\text{av}} = \frac{1}{2}\mathscr{E}_{\text{av}}$. Eq. (32.50) is consistent with this, for to the order of approximation involved, $\mathscr{T}_{\text{av}} = \mathscr{V}_{\text{av}}$ and $c_1^2 = c^2$. However, for waves of finite height it was shown in Sect. 27α [cf. Eqs. (27.42), (27.43)] that to the second order of approximation $\mathscr{T}_{\text{av}} > \mathscr{V}_{\text{av}}$. PLATZMAN (1947) has verified that this remains true when 4th-order terms are kept.

Several of the above results have analogues for steady motion of nonperiodic waves, provided that $\eta(x)\to 0$ as $x\to\pm\infty$ in such a way that $\int_{-\infty}^{\infty}\eta\,dx$ is finite. Under such circumstances $c_1^2 = c^2$ and the following results may be established [the notation is that of (15.31) with obvious extensions]:

$$\left.\begin{aligned}
&\mathscr{M}_{\text{total}} = c\int_{-\infty}^{\infty}\eta\,dx = c\,\mathscr{A}_{\text{total}},\\
&\mathscr{T}_{x\,\text{total}} - \mathscr{T}_{y\,\text{total}} = \mathscr{V}_{\text{total}},\\
&\mathscr{T}_{x\,\text{total}} - \mathscr{T}_{y\,\text{total}} + 2\mathscr{V}_{\text{total}} + (g\,h - c^2)\,\mathscr{A}_{\text{total}} = 0.
\end{aligned}\right\} \tag{32.51}$$

For details of the proof one may refer to STARR (1947b). From the last two equations follows

$$c^2 = g\,h + 3\mathscr{V}_{\text{total}}/\mathscr{A}_{\text{total}} > g\,h. \tag{32.52}$$

We note that the second equation of (32.51) is a special case of a more general one applying to any steady motion:

$$\mathscr{T}_x(x) - \mathscr{T}_y(x) - \mathscr{V}(x) = \text{const} \tag{32.53}$$

where the constant is zero under the conditions of (32.51). The proof is analogous to that of (8.6). Here

$$\mathscr{T}_x(x) = \int_{-h}^{\eta(x)} \tfrac{1}{2}\varrho\,u^2(x,y)\,dy, \qquad \text{etc.}$$

β) Waves in heterogeneous fluids. The first two theorems proved below are also true for homogeneous fluids and were first proved for this case. The last theorems deal specifically with heterogeneous fluids. In the extended form they are all due to DUBREIL-JACOTIN (1932).

A flow will be called barotropic if both the pressure and density are constant along streamlines. We first derive the energy integral for such flows. The Eqs. (2.6) may be written in the following form in two dimensions:

$$-\frac{1}{\varrho}\frac{\partial p}{\partial x}=\frac{\partial E}{\partial x}-v\zeta+\frac{\partial u}{\partial t}, \qquad -\frac{1}{\varrho}\frac{\partial p}{\partial y}=\frac{\partial E}{\partial y}+u\zeta+\frac{\partial v}{\partial t}, \tag{32.54}$$

where

$$E=g\,y+\frac{1}{2}(u^2+v^2), \qquad \zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}.$$

Since p is assumed constant on a streamline, $u p_x + v p_y = 0$; it follows from (32.27) and the definition of E that

$$0=u\frac{\partial E}{\partial x}+v\frac{\partial E}{\partial y}+\frac{1}{2}\frac{\partial}{\partial t}q^2=g\,v+\frac{1}{2}\frac{D}{Dt}q^2=\frac{D}{Dt}E. \tag{32.55}$$

In particular, if the flow is steady, E is also constant along a streamline. For steady flow it is a consequence of the incompressibility condition that ϱ is also constant along a stream-line.

The following theorem was proved by BURNSIDE (1915) for a homogeneous fluid. He gives two proofs, of which the second can be carried over to the present more general situation with no change. It will perhaps give more substance to the theorem if we remark that GERSTNER'S wave (see subsection 34β), which is not irrotational, satisfies the other conditions of the theorem.

Theorem. The only steady two-dimensional irrotational motion of a fluid subject to gravity for which all streamlines are also lines of constant pressure is a uniform flow.

Let the streamlines be given by $\psi(x, y)=\text{const}$. Since, from the remark following (32.55), $E=\text{const}$ along a streamline, we may write

$$\tfrac{1}{2}(\psi_x^2+\psi_y^2)+g\,y=E(\psi). \tag{32.56}$$

[BURNSIDE shows that one may generalize (32.56) by replacing $g y$ by a function $g(y)$.] Since the motion is irrotational, $\Delta\psi=0$ and hence also

$$\Delta \log(\psi_x^2+\psi_y^2)=0.$$

But then

$$\Delta \log[E(\psi)-g\,y]=0,$$

which yields after some computation

$$2y\,E'(\psi)\,\psi_y=2(E-g\,y)\,[E'^2-(E-g\,y)\,E'']+g^2. \tag{32.57}$$

We write this in the form

$$\psi_y(x, y)=G(\psi, y). \tag{32.58}$$

It then follows from (32.56) and (32.58) that

$$\psi_{xx}=E'-G\,G_\psi, \qquad \psi_{yy}=G_\psi\,\psi_y+G_y$$

or

$$E'(\psi)+G_y(\psi, y)=0.$$

But then

$$\psi_y = -y\,E'(\psi) + \text{const}$$

and ψ is a function of y only. Hence, since $\Delta\psi = \psi_{yy} = 0$, ψ_y is a constant and the flow is uniform.

The next theorem was first proved by LEVI-CIVITA (1925) for homogeneous fluids. FENCHEL (1931) showed that his hypotheses could be weakened and DUBREIL-JACOTIN (1932) extended FENCHEL'S proof to heterogeneous fluids. The gist of the theorem is that if the surface profile moves without change of form, then the whole velocity field is steady in a coordinate system moving with the surface. The theorem will be formulated in the moving coordinate system.

Theorem. Let a possibly heterogeneous fluid, bounded below by a horizontal plane $y = -h$, be flowing irrotationally in the x-direction with discharge rate $Q(t)$ and with a fixed surface profile $y = \eta(x)$. If η and u satisfy the conditions

$$-h < b_1 < \eta < b_2, \qquad u > \varepsilon > 0, \tag{32.59}$$

then the velocity potential $f(z)$ is independent of t.

First we derive a boundary condition at the free surface. From the condition of constant pressure and the assumption that the surface profile is an invariant streamline it follows that

$$u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} = 0 \qquad \text{on } \psi = 0.$$

It then follows as in (32.55) that

$$\frac{DE}{Dt} = g\,v + \frac{Dq^2}{Dt} = 0 \qquad \text{on } \psi = 0. \tag{32.60}$$

However, this conclusion holds now only on this one streamline.

The complex potential $f(z, t) = \varphi + i\psi$ maps the region of the z-plane occupied by fluid onto the strip $-Q(t) \leqq \psi \leqq 0$, where the free surface corresponds to $\psi = 0$, the bottom to $\psi = -Q$ and $x = \pm\infty$ to $\varphi = \pm\infty$. Let $F(z) = \Phi + i\Psi$ be the mapping, unique up to an additive real constant, of the fluid region onto the strip $-1 \leqq \Psi \leqq 0$ with $x = \pm\infty$ corresponding to $\Phi = \pm\infty$. Then

$$f(z, t) = Q(t)\,F(z) \tag{32.61}$$

evidently satisfies the requirements for $f(z, t)$ and, in fact, is determined uniquely, up to the added constant in F, by $Q(t)$ and $\eta(x)$. Now substitute $\varphi(x, y, t) = Q(t)\,\Phi(x, y)$ into (32.60):

$$g\,Q\,\Phi_y + Q\,Q'\,[\Phi_x^2 + \Phi_y^2] + Q^2\,[\Phi_{xx}\,\Phi_x^2 + 2\,\Phi_{xy}\,\Phi_x\,\Phi_y + \Phi_{yy}\,\Phi_y^2] = 0, \tag{32.62}$$

which we may write in the form

$$Q' + A\,Q + B = 0 \qquad \text{on } \Psi = 0 \tag{32.63}$$

where A and B are independent of t. Division by $\Phi_x^2 + \Phi_y^2$ is possible since (32.59) implies that this does not vanish. Note also that

$$B = g\,\frac{\Phi_y}{\Phi_x^2 + \Phi_y^2} = g\,y_\Phi\big|_{\Psi=0} = g\,\frac{d}{d\Phi}\,\eta, \tag{32.64}$$

and that both A and B may be considered as functions of Φ. Consider two cases: (a) $A = \text{const}$, (b) $A \neq \text{const}$. (a) In this case, since B is independent of t and

$Q'+AQ$ is independent of Φ, it follows from (32.63) that both equal constants. It now follows from (32.64) that, unless this constant is zero, the profile $\eta(x)$ will be unbounded and the first part of (32.59) will be contradicted. Hence, in case (a) $\eta=$const and the mapping F must be of the form $F=az+b$, a and b real. It then follows that $A=0$ and hence $Q'=0$, i.e. the flow is uniform. (b) Let A_1, A_2 be two different values of A, $A_1 \neq A_2$. Write Eq. (32.63) for each value and subtract. This yields

$$Q = -\frac{B_1 - B_2}{A_1 - A_2}. \tag{32.65}$$

But then Q is evidently independent of t. Hence also $f(z,t) = QF(z)$ is also independent of t. This completes the proof.

The next theorem, due to DUBREIL-JACOTIN (1932), specifically requires that the fluid be heterogeneous.

Theorem. Suppose the motion of an incompressible heterogeneous fluid to be irrotational, the free surface to move without change of form, and that, in a coordinate system moving with the surface, conditions (32.59) are satisfied. Then not only is the velocity field steady, but also E, p and ϱ are constant along the streamlines.

It follows from the preceding theorem that the velocity is steady, hence that $E=E(x,y)$. However, we may still conceivably have $p=p(x,y,t)$, $\varrho=\varrho(x,y,t)$. The Eqs. (32.54) may now be written in the form

$$-\frac{1}{\varrho}\frac{\partial p}{\partial x} = \frac{\partial E}{\partial x}, \qquad -\frac{1}{\varrho}\frac{\partial p}{\partial y} = \frac{\partial E}{\partial y}. \tag{32.66}$$

Elimination of first ϱ, then p between these two equations yields

$$\frac{\partial(p,E)}{\partial(x,y)} = 0, \qquad \frac{\partial(\varrho,E)}{\partial(x,y)} = 0.$$

We assume that the corresponding functional relations may be solved and write

$$p = p(E,t), \qquad \varrho = \varrho(E,t), \tag{32.67}$$

where, from (32.66)

$$\varrho = -\frac{\partial p}{\partial E}. \tag{32.68}$$

From the equation expressing incompressibility, namely,

$$\frac{D\varrho}{Dt} = \frac{\partial \varrho}{\partial t} + u\frac{\partial \varrho}{\partial x} + v\frac{\partial \varrho}{\partial y} = 0,$$

follows

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial E}\left(u\frac{\partial E}{\partial x} + v\frac{\partial E}{\partial y}\right) = \frac{\partial \varrho}{\partial t} + \frac{\partial(E,\psi)}{\partial(x,y)}\frac{\partial \varrho}{\partial E} = 0. \tag{32.69}$$

We shall assume $\partial\varrho/\partial E \neq 0$ everywhere, and may thus write

$$\frac{\partial(E,\psi)}{\partial(x,y)} = -\frac{\varrho_t(E,t)}{\varrho_E(E,t)}. \tag{32.70}$$

Since the left-hand side is independent of t, it follows from the form of the right-hand side that we may set both sides equal to $k(E)$, i.e.

$$\frac{\partial \varrho}{\partial t} + k(E)\frac{\partial \varrho}{\partial E} = 0. \tag{32.71}$$

Let us suppose $k(E) \not\equiv 0$, e.g. $k(E_1) \neq 0$. Then ϱ must be a function of the form

$$\varrho = \varrho\left(t - \int_{E_1}^{E} \frac{dE}{k(E)}\right) \tag{32.72}$$

in some neighborhood of E_1. If $k(E)$ vanishes for some values of E, let E_0 be the first zero larger than E_1. Then from (32.71) and (32.72)

$$\varrho_t(E, t) = \varrho'\left(t - \int_{E_1}^{E} \frac{dE}{k(E)}\right) \to 0 \quad \text{as } E \to E_0. \tag{32.73}$$

But (32.73) can hold for all t only if $\varrho' = 0$, i.e. if $\varrho = \text{const}$, which is contrary to the assumed heterogeneity. Moreover, at least one such zero of k exists, for we already know from (32.60) that E is constant along the free surface, so that in steady motion the Jacobian in (32.70) vanishes for $\psi = 0$. Hence $k(E) = 0$ for the corresponding value of E. We must conclude that $k(E) \equiv 0$. This implies, from (32.70) that $E = E(\psi)$ and $\varrho = \varrho(E)$. From (32.68) and the condition $p_t = 0$ on the free surface, it follows that also $p = p(E)$. Hence p, ϱ and E are all constant on streamlines.

The last in this complex of theorems is also due to DUBREIL-JACOTIN (1932).

Theorem. There cannot exist irrotational waves in a heterogeneous fluid such that the profile is propagated without change of form.

This follows immediately from the first and last theorems proved above, and is, of course, subject to condition (32.59). This striking result is all the more so in view of the fact that GERSTNER'S wave (subsection 34β) does provide a steadily propagating wave, even in a heterogeneous fluid. The theorem also casts some doubt upon the significance of the linearized theory of irrotational wave motion in a heterogeneous fluid as developed, for example, in LAMB (1932, § 235). Such a wave evidently cannot be considered as a first approximation to an exact steady solution.

see errata

γ) *Some transformations of the boundary-value problem.* By means of introduction of new variables or other devices, it is possible to formulate the boundary-value problem for exact solutions in a variety of ways. Several such formulations will be considered in subsection 34α on inverse methods. Here we give a few which seem to be of general interest.

Inversion of $f(z)$. One elementary but important transformation has already been introduced in subsection 32α in the discussion of mass transport. This is the inversion of the velocity potential $f(z)$ when $|f'|$ vanishes nowhere within the fluid, and treatment of f as the independent variable. This has the advantage that under certain circumstances the domain of definition of the independent variable can be given exactly; when z is the independent variable, the domain of definition is one of the unknowns of the problem. For example, if the motion is reducible to a steady flow with discharge rate Q, one may take the surface profile to correspond to $\psi = 0$ and the bottom streamline to correspond to $\psi = -Q$. Hence the domain of definition of $z(f)$ is the strip $0 \geqq \psi \geqq -Q$; if the fluid is infinitely deep, the domain is the half-plane $\psi \leqq 0$. Whenever f can be taken as the independent variable, then one can also express $w = f'$ as a function of f. It has been established independently by GERBER (1951) and LEWY (1952a) that the equation describing the free surface, $z = z(\varphi)$, is an analytic function of φ at all points for which $w \neq 0$.

STOKES' "second method". In the introduction to Sect. 27 it was mentioned that STOKES (1880), in a supplement to an earlier paper in his collected works, developed a method for approximating exact periodic waves which is different from the straightforward generalization of infinitesimal-wave theory expounded in that section. This method is based upon use of f as the independent variable and expansion of z as a Fourier series in f:

$$c\,z = f + i\,\frac{c\,\lambda}{2\pi}\sum_{n=0}^{\infty} a_n\,e^{-i n 2\pi f/c\lambda} \tag{32.74}$$

or

$$c\,z = f + i\,\frac{c\,\lambda}{2\pi}\,a_0 + \frac{c\,\lambda}{2\pi}\sum_{n=1}^{\infty} a_n \sin n\,\frac{2\pi}{c\,\lambda}\,(f + i\,Q) \tag{32.75}$$

for infinite and finite depth respectively; the a_n may be taken to be real. Here ψ is taken as in the preceding paragraph. The coefficients a_n are to be determined from the condition that the pressure be constant on the surface, i.e. from

$$q^2 + 2g\,y = C \qquad \text{for } \psi = 0. \tag{32.76}$$

If the mean water level is taken at $y=0$ and the fluid is infinitely deep, then $C = c^2$; we shall consider only this case here. Then Eq. (32.76) may be expressed as

$$(c^2 - 2g\,y)\,|z'|^2 = 1. \tag{32.77}$$

Substitution of (32.74) in (32.77) yields

$$\left.\begin{aligned} &\left(1 - \frac{g\,\lambda}{\pi\,c^2}\sum_{n=0}^{\infty} a_n \cos\frac{2\pi\,n\,\varphi}{c\,\lambda}\right)\times\\ &\quad\times\left(1 + 2\sum_{n=1}^{\infty} n\,a_n \cos\frac{2\pi\,n\,\varphi}{c\,\lambda} + \sum_{n,m=1}^{\infty} n\,m\,a_n\,a_m \cos(n-m)\,\frac{2\pi\,\varphi}{c\,\lambda}\right) = 1. \end{aligned}\right\} \tag{32.78}$$

After multiplying the two factors and reducing the cosine products to cosines of sums and differences, the resulting expression may be put into the form

$$\sum_{n=0}^{\infty}\left(\frac{g\,\lambda}{\pi\,c^2}\,b_n + c_n\right)\cos\frac{2\pi\,n\,\varphi}{c\,\lambda} = 0, \tag{32.79}$$

where the b_n's and c_n's are forms of the third degree in the a_n's. The coefficients of the individual cosine terms must then be equated to zero. This results in an infinite sequence of equations, each involving all the a_n's and $g\,\lambda/\pi c^2$. In order to proceed further, one must devise some method for approximate determination of the a_n's. STOKES' procedure was to assume that each a_n could be expanded in a power series in some parameter, the initial term in the series having the power n. This allows one to carry through a step-by-step improvement in the approximation of the a_n's by including successively higher powers of the parameter. We shall not pursue the matter further, but remark that the most systematic arrangement of such computations seems to have been devised by SRETENSKII (1952).

LEVI-CIVITA's differential-difference equation. The following theorem, due to LEVI-CIVITA (1907), reduces determination of $w(f)$ for steady flow over a horizontal bottom to solution of a differential-difference equation.

Theorem. The complex velocity $w=u-iv$ of an irrotational gravity flow with constant discharge rate Q and with $u\geqq\varepsilon>0$ must satisfy the differential-difference equation

$$\frac{d}{df}[w(f+iQ)\,w(f-iQ)]-ig\left[\frac{1}{w(f+iQ)}-\frac{1}{w(f-iQ)}\right]=0. \tag{32.80}$$

Conversely, any function $w(f)$ satisfying (32.80) which is regular in the strip $-2Q\leqq \operatorname{Im} f\leqq 0$, finite at ∞, real on $\operatorname{Im} f=-Q$ and has $u>\varepsilon>0$ represents such a flow.

In order to derive (32.80), we note first that the functions $w(f)$ and $z(f)+ih$ both have vanishing real parts for $\psi=-Q$ and consequently can be extended by reflection to the strip $-Q\geqq\psi\geqq-2Q$:

$$w(\bar f-2iQ)=\overline{w(f)},\qquad z(\bar f-2iQ)+ih=\overline{z(f)}-ih. \tag{32.81}$$

The free-surface condition may be expressed by the equation

$$\frac{\partial}{\partial\varphi}w\overline{w}+2g\frac{\partial}{\partial\varphi}y=0\qquad\text{for }\psi=0, \tag{32.82}$$

or, by making use of the extended definitions of w and z, by

$$\frac{\partial}{\partial\varphi}\{w(\varphi)\,w(\varphi-2iQ)-ig[z(\varphi)-z(\varphi-2iQ)]\}=0. \tag{32.83}$$

Consider the function

$$H(f)=w(f+iQ)\,w(f-iQ)-ig[z(f+iQ)-z(f-iQ)]. \tag{32.84}$$

Evidently, H is defined and is regular on the line $\psi=-iQ$ and thus in some neighborhood of this line. From (32.83) it follows that $H'(\varphi-iQ)=0$, hence that $H'(f)\equiv 0$ in its region of definition. Eq. (32.80) follows from the fact that $z'(f\pm iQ)=1/w(f\pm iQ)$. For proof of the converse we refer to LEVI-CIVITA's paper. LEVI-CIVITA also gives a special form of (32.80) appropriate to a space-periodic flow. CISOTTI (1919) generalized the preceding theorem to include a variable discharge rate. The Eq. (32.80) may be considered to contain the Eq. (22.30), when in that equation $f(z,t)=f(z-ct)$, in the sense that linearization of (32.80) by assuming

$$w=c(1+\varepsilon w_1+\cdots)$$

yields (22.30).

RUDZKI's transformation. The following transformation was apparently first introduced by RUDZKI (1898). It has later been used by many others in the investigation of exact water waves. The validity of the reformulated boundary condition is not limited to periodic waves. However, it is assumed that a coordinate system has been selected with respect to which the flow is steady. It is again assumed that $u>\varepsilon>0$. Let ϑ be the angle between the velocity vector $\overline{w}=u+iv$ and the positive x-axis. Then one may write

$$w=u-iv=q\,e^{-i\vartheta}=c\,e^{-i\omega} \tag{32.85}$$

where

$$\omega=\vartheta+i\tau,\qquad q=c\,e^{\tau}. \tag{32.86}$$

Here c is some typical velocity, say the wave velocity as defined in Sect. 7. We consider ω as a function of f and let $\psi=0$ correspond to the free surface. The free-surface condition may then be expressed by

$$g\frac{\partial y}{\partial\varphi}+q\frac{\partial q}{\partial\varphi}=0\quad\text{for }\psi=0. \tag{32.87}$$

But [see (32.16)]

$$\frac{\partial y}{\partial \varphi} = \frac{1}{q^2}\frac{\partial \varphi}{\partial y} = \frac{1}{q}\sin\vartheta$$

and, from (32.86),

$$\frac{\partial q}{\partial \varphi} = c\,e^{\tau}\frac{\partial \tau}{\partial \varphi} = q\frac{\partial \tau}{\partial \varphi}.$$

Hence (32.87) becomes

$$\frac{\partial \tau}{\partial \varphi} = -g\frac{1}{q^3}\sin\vartheta = -\frac{g}{c^3}e^{-3\tau}\sin\vartheta \quad \text{for} \quad \psi = 0, \tag{32.88}$$

or, since $\partial\tau/\partial\varphi = -\partial\vartheta/\partial\psi$ from the Cauchy-Riemann equations,

$$\frac{\partial \vartheta}{\partial \psi} = \frac{g}{c^3}e^{-3\tau}\sin\vartheta \qquad \text{for } \psi = 0. \tag{32.89}$$

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It one can find a function $\omega(f)$ regular in the strip $0 \geqq \psi \geqq -Q$, with $|\vartheta| < \frac{1}{2}\pi - \varepsilon'$, and with its real and imaginary parts satisfying (32.88) or (32.89) on $\psi = 0$, one may then construct from it a free-surface flow with gravity. Of course, further conditions must be imposed at $\psi = -Q$ or as $\psi \to -\infty$.

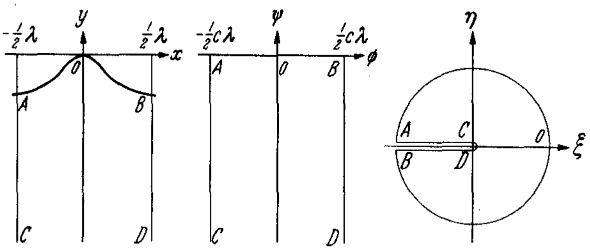

Fig. 50.

NEKRASOV'S transformation. The following transformation is due to NEKRASOV (1921, 1951). It will be assumed that the surface is periodic with period λ, symmetric about a crest and that the fluid is infinitely deep and $\lim_{y\to-\infty} w = c$. Let the origin in the z-plane be taken at a crest, $\psi = 0$ be the free surface, and assume $u > \varepsilon > 0$. In addition to the z- and f-planes, we introduce a ζ-plane,

$$\zeta = \xi + i\eta = \varrho\,e^{i\gamma}, \tag{32.90}$$

related to the f-plane through

$$\zeta = e^{-\frac{2\pi i}{\lambda c}f}. \tag{32.91}$$

With a cut along the negative ξ-axis there is a one-to-one correspondence between the various domains $CAOBD$ shown in Fig. 50.

The relation between the z- and ζ-planes will be determined by

$$\frac{dz}{d\zeta} = -\frac{\lambda}{2\pi i}\frac{h(\zeta)}{\zeta}, \quad h(\zeta) = 1 + a_1\zeta + a_2\zeta^2 + \cdots, \quad a_k \text{ real}, \tag{32.92}$$

where $h(\zeta)$ is regular in the disc and is related to w by

$$w = \frac{df}{d\zeta}\frac{d\zeta}{dz} = \frac{c}{h(\zeta)}. \tag{32.93}$$

The form of h shown in (32.92) follows from the assumed properties of the motion. Since $\varrho = 1$ on the free surface, the condition of constant pressure may be expressed by

$$2g\frac{\partial y}{\partial \gamma} + \frac{\partial q^2}{\partial \gamma} = 0 \quad \text{for} \quad \varrho = 1. \tag{32.94}$$

But

$$\left.\frac{\partial y}{\partial \gamma}\right|_{\varrho=1} = \operatorname{Im} \frac{dz}{d\zeta}\frac{d\zeta}{d\gamma}\bigg|_{\varrho=1} = \operatorname{Im} \frac{-\lambda}{2\pi i}\frac{h(\zeta)}{\zeta} i\zeta\bigg|_{\varrho=1} = \frac{-\lambda}{2\pi} \operatorname{Im} h(e^{i\gamma}). \tag{32.95}$$

It then follows from (32.93) and (32.95) that

$$\frac{d}{d\gamma}\frac{1}{h(e^{i\gamma})\,h(e^{-i\gamma})} = \frac{\lambda g}{\pi c^2} \operatorname{Im} h(e^{i\gamma}). \tag{32.96}$$

In this formulation of the problem one seeks a function $h(\zeta)$, regular in the disc $|\zeta| \leqq 1$, real on the real axis, $h(0) = 1$, and satisfying (32.96). From such a function one can easily construct a periodic gravity flow with free surface.

NEKRASOV's integral equation. NEKRASOV also considers the function ω of (32.92), but as a function of ζ. Let us start from (32.88) and compute

$$\frac{\partial \tau}{\partial \gamma} = \frac{\partial \tau}{\partial \varphi}\frac{\partial \varphi}{\partial \gamma} = -\frac{g}{c^3} e^{-3\tau}\sin\vartheta \cdot \frac{-\lambda c}{2\pi} = \frac{g\lambda}{2\pi c^2} e^{-3\tau} \sin\vartheta \quad \text{for} \quad \varrho = 1. \tag{32.97}$$

One may formally integrate this equation and obtain

$$e^{3\tau} = \frac{3}{2\pi}\frac{g\lambda}{c^2\mu}\left[1 + \mu\int_0^\gamma \sin\vartheta(\alpha)\,d\alpha\right], \tag{32.98}$$

where $1/\mu$ is the integration constant; μ is related to the velocity at the crest, $q_0 = \tau(1) = c/h(1)$, by

$$\mu = \frac{3}{2\pi}\frac{g\lambda c}{q_0^3} > 0. \tag{32.99}$$

Substitution of (32.98) into (32.99) yields the following equation for the relation between τ and ϑ on the boundary:

$$\frac{d\tau(\gamma)}{d\gamma} = \frac{1}{3}\,\frac{\mu\sin\vartheta(\gamma)}{1 + \mu\int_0^\gamma \sin\vartheta(\alpha)\,d\alpha} \tag{32.100}$$

[it follows from (32.98) that the denominator does not vanish]. It is known from the theory of functions of a complex variable (see, e.g., CARATHÉODORY, Funktionentheorie, Bd. 1, § 147—149, Birkhäuser, Basel, 1950) that, if a function is regular within and on a closed Jordan curve, it is determined up to an additive constant by giving either its real or imaginary part on the boundary. In particular, in the case at hand we may express the value of ϑ on the boundary $|\zeta| = 1$ in terms of τ on the boundary:

$$\vartheta(\gamma) = \text{const} - \frac{1}{2\pi}\,\mathrm{PV}\int_0^{2\pi} \tau(\beta)\cot\frac{1}{2}(\gamma - \beta)\,d\beta, \tag{32.101}$$

where the constant $= i\vartheta|_{\zeta=0} = 0$. An integration by parts gives

$$\vartheta(\gamma) = -\frac{1}{\pi}\int_0^{2\pi} \frac{d\tau}{d\beta}\log\left|\sin\frac{1}{2}(\gamma - \beta)\right| d\beta. \tag{32.102}$$

From the assumed symmetry about a crest follows $\tau'(-\beta) = -\tau'(\beta)$, so that (32.102) may be expressed as follows:

$$\vartheta(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\tau}{d\beta}\log\left|\frac{\sin\frac{1}{2}(\gamma + \beta)}{\sin\frac{1}{2}(\gamma - \beta)}\right| d\beta. \tag{32.103}$$

Substitution of (32.100) into (32.103) yields NEKRASOV's nonlinear integral equation for $\vartheta(\gamma)$:

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$$\vartheta(\gamma) = \frac{1}{\sigma\pi} \int_0^{2\pi} \frac{\mu \sin\vartheta(\beta)}{1+\mu \int_0^\beta \sin\vartheta(\alpha)\, d\alpha} \log\left|\frac{\sin\frac{1}{2}(\gamma+\beta)}{\sin\frac{1}{2}(\gamma-\beta)}\right| d\beta. \tag{32.104}$$

If one can find ϑ satisfying (32.104), one can then reconstruct $\omega(\zeta)$ and hence the whole flow.

NEKRASOV (1928, 1951) carried through a similar analysis when the depth is finite. We shall only sketch it. In Fig. 50 suppose that $y=-h_1$ represents the bottom (h_1 is not the mean depth) and $\psi=-Q$ the corresponding streamline. In the ζ-plane this maps into a circle of radius $\varrho_0<1$, where

$$\varrho_0 = e^{-\frac{2\pi Q}{\lambda c}}. \tag{32.105}$$

In (32.92) $h(\zeta)$ becomes a Laurent series. The integral equation for $\vartheta(\gamma)$ remains the same in form as (32.104), but the kernel function $\log|\ldots|$ is now replaced by

$$\sum_{n=1}^{\infty} \frac{2}{n} \tanh\frac{2\pi Q}{\lambda c} \sin n\gamma \sin n\beta. \tag{32.106}$$

MOISEEV (1957b) has further generalized NEKRASOV's equation so as to allow a wavy bottom.

The solution $\vartheta(\gamma)$ of (32.104) will, of course, depend upon the parameter μ, except for the trivial solution $\vartheta\equiv 0$ corresponding to a uniform flow. It is possible to show that not all μ's are allowable. Let

$$M = \max|\vartheta(\gamma)|. \tag{32.107}$$

It then follows from (32.102) that

$$0 \leqq |\vartheta(\gamma)| < \frac{1}{6\pi} \frac{\mu\sin M}{1-\pi\mu\sin M} \int_0^{2\pi} -\log\left|\sin\frac{1}{2}(\gamma-\beta)\right| d\beta < \frac{1}{3} \frac{\mu\sin M}{1-\pi\mu\sin M}, \tag{32.108}$$

hence that

$$0 \leqq M < \frac{1}{3} \frac{\mu\sin M}{1-\pi\mu\sin M}. \tag{32.109}$$

From this follows

$$\frac{1}{\pi\sin M} > \mu > \frac{1}{\pi\sin M + \frac{1}{3}M^{-1}\sin M} > \frac{3}{1+3\pi}. \tag{32.110}$$

VILLAT's integral equation. Even though we shall not consider its contents in any detail, it would be improper not to mention an important paper of VILLAT (1915). VILLAT wished to find the steady motion of a fluid in a canal of given bottom profile and also with a given top profile over the part of the fluid upstream of some point. Downstream of this point the top profile is one of constant pressure. The boundary condition on the free surface, (32.89), is modified by introduction of new variables, and a pair of integral equations, one of them nonlinear, is derived. The method is also applicable if the upstream "cover" is absent and, in fact, becomes a little simpler. The chief use made of the procedure by VILLAT is as an inverse method in which the free surface is given and the corresponding bottom and cover determined.

33. Waves of maximum amplitude. In the higher-order theory of infinitesimal waves one of the important effects of including higher-order terms was to make the profile more peaked at the crests and flatter in the troughs. The effect was the same for either steady progressive waves or standing waves. Since the peakedness increased with increase of the amplitude-to-wavelength ratio, it seems reasonable to conjecture that there is some bound to this ratio and that, if a wave of maximum amplitude-to-length ratio exists, it will be characterized by a corner or a cusp at the crest, at least if capillarity is neglected. It has never been proved that such waves exist. However, if one assumes their existence, it is possible to prove some necessary properties. This will be done below.

Following an earlier erroneous investigation of RANKINE (1865), STOKES (1880, p. 225) showed that, if a corner occurs in steady motion, the angle included between the tangents must be 120°. MICHELL (1893) assumed that a periodic highest progressive wave exists and showed how to compute the coefficients of an associated series, but without proving convergence. HAVELOCK (1919) made MICHELL'S procedure the basis of a general method of approximation to periodic progressive waves, again with no proof of convergence. MICHELL'S wave was later investigated by a different procedure by NEKRASOV (1920). However, NEKRASOV did not carry his computations to the same degree of accuracy as MICHELL and HAVELOCK, so that the numerical results are discrepant. More recently YAMADA (1957) rediscovered NEKRASOV'S method and carried through the calculations with the necessary accuracy; the results are now in substantial agreement with those of HAVELOCK and MICHELL.

PENNEY and PRICE (1952b), in their work on standing waves of finite amplitude, include an analysis intended to show that, if there exists a standing wave of maximum amplitude with a corner at the crest, then the angle must be 90°. G.I. TAYLOR (1953) has questioned the validity of the proof, and it appears, in fact, to be untenable. On the other hand, in the same paper TAYLOR reports the results of experiments which appear to confirm PENNY and PRICE'S prediction. In view of the present unsatisfactory state of the theory, it will not be further discussed here.

STOKES' theorem. We prove first STOKES' theorem on the angle at a corner in steady flow. Let the corner be at the origin $z=0$, the free surface be the streamline $\psi=0$, and $\varphi=0$ at the corner. Since $z=0$ is assumed to be a corner, it must also be a stagnation point and the constant-pressure condition on the surface may be taken in the form

$$q^2+2g\eta(x)=0. \tag{33.1}$$

In the mapping from the z- to the f-plane the point $z=0$ must be a branch point, so that in the neighborhood of $z=0$ the complex velocity potential will take the form

$$f=Az^n. \tag{33.2}$$

If $\alpha_+<0$ is the angle between the right-hand tangent to the corner and OX, then near $z=0$ Eq. (33.1) can be written

$$|A|^2 n^2 r^{2n-2}+2gr\sin\alpha_+=0.$$

This can hold for all small r only if

$$n=\tfrac{3}{2}. \tag{33.3}$$

It also follows that, if α_- is the angle between the left-hand tangent and OX, then $\sin\alpha_-=\sin\alpha_+$ and $\alpha_-=-180°-\alpha_+$ so that the surface is symmetrical about

OX near the corner. If $\psi \leq 0$ corresponds to the region occupied by fluid and if the branch of $z = r\,e^{i\alpha}$ with $-\frac{3}{2}\pi < \alpha < \frac{1}{2}\pi$ is taken, then the complex velocity potential has the following form near $z=0$:

$$\left.\begin{aligned} f(z) &= -\tfrac{2}{3}\sqrt{g}\,(-iz)^{\frac{3}{2}} \\ &= -\tfrac{2}{3}\sqrt{g}\,r^{\frac{3}{2}}\left[\cos\tfrac{3}{2}(\alpha - \tfrac{1}{2}\pi) + i\sin\tfrac{3}{2}(\alpha - \tfrac{1}{2}\pi)\right]. \end{aligned}\right\} \tag{33.4}$$

The streamline $\psi = 0$ has a corner at $z=0$ with included angle 120°. In this case the flow is to the right. The inversion of (33.4) gives

$$z(f) = \left[\frac{3}{2\sqrt{g}}\right]^{\frac{2}{3}} e^{-i\pi/6} f^{\frac{2}{3}} \tag{33.5}$$

for f near 0.

α) Periodic wave of maximum height. Let us suppose that a periodic progressive wave of maximum amplitude-length ratio exists. We may take this as a steady flow with complex velocity potential $f(z) = \varphi + i\psi$ and with

$$\lim_{y \to -\infty} f'(z) = c. \tag{33.6}$$

Let the origin of the z-plane be at one of the crests, the surface profile correspond to $\psi = 0$, and the origin of the f-plane to that of the z-plane. Then the free surface condition may be taken in the form (33.1).

MICHELL's method. First we give MICHELL's procedure for finding $f'(z)$. As we have done earlier, we shall write

$$f'(z) = q\,e^{-i\vartheta}, \qquad z'(f) = \frac{1}{q}\,e^{i\vartheta}. \tag{33.7}$$

From the assumed periodicity and symmetry, ϑ is an odd periodic function of φ with period $c\lambda$ for $\psi = 0$. From (33.7) follows

$$\frac{d}{df}\log z'(f) = -\frac{\partial}{\partial\varphi}\log q + i\,\frac{\partial\vartheta}{\partial\varphi}. \tag{33.8}$$

For $\psi = 0$, $\partial\vartheta/\partial\varphi$ is an even periodic function of φ with removable singularities at the crests; we expand it in a Fourier series:

$$\frac{\partial\vartheta}{\partial\varphi} = \frac{\pi}{c\lambda}\left[a_0 + a_1\cos\frac{2\pi\varphi}{c\lambda} + a_2\cos\frac{4\pi\varphi}{c\lambda} + \cdots\right]. \tag{33.9}$$

The a_k are real. Substitute (33.9) into (33.8) and rewrite it in the following way:

$$\left[\frac{d}{df}\log z'(f) - i\,\frac{\pi}{c\lambda}\sum_{n=0}^{\infty} a_n\,e^{-i2n\pi f/c\lambda}\right]_{\psi=0} = -\frac{\partial}{\partial\varphi}\log q - \frac{\pi}{c\lambda}\sum_{n=1}^{\infty} a_n \sin\frac{2n\pi\varphi}{c\lambda}. \tag{33.10}$$

Now consider the function

$$Z(f) = \frac{d}{df}\log z'(f) - i\,\frac{\pi}{c\lambda}\sum_{n=0}^{\infty} a_n\,e^{-i2n\pi f/c\lambda}. \tag{33.11}$$

$Z(f)$ is defined in the whole lower half-plane, is regular for $\psi < 0$, and, as $\psi \to -\infty$, $Z(f) \to -i\pi a_0/c\lambda$. Moreover, from (33.10) Z is also real on the real axis and hence may be extended by reflection to the upper half-plane. Z is then a function with only singularities on the real axis at the points $\varphi = nc\lambda$ associated with the crests. The form of the singularity may be determined from (33.5). In fact, near $f=0$

$$\frac{d}{df}\log z' = -\frac{1}{3}\,\frac{1}{f}. \tag{33.12}$$

Hence $Z(f)$ has singularities of the form

$$-\frac{1}{3}\frac{1}{f-nc\lambda}, \quad n=0, \pm 1, \pm 2, \ldots, \tag{33.13}$$

along the real axis, and only these, so that it must have the form

$$Z(f) = -\frac{\pi}{3c\lambda}\cot\frac{\pi f}{c\lambda} + b, \quad b = \text{const}. \tag{33.14}$$

From (33.14) $Z \to b - i\pi/3c\lambda$ as $\psi \to -\infty$. Then from (33.11)

$$b = -i\frac{\pi}{c\lambda}\left(a_0 - \frac{1}{3}\right).$$

Since Z must be real for real f, it follows that

$$a_0 = \tfrac{1}{3} \tag{33.15}$$

and

$$Z(f) = -\frac{\pi}{3c\lambda}\cot\frac{\pi f}{c\lambda}. \tag{33.16}$$

It now follows from the definition of $Z(f)$ that

$$\frac{d}{df}\log z'(f) = -\frac{\pi}{3c\lambda}\cot\frac{\pi f}{c\lambda} + i\frac{\pi}{3c\lambda} + i\frac{\pi}{c\lambda}\sum_{n=1}^{\infty} a_n\,\mathrm{e}^{-i2n\pi f/c\lambda}, \tag{33.17}$$

which yields, after integration, inversion of the logarithm and use of (33.6) to evaluate a multiplicative constant,

$$\left.\begin{aligned} z'(f) &= \frac{1}{c\sqrt[3]{2}}\,\mathrm{e}^{\frac{1}{3}i\pi f/c\lambda}\left(i\sin\frac{\pi f}{c\lambda}\right)^{-\frac{1}{3}}\prod_{n=1}^{\infty}\exp\left(\frac{-c\lambda}{2n\pi}a_n\,\mathrm{e}^{-i2n\pi f/c\lambda}\right) \\ &= \frac{1}{c\sqrt[3]{2}}\,\mathrm{e}^{\frac{1}{3}i\pi f/c\lambda}\left(i\sin\frac{\pi f}{c\lambda}\right)^{-\frac{1}{3}}\sum_{n=0}^{\infty} b_n\,\mathrm{e}^{-i2n\pi f/c\lambda}, \end{aligned}\right\} \tag{33.18}$$

where $b_0 = 1$ and the b_n are real. The branch of the root must be chosen so that its argument lies between $\pm\frac{1}{2}\pi$ for $\psi = 0$. From (33.18) one finds immediately also

$$f'(z) = c\sqrt[3]{2}\,\mathrm{e}^{-\frac{1}{3}i\pi f/c\lambda}\left(i\sin\frac{\pi f}{c\lambda}\right)^{\frac{1}{3}}\sum_{n=0}^{\infty} c_n\,\mathrm{e}^{-i2n\pi f/c\lambda}, \quad c_0 = 1,\ c_n \text{ real}. \tag{33.19}$$

Aside from the first one, the coefficients in (33.18) or (33.19) are still to be determined. The constant-pressure condition for the surface profile is still available for this purpose, for we have made use of the Eq. (33.4) or (33.5) only through the value of the exponent. The value of the gravitation constant has not entered into (33.18) or (33.19). In fact, a comparison of (33.5) after differentiation and (33.18) in the neighborhood of $f = 0$ yields immediately

$$\frac{c^2}{g\lambda} = \frac{3}{4\pi}[1 + b_1 + b_2 + \cdots]^3 = \frac{3}{4\pi}[1 + c_1 + c_2 + \cdots]^{-3}, \tag{33.20}$$

so that, once the b_n or c_n are determined, the relation between wavelength and velocity may be found. This method could presumably be pursued to obtain a sequence of further equations for determination of the b_n. However, MICHELL proceeds somewhat differently. If we differentiate (33.1) with respect to φ, we may write the free surface condition as follows [cf. (32.54) and following]:

$$\left.\begin{aligned} \frac{\partial}{\partial\varphi} q^2 &= -\frac{g}{q^2}\frac{\partial\varphi}{\partial y} \qquad \text{for } \psi = 0 \\ &\text{or} \\ \frac{\partial}{\partial\varphi}|f'|^4 &= 4g \operatorname{Im} f' \qquad \text{for } \psi = 0. \end{aligned}\right\} \tag{33.21}$$

Substitution of (33.19) in (33.21) yields an equation of the following form

$$\left.\begin{aligned} &\frac{4}{3}\pi\, 2^{\frac{4}{3}} \frac{c^3}{\lambda} \sin^{\frac{1}{3}} \frac{\pi\varphi}{c\lambda} \left\{A_1 \cos\frac{\pi\varphi}{c\lambda} + A_3 \cos\frac{3\pi\varphi}{c\lambda} + \cdots\right\} \\ &\qquad = \frac{4}{3} g c \sin^{\frac{1}{3}}\frac{\pi\varphi}{c\lambda}\left\{B_1\cos\frac{\pi\varphi}{c\lambda} + B_3\cos\frac{3\pi\varphi}{c\lambda} + \cdots\right\}, \end{aligned}\right\} \tag{33.22}$$

where the B_n's depend linearly upon the c_n's, and the A_n's depend upon them in a more complicated manner. The derivation of (33.22), especially of the right-hand part, and of the particular dependence of the A_n's and B_n's upon the c_n's is rather tedious and we refer to either MICHELL's original paper or preferably to HAVELOCK's more general and systematic treatment. Equating coefficients of the individual cosine terms leads to a set of equations relating $c^2/g\lambda$, c_1, c_2, ... The values as computed by HAVELOCK, which we assume to be somewhat more accurate than MICHELL's own, are as follows:

$$\frac{g\lambda}{c^2} = 0.833 \cdot 2\pi, \quad c_1 = 0.0414, \quad c_2 = 0.0114, \quad c_3 = 0.0042, \quad c_4 = 0.0014. \tag{33.23}$$

The value for $g\lambda/c^2$ should be compared with that for infinitesimal waves, namely 2π. Substitution of $\frac{1}{2}c\lambda$ for f in (33.19) yields the velocity at a trough:

see errata

$$u = c\sqrt[3]{2}\,[1 - c_1 + c_2 - c_3 + \cdots] \approx 1.219c. \tag{33.24}$$

From $q^2 + 2g\eta = 0$ one may now find η for the trough and hence the amplitude-wavelength ratio:

$$\left|\frac{\eta}{\lambda}\right| = \frac{1}{\sqrt[3]{2}}\frac{c^2}{g\lambda}[1 - c_1 + c_2 - \cdots]^2 \approx 0.1418. \tag{33.25}$$

H. JEFFREYS (1951) has recently reexamined the basis of the Michell-Havelock method of approximation and concludes that an apparent discrepancy between the values in (33.23) and Eq. (33.20) does not really indicate a numerical error in the computations.

We note in passing that MICHELL also gave the form of $f'(z)$ analogous to (33.19) which must hold if a highest wave with corner exists in a fluid of finite depth.

Method of NEKRASOV and YAMADA. This method makes use of the ζ-plane introduced in (32.57) and related to the f-plane by (32.58). We may again make use of Fig. 50 but must keep in mind that in the z-plane there is now a corner at 0 with an included angle of 120°. Hence (32.59), the equation relating the z- and ζ-planes, must be replaced by

$$\frac{dz}{d\zeta} = -\frac{\lambda}{2\pi i}\frac{h(\zeta)}{\zeta(1-\zeta)^{\frac{1}{3}}}, \qquad h(\zeta) = 1 + a_1\zeta + a_2\zeta^2 + \cdots \tag{33.26}$$

and (32.60) by

$$w = \frac{df}{dz} = c\,\frac{(1-\zeta)^{\frac{1}{3}}}{h(\zeta)}. \tag{33.27}$$

The coefficients a_n are now to be determined by the constant-pressure condition on the free surface taken in the form (32.94). From

$$q^2|_{\varrho=1} = c^2 \frac{[(1-e^{i\gamma})(1-e^{-i\gamma})]^{\frac{1}{3}}}{h(e^{i\gamma})\,h(e^{-i\gamma})} = c^2 \frac{(2\sin\frac{1}{2}\gamma)^{\frac{2}{3}}}{h(e^{i\gamma})\,h(e^{-i\gamma})} \tag{33.28}$$

and

$$\left.\frac{dz}{d\gamma}\right|_{\varrho=1} = -\frac{\lambda}{2\pi}\frac{h(e^{i\gamma})}{(1-e^{i\gamma})^{\frac{1}{3}}} = -\frac{\lambda}{2\pi}\left(2\sin\frac{1}{2}\gamma\right)^{-\frac{1}{3}} e^{-i(\gamma-\pi)/6}\,h(e^{i\gamma}) \tag{33.29}$$

one obtains as the equation analogous to (32.96)

$$\frac{d}{d\gamma}\frac{(2\sin\frac{1}{2}\gamma)^{\frac{2}{3}}}{h(e^{i\gamma})\,h(e^{-i\gamma})} = \frac{g\lambda}{\pi c^2}\left(2\sin\frac{1}{2}\gamma\right)^{-\frac{1}{3}} \operatorname{Im}\{e^{-i(\gamma-\pi)/6}\,h(e^{i\gamma})\}. \tag{33.30}$$

This yields a set of equations for determination of $g\lambda/c^2$, a_1, a_2, … . The actual computation appears to be as tedious as that of MICHELL's method and, in fact, NEKRASOV's (1920) computations do not seem to have yielded as accurate results as MICHELL's. However, as mentioned earlier, YAMADA (1957) has set up a systematic computation procedure and has obtained results in substantial agreement with those of MICHELL and HAVELOCK. Once $g\lambda/c^2$ and the a_n have been determined, the surface profile can be found in parametric form by integrating (33.29) with respect to γ from 0 to γ. Fig. 51, reproduced from YAMADA's cited paper, shows the form of the profile.

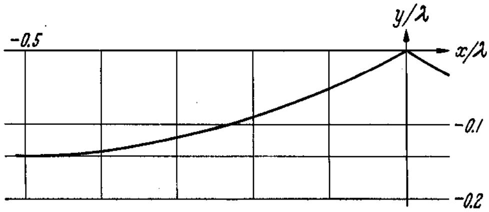


Fig. 51.

β) Havelock's approximation for gravity waves. In a paper already cited several times above HAVELOCK (1919) extended MICHELL's method of construction of periodic waves of maximum amplitude, outlined in the preceding section, to one for construction of periodic waves of any allowable amplitude-length ratio. Up to the present, no one has proved the series involved to converge. However, as HAVELOCK points out, the method has attractive theoretical features: the parameter describing the family of waves occurs in the form $e^{-\beta}$ where β varies from 0, corresponding to the highest wave, to ∞, corresponding to infinitesimal waves.

The method starts out exactly like MICHELL's up to Eq. (33.19) except that it is not assumed that $\psi=0$ corresponds to the free surface. We recall that in MICHELL's analysis the constant-pressure condition did not enter completely until after (33.19), in particular, in (33.21). HAVELOCK assumes instead that this condition is to be satisfied on some other streamline, $\psi=-\alpha$, which will then be taken to correspond to the free surface. The condition may still be written in the form (33.21) provided that one replaces $\psi=0$ by $\psi=-\alpha$. For $\psi=-\alpha$ one may write

$$f' = c\sqrt[3]{2}\, e^{-\frac{1}{3}i\pi(\varphi-i\alpha)/c\lambda}\left(i\sin\pi(\varphi-i\alpha)/c\lambda\right)^{\frac{1}{3}}\sum_{n=0}^{\infty}\gamma_n\, e^{-i2n\pi\varphi/c\lambda}, \tag{33.31}$$

where $\gamma_n = c_n\, e^{-2n\pi\alpha/c\lambda}$, the c_n being the same as those in (33.19). HAVELOCK shows that one may express $\partial|f'|^4/\partial\varphi$ in the following form

$$\left.\begin{aligned} \frac{\partial}{\partial\varphi}|f'|^4 &= \frac{4}{3}\pi\, 2^{\frac{4}{3}}\frac{c^3}{\lambda}\sin\frac{\pi\varphi}{c\lambda}\left[\sinh^2\frac{\pi\alpha}{c\lambda}+\sin^2\frac{\pi\varphi}{c\lambda}\right]^{-\frac{1}{3}}\times \\ &\quad\times\left[A_1\cos\frac{\pi\varphi}{c\lambda}+A_3\cos\frac{3\pi\varphi}{c\lambda}+\cdots\right] \end{aligned}\right\} \quad (33.32$$

and Im f' in the form

$$\left.\begin{aligned}\operatorname{Im} f' = \frac{1}{3} c\, e^{-4\pi\alpha/3c\lambda} \sin\frac{\pi\varphi}{c\lambda}\left[\sinh^2\frac{\pi\alpha}{c\lambda} + \sin^2\frac{\pi\varphi}{c\lambda}\right]^{-\frac{1}{3}} \times \\ \times\left[B_1\cos\frac{\pi\varphi}{c\lambda} + B_3\cos\frac{3\pi\varphi}{c\lambda} + \cdots\right].\end{aligned}\right\} \tag{33.33}$$

Here the A_n's are rather complicated expressions in the γ_n's but also involve $\cosh \pi\alpha/c\lambda$ linearly; the B_n's are linear expressions in the γ_n with coefficients which are functions of $e^{-2\pi\alpha/c\lambda}$. HAVELOCK finds complete expressions for the B_n's; for A_1, A_3, A_5, A_7 he finds the dependence upon the first few γ_n's. One must refer to the original for details, especially for the scheme for approximate solution for the γ_n's.

When $\alpha = 0$ the above analysis is precisely that for the highest wave. The numerical results of HAVELOCK's computations for this case were given in the last section. He also computes $g\lambda/c^2$, γ_1, γ_2 (also γ_3 for the first) for two further cases: $e^{-2\pi\alpha/c\lambda} = 0.75$ and 0.3. The agreement with results computed by other methods, either those of subsection 27α or similar ones, is very close. However, to establish the validity of the method, one must prove convergence of the series $\Sigma|\gamma_n|$.

The relation of this method of approximation to STOKES' "second method" (see subsection 32γ) is also clarified by HAVELOCK. For this we refer to the original paper.

34. Explicit solutions. Although it is not in general possible to give an explicit exact solution to a particular problem of interest, it is possible to give various classes of exact solutions and then to determine the associated solid boundaries. This is sometimes referred to as an "inverse method". Several such methods for constructing exact solutions will be discussed in subsection 34α. In addition, there is one periodic wave in infinitely deep fluid which satisfies the boundary conditions exactly, the Gerstner wave. This will be discussed in subsection 34β. In subsection 34γ we shall discuss briefly what may be called pseudo-exact solutions due to DAVIES and PACKHAM. In these the exact boundary condition is replaced by a closely related one which allows exact solution. They derive their interest from the fact that they contain in one family waves ranging from the smallest amplitude-length ratio up to a counterpart of the Michell wave. Furthermore, the procedure also can be used for pseudo solitary and cnoidal waves. Subsection 34δ will be devoted to an exact solution for pure capillary waves recently discovered by CRAPPER (1957).

α) Inverse methods. SAUTREAUX's method. Possibly the earliest method capable of generating a wide class of steady irrotational solutions is due to C. SAUTREAUX (1893, 1894, 1901). It has been rediscovered several times subsequently, e.g., by BLASIUS (1910), WILTON (1913), RICHARDSON (1920) and LEWY (1952). F. AIMOND (1929) has given a very comprehensive treatment of the method and of various related ones. The method may be easily generalized to include an arbitrary impressed pressure distribution on the free surface (see the papers of RICHARDSON or AIMOND).

Let $z = x + iy$, $f = \varphi + i\psi$, and take f as the independent variable. The free surface will be represented by $\psi = 0$. We further assume $q^2 > \varepsilon > 0$. In the constant-pressure condition on the surface, $\frac{1}{2}q^2 + g\eta = \text{const}$, it will be convenient to take the position of the x-axis so that the constant is zero, and hence $\eta \leqq 0$. This condition may then be expressed in terms of $z(f)$ as follows [cf. (32.56)]:

$$z'(\varphi)\,\overline{z'(\varphi)}\left[z(\varphi) - \overline{z(\varphi)}\right] = -ig. \tag{34.1}$$

Define

$$\mu(f) = \tfrac{1}{2} i \left[z(f) - \overline{z(f)}\right]. \tag{34.2}$$

Then $-\mu(\varphi) = y(\varphi)$, the y-coordinate of the free surface. Hence, from (34.1)

$$\mu(\varphi) = \frac{1}{2g} \frac{1}{z'(\varphi)\,\overline{z'(\varphi)}}, \tag{34.3}$$

From (34.2) and (34.3) one may now derive

$$2\,[g\,\mu(\varphi)]^{-1} - 4\mu'^2(\varphi) = \left[z'(\varphi) + \overline{z'(\varphi)}\right]^2. \tag{34.4}$$

Elimination of $\overline{z'}$ between (34.2) and (34.4) yields

$$z'(\varphi) = -i\,\mu'(\varphi) + \sqrt{(2g\,\mu)^{-1} - \mu'^2}, \tag{34.5}$$

where

$$\mu(\varphi) > 0, \qquad 2g\,\mu(\varphi)\,\mu'^2(\varphi) \leqq 1. \tag{34.6}$$

But then, since z' is an analytic function of f, at least near $\psi = 0$,

$$z'(f) = -i\,\mu'(f) + \sqrt{(2g\,\mu)^{-1} - \mu'^2} \tag{34.7}$$

and

$$z(f) = -i\,\mu(f) + \int \sqrt{(2g\,\mu)^{-1} - \mu'^2}\,df. \tag{34.8}$$

One may now reverse the procedure, select an arbitrary analytic function $\mu(f)$ satisfying (34.6) and construct the function $z(f)$ by means of (34.8). The resulting function will describe a flow for which $z(\varphi)$ is the free surface. If (34.6) is satisfied only for some range of φ, then for the remaining range one must treat the streamline $\psi = 0$ as a solid boundary.

One can use the preceding result to construct a flow if the form of the free surface is given. Let the surface be given in the form $x = \xi(y)$ in a neighborhood of some point of the surface. Since $y(\varphi) = -\mu(\varphi)$ on the surface, we may define $\sigma(\mu) = \xi'(y) = x'(\varphi)/y'(\varphi)$; σ is an analytic function of μ for real μ as follows from the theorem of LEWY and GERBER cited near the beginning of subsection 32γ. Hence, from (34.7),

$$\sigma(\mu) = -\,[(2g\,\mu)^{-1} - \mu'^2]^{\frac{1}{2}}/\mu'(\varphi). \tag{34.9}$$

Solving for $1/\mu'$, one finds

$$\frac{d\varphi}{d\mu} = \sqrt{2g\,\mu\,(1 + \sigma^2)}. \tag{34.10}$$

Since μ is also an analytic function of φ, the same relation holds for $df/d\mu$ when μ is complex, and consequently

$$f = \int \sqrt{2g\,\mu\,(1 + \sigma^2(\mu))}\,d\mu. \tag{34.11}$$

It follows similarly from (34.8) and (34.9), first for real μ, then for complex μ that

$$z = -i\,\mu - \int \sigma(\mu)\,d\mu. \tag{34.12}$$

Eqs. (34.11) and (34.12) thus provide a relation between f and z determined by the form of $\sigma(\mu)$ for real μ.

RUDZKI's method. RUDZKI (1898) has given a different formula for deriving exact solutions. The derivation and statement of the formula below differ somewhat from RUDZKI's, but the result is equivalent.

Let $z = z(f)$ and write

$$z' = \frac{1}{q} e^{i\vartheta}, \quad q = q(\varphi, \psi), \quad \vartheta = \vartheta(\varphi, \psi). \tag{34.13}$$

The free-surface condition may be expressed as follows, from (32.61) and the equation preceding it,

$$q^2 \frac{\partial q}{\partial \varphi} = -g \sin\vartheta \quad \text{for} \quad \psi = 0. \tag{34.14}$$

Hence

$$q = \left[-3g \int \sin\vartheta(\varphi, 0)\, d\varphi\right]^{\frac{1}{3}} \quad \text{for} \quad \psi = 0, \tag{34.15}$$

where the branch of the cube root is taken which is real for real numbers. Combining (34.15) with (34.13) gives

$$z'(\varphi) = e^{i\vartheta(\varphi,0)} \left[-3g \int \sin\vartheta(\varphi, 0)\, d\varphi\right]^{-\frac{1}{3}}. \tag{34.16}$$

This relation must then hold also for $\psi \neq 0$, i.e.

$$z'(f) = e^{i\vartheta(f,0)} \left[-3g \int \sin\vartheta(f, 0)\, df\right]^{-\frac{1}{3}}. \tag{34.17}$$

As in SAUTREAUX's method, we may now reverse the above procedure, take $\vartheta(f)$ as an arbitrary analytic function of f such that ϑ is real for f real, and construct $z'(f)$ from (34.17).

RICHARDSON's method. From (34.17) one can derive immediately a formula due to RICHARDSON (1920) for constructing exact solutions. Let $G'(f) = -\sin\vartheta(f)$. Then $e^{i\vartheta} = \sqrt{1 - G'^2} - iG'$ and (34.17) becomes

$$z'(f) = [3g\, G(f)]^{-\frac{1}{3}} \left[-i\, G'(f) + \sqrt{1 - G'^2}\right]. \tag{34.18}$$

Again, inversely, if $G(f)$ is any analytic function such that, for real f, G', $\sqrt{1-G'^2}$ and G are real, (34.18) gives a corresponding exact free-surface flow.

Examples. The largest collections of specific flows constructed by one of the preceding methods are in the paper of RICHARDSON (1920) and a report of VITOUSEK (1954). Several examples are given below.

1. In (34.17) let $\vartheta(f) = \text{const} = \alpha < 0$. Further, take the constant of integration as zero even though this results in a singularity in z' on the surface. One finds easily

$$f = \tfrac{2}{3} \sqrt{-2g \sin\alpha}\, (z\, e^{-i\alpha})^{\frac{3}{2}}. \tag{34.19}$$

The free surface will consist of only the ray $z = r\, e^{i\alpha}$ unless $\alpha = \pi/6$. However, the ray $z = r\, e^{i(\alpha - \frac{2}{3}\pi)}$ is also a streamline, but not one along which the pressure is constant unless $\alpha = -\pi/6$. Hence it must be taken as a solid boundary in general. The special case $\alpha = -\pi/6$ is just the flow (33.4) considered earlier and has a corner. One may, of course, take any other streamline $\psi = \psi_0 < 0$ as another solid boundary representing a bottom. The pressure remains positive everywhere only if $-\pi/6 < \alpha < 0$. This special family of flows was discussed by WEINGARTEN (1904).

2. If in (34.8) one takes $\mu(f) = f/c$ or in (34.18) takes $G(f) = \frac{2}{3} \sqrt{2g/c^3}\, f^{\frac{3}{2}}$, where c is some fixed velocity, one finds

$$c\, z'(f) = -i + \sqrt{(2g\, f/c^3)^{-1} - 1}. \tag{34.20}$$

This yields a flow of the sort shown in Fig. 52c, taken from RICHARDSON. The internal solid boundary represents some streamline $\psi = \psi_0 < 0$. The free surface corresponds to the segment $\psi = 0$, $0 < \varphi < c^3/2g$ in the f-plane.

3. Let c be some fixed velocity and let

$$G(f) = \frac{3c^3}{g}\left[B + \tanh\left(\alpha \frac{g}{3c^3} f\right)\right], \qquad B > 1,\ \alpha < 1,$$

in (34.18). Then

$$c\,z'(f) = \left[B + \tanh\left(\alpha \frac{g}{3c^3} f\right)\right]^{-\frac{1}{3}} \left\{-i\,\alpha\,\mathrm{sech}^2\left(\alpha \frac{g}{3c^3} f\right) + \sqrt{1 - \alpha^2\,\mathrm{sech}^4\left(\alpha \frac{g}{3c^3} f\right)}\right\}. \tag{34.21}$$

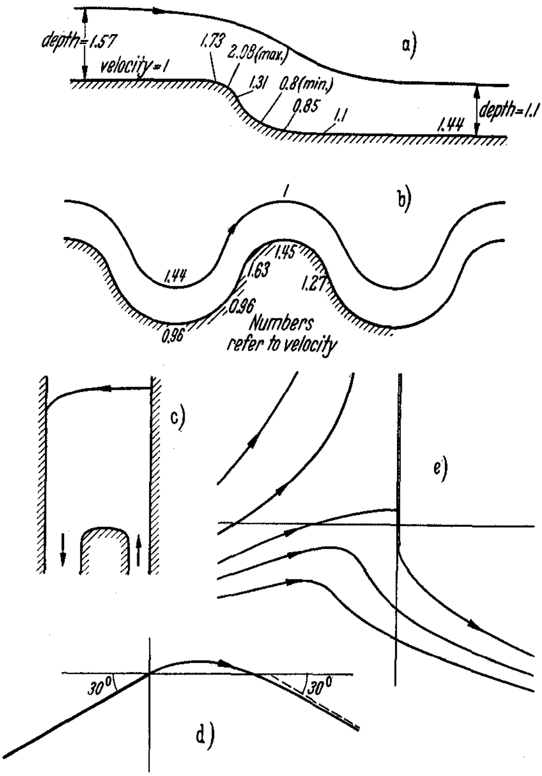


Fig. 52a—d.

Here $\psi = 0$ corresponds to the free surface. The choice of the bottom streamline is restricted by the necessity of avoiding having the singularity at $B = \tanh \frac{1}{3}\alpha g c^{-3} f$ within the fluid. Fig. 52a, also from Richardson, shows a flow computed from (34.21) for $B = 2$, $\alpha = \frac{1}{2}$ and $c = 1$.

4. Flow over a corrugated bottom has been investigated by both Richardson and Rudzki by essentially the same method. Following Richardson, we let

$$G(f) = \frac{3c^3}{g}\left[B - \cos\alpha \frac{g}{3c^3} f\right], \qquad B > 1,\ \alpha < 1.$$

Then

$$c\,z'(f) = \left[B - \cos\alpha \frac{g}{3c^3} f\right]^{-\frac{1}{3}} \left\{-i\,\alpha \sin\alpha \frac{g}{3c^3} f + \sqrt{1 - \alpha^2 \cos^2\alpha \frac{g}{3c^3} f}\right\}. \tag{34.22}$$

Fig. 52b shows a flow computed from this formula for $B = 2$, $\alpha = 0.9$.

5. Flows similar to flows over a weir, under a sluice gate, through an opening, etc. have been considered by a number of the cited authors. Sautreaux (1901)

applied his formula (34.8) with $\mu=(c^2/2g)\,e^{-2gf/c^2}$ to obtain a number of different flows of this nature. Fig. 52d shows one of them. LAUCK (1925) has also constructed such flows. RICHARDSON obtained a flow through an opening by selecting

$$G(f)=\frac{3c^3}{g}\left[B-e^{gf/3c^2}\right].$$

Possibly the simplest such flow, studied by both BLASIUS and VITOUSEK, is obtained by taking $\mu=\sqrt{cf/g}$ in (34.8); this yields

$$\frac{g}{c^2}z=-i\sqrt{\frac{gf}{c^3}}+\frac{1}{3}\left[2\sqrt{\frac{gf}{c^3}}-1\right]^{\frac{3}{2}}. \tag{34.23}$$

The flow is shown in Fig. 52e.

FRITZ JOHN'S method. FRITZ JOHN (1953) has devised a method for constructing exact irrotational two-dimensional free-surface flows which may be time-dependent. Let $F(z,t)=\Phi+i\Psi$ denote the complex velocity potential. The flow of particles on the free surface, $y=\eta(x,t)$, will also be described in a Lagrangian system:

$$z=e(\alpha,t), \tag{34.24}$$

where α is a real number associated with a particular particle. Then

$$\frac{dz}{dt}=\frac{\partial e}{\partial t}=F'\big(x+i\eta(x,t),t\big), \tag{34.25}$$

where F' denotes the partial derivative with respect to z. The equations of motion (2.7), reduced to two dimensions and to motion along the free surface, give

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial\alpha}+\left(g+\frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial\alpha}=-\frac{1}{\varrho}\frac{\partial p}{\partial\alpha}. \tag{34.26}$$

Since $p=\text{const}$ on the surface, $\partial p/\partial\alpha=0$ and (34.26) states that $\partial^2 z/\partial t^2+ig$ is perpendicular to $\partial z/\partial\alpha$, or that

$$e_{tt}+ig=i\,r(\alpha,t)\,e_\alpha, \tag{34.27}$$

where $r(\alpha,t)$ is a real function. Thus $e(\alpha,t)$ must satisfy the parabolic partial differential equation (34.27).

If $e(\alpha,t)$ is a solution of (34.27) for some $r(\alpha,t)$ which is real for real α and if e and e_t are analytic functions of α and real for real α, then one may construct the velocity potential $F(z,t)$ for a free-boundary flow as follows. Actually, we shall construct F as a function of α and t, i.e. we shall construct a function G related to F by $G(\alpha,t)=F\big(e(\alpha,t),t\big)$. For real α it follows from (34.25) that

$$G_\alpha=F'\frac{\partial z}{\partial\alpha}=\overline{e_t(\alpha,t)}\,e_\alpha(\alpha,t)=\overline{e_t(\bar\alpha,t)}\,e_\alpha(\alpha,t). \tag{34.28}$$

One may now use the last expression in (34.28) to extend analytically G_α, and hence G from real to complex α's. By inverting $z=e(\alpha,t)$, one may now construct $F(z,t)$ [invertibility follows from (2.4) which implies $e_\alpha\bar e_\alpha=1$].

It is possible to prescribe $\eta(x,t)$ and then construct the associated function $r(\alpha,t)$. For it follows from (34.26) with $y=\eta\big(x(\alpha,t),t\big)$ that

$$\frac{\partial^2 x}{\partial t^2}+\eta_x\left[\eta_x\frac{\partial^2 x}{\partial t^2}+\eta_{xx}\left(\frac{\partial x}{\partial t}\right)^2+2\eta_{xt}\frac{\partial x}{\partial t}+\eta_{tt}+g\right]=0. \tag{34.29}$$

Any set of solutions $x(\alpha, t)$ depending upon a parameter α yields a function $e(\alpha, t)$ defined by

$$e(\alpha, t) = x(\alpha, t) + i\,\eta\big(x(\alpha, t), t\big). \tag{34.30}$$

The function $r(\alpha, t)$ for real α is given by

$$r(\alpha, t) = \frac{e_{tt} + i g}{i\, e_\alpha} = -\frac{x_{tt}}{\eta_x\, x_\alpha}, \tag{34.31}$$

where (34.29) has been used in obtaining the last expression.

We shall suppose now that the motion is steady and make the following special choice of Lagrangian parameter α. Select some fixed point z_0 of the surface $y = \eta(x)$ and for any particle on the surface let $-\alpha$ be the time at which the particle was at z_0. Since the motion is steady, all particles take the same amount of time to travel from z_0 to any given point z and hence

$$e(\alpha, t) = e(0, t + \alpha) \equiv e(t + \alpha). \tag{34.32}$$

It then follows from (34.27) that also

$$r(\alpha, t) = r(\alpha + t). \tag{34.33}$$

Hence (34.22) becomes an ordinary differential equation in a single variable, say $\tau = t + \alpha$:

$$e''(\tau) - i\, r(\tau)\, e'(\tau) + i g = 0. \tag{34.34}$$

It follows next from (34.28) that $G(\alpha, t) = G(\alpha + t)$ and thus, if $e(\tau)$ is an analytic solution of (34.34), real for real τ,

$$G'(\tau) = \overline{e'(\overline{\tau})}\, e'(\tau). \tag{34.35}$$

In this case each choice of a function $r(\tau)$, real for real τ, results in a function $e(\tau)$ as a solution of (34.34), and then in a function $G(\tau)$ obtained by a quadrature of (34.35). One may invert $z = e(\alpha + t)$ and find F as a function of z as in the last paragraph or else regard

$$z = e(\tau), \qquad F = G(\tau) \tag{34.36}$$

as parametric equations with complex parameter τ.

Several examples are considered by John, two of which are time-dependent. A simple and interesting steady flow is obtained by taking $r(\tau) = \nu$, a constant, in (34.34). Then (34.34) and (34.35), after setting the constants of integration equal to zero, yield

$$z = \frac{g}{\nu}\tau + A\, e^{i\nu\tau}, \qquad F = \left(\frac{g^2}{\nu^2} + \nu^2 A^2\right)\tau - 2\frac{g}{\nu} A \cos \nu\tau. \tag{34.38}$$

The free surface, obtained by taking τ real in the first formula, is a trochoidal curve without self-intersections if $A < g/\nu^2$; the wavelength is $\lambda = 2\pi g/\nu^2$ and the amplitude is A. If $A < g/\nu^2$, then $|dF/dz| > 0$ and $A/\lambda < 1/2\pi$. However, dF/dz can become infinite if $dz/d\tau = 0$. Such points occur at

$$z = \left(n + \frac{1}{4}\right)\lambda + i\frac{\lambda}{2\pi}\left(1 - \log\frac{\lambda}{2\pi A}\right). \tag{34.39}$$

In order to avoid having them within the fluid, the bottom must be chosen as a streamline which passes above or through these points. Fig. 53, taken from John's paper, shows several profiles and the associated streamlines through the branch points (34.39) computed for various values of the constant A when

$\lambda = 2\pi$ (this is equivalent to graphing $2\pi z/\lambda$ for various values of $2\pi A/\lambda$). The surface profile and bottom come closer together as $A \to 1$ and draw further apart as $A \to 0$. For $A = 0.9$ they are so close that they cannot be conveniently separated in the figure; in such cases one may, of course have reservations about the applicability of the perfect-fluid model.

The surface profile in this example is exactly the same as in the Gerstner wave treated in the next section. However, the Gerstner wave is defined for infinite depth and is not irrotational. The flow described above may also be obtained by SAUTREAUX's method. VITOUSEK (1954) has studied it by this procedure.

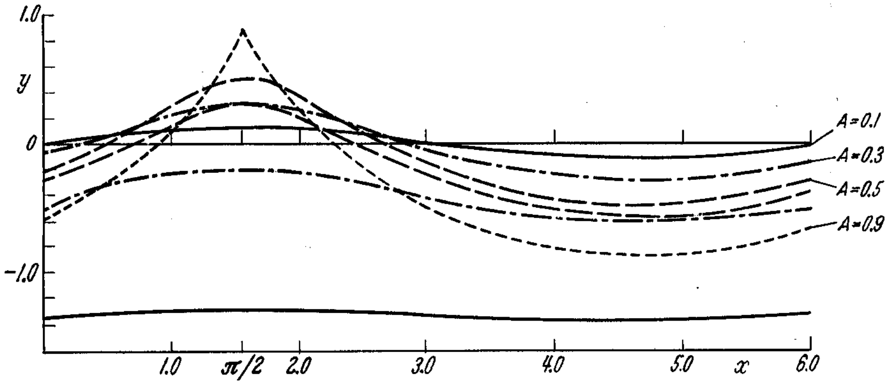


Fig. 53.

Methods of VILLAT and PONCIN. At the end of Sect. 32γ brief mention was made of a pair of integral equations derived by VILLAT (1915) for the determination of flows over some given bottom profile and with the top profile also given upstream of some point. The method seems to be chiefly useful as an inverse method in which the free surface is given and the other profiles sought. VILLAT has worked out one case, but not in complete detail, where the top cover is missing and the bottom has a declivity.

PONCIN (1932) has further elaborated VILLAT's method in the direction of starting with the fixed profile and finding the free profile. Actually, he does not really achieve this. Instead, he is able to construct a flow for a fixed profile of the same general behavior as the given one, but not identical with it. The method is applied to a number of interesting special cases, including flow over wavy bottoms and over bottoms with a declivity. The solutions are generally for large values of the velocity. The method and results do not lend themselves to a brief summary.

β) Gerstner's wave. GERSTNER's wave (1802) is apparently the first flow to have been discovered which satisfies exactly the condition of constant pressure on the surface, and is, in fact, one of the earliest papers on water-wave theory. It was subsequently rediscovered by RANKINE (1863). As will be shown below, the motion is not irrotational. This fact itself would not be enough to rule it out as a mathematical model for real periodic waves. However, the direction of the vorticity is such that it is difficult to conceive of a scheme whereby such a wave could be generated in nature.

The motion is most easily described in Lagrangian coordinates. Each particle is associated with a pair of parameters (a, b), $b \leqq 0$. However, (a, b) does not represent a particular position of the particle at some time t_0, but instead a mean position. Hence, instead of (2.3) and (2.4) we need require instead only that the determinant D of those formulas be independent of t. The motion is described by the equations

$$x = a + A\,e^{mb}\sin(m\,a + \sigma\,t), \qquad y = b - A\,e^{mb}\cos(m\,a + \sigma\,t). \qquad (34.40)$$

If $b=0$ is taken as the free surface, the motion evidently represents a wave moving to the left with velocity $c=\sigma/m$, while the particles themselves describe in a counter-clockwise direction circular paths about the points (a, b) associated with the particles. The surface $b=0$ is a trochoid and, in fact, each curve $b=\text{const}<0$ is also a trochoid. In order that there shall be no self-intersections, one must have

$$A \leqq \frac{1}{m}. \qquad (34.41)$$

In order to verify that the motion is kinematically possible, it is necessary to show, as noted above, only that the Jacobian $\partial(x, y)/\partial(a, b)$ is independent of t. An easy computation shows

$$\frac{\partial(x, y)}{\partial(a, b)} = 1 - m^2 A^2 e^{-2mb}, \qquad (34.42)$$

so that the continuity condition is satisfied. Next one must show that the pressure is constant along the free surface. We shall, in fact, show more, namely, that it is constant along any line $b=\text{const}<0$, provided $\sigma^2=gm$. To see this, introduce the Eq. (34.40) into the first of Eqs. (2.7). A straightforward computation yields

$$A(g\,m - \sigma^2)\,e^{mb}\sin(m\,a + \sigma\,t) = -\frac{1}{\varrho}\frac{\partial p}{\partial a}. \qquad (34.43)$$

If the pressure is constant along the surface, then $\partial p/\partial a=0$. This can only hold if

$$\sigma^2 = g\,m. \qquad (34.44)$$

However, if $\sigma^2=gm$, then $\partial p/\partial a=0$ for all b, so that each curve $b=\text{const}$ is an isobaric curve. Although we shall verify this fact directly, it now follows immediately from BURNSIDE's theorem in subsection 32β that the motion cannot be irrotational. A direct computation of the vorticity vector is facilitated by noting that

$$\left.\begin{aligned} u &= \frac{\partial x}{\partial t} = A\,\sigma\,e^{mb}\cos(m\,a + \sigma\,t) = -\sigma(y - b), \\ v &= \frac{\partial y}{\partial t} = A\,\sigma\,e^{mb}\sin(m\,a + \sigma\,t) = \sigma(x - a). \end{aligned}\right\} \qquad (34.45)$$

Hence

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \sigma\left(2 - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right). \qquad (34.46)$$

The right-hand side of (34.46) may be computed from (34.40) by application of the rules of inversion for partial derivatives. The final result is

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{2\sigma\,m^2 A^2 e^{2mb}}{1 - m^2 A^2 e^{2mb}}; \qquad (34.47)$$

the negative sign indicates that the vorticity is directed oppositely to the orbital motion of the particles. The relatively strong vorticity of Gerstner waves when mA is not quite small has been pointed out by TRUESDELL (1953), who measures it with a dimensionless vorticity number (see Sect. 27 of SERRIN's article in Vol. VIII/1).

We shall omit a discussion of the construction of the curves $b=\text{const}$, streamlines in a coordinate system moving with the waves; it may be found in LAMB (1932, § 251), MILNE-THOMSON (1956, § 14.81) and in KOCHIN, KIBEL and ROZE (1948, Chap. 8, § 16) together with reproductions of GERSTNER'S original curves. It is, however, of interest to note that there is, according to (34.41), a "highest" wave of ratio $2A/\lambda = 1/\pi = 0.318$, a figure which may be compared with the value 0.142 for MICHELL'S wave. The highest Gerstner wave has a cusp at the crests, a further indication that the motion cannot be irrotational.

The pressure distribution may be found by substituting (34.40) in the second equation of (2.7), using (34.44), and integrating. The result is

$$p = p_0 - \varrho g b - \tfrac{1}{2}\varrho \sigma^2 A (1 - e^{2mb}). \tag{34.48}$$

A computation of the potential and kinetic energies over a wave length yields

$$T = V = \frac{1}{4}\lambda g \varrho A^2 \left[1 - \frac{2\pi^2}{\lambda^2} R^2\right]. \tag{34.49}$$

Finally, we note that nowhere in the preceding analysis have we made use of homogeneity of the fluid, i.e. the Gerstner wave also represents an exact solution for an arbitrary heterogeneous fluid (with ϱ constant along streamlines). Moreover, DUBREIL-JACOTIN (1935) has shown that the Gerstner wave is unique in this respect.

see errata

GERSTNER'S wave is defined only for infinite depth. One may ask if a similar wave exists for finite depth. "Similar", in this context, will be taken to mean a periodic wave which satisfies exactly the constant-pressure condition on the free surface and for which the particle orbits are closed. DUBREIL-JACOTIN (1934) has proved the existence of such a wave and showed that it is unique when the period is fixed. However, this motion cannot be given explicitly except in the case of infinite depth. Methods of approximate computation of the wave have been given by KRAVTCHENKO and DAUBERT (1957) and GOUYON (1958).

γ) Pseudo-exact solutions. Although the solutions of this section are not really solutions satisfying the exact boundary conditions formulated earlier, they are exact solutions to a closely related problem, also with a nonlinear boundary condition. Their interest derives from the fact that it is possible to encompass within one explicit formula waves of all amplitudes up to a highest wave analogous to MICHELL'S wave. The procedure also allows explicit construction of solitary and cnoidal waves. It is possibly a misnomer to call these solutions pseudo-exact, for one may also interpret them as the first term in a certain series solution of the correctly posed problem. In this sense they are analogous to HAVELOCK'S approximation procedure described in subsection 33 β. The work to be described has appeared in a series of papers by DAVIES (1951, 1952), PACKHAM (1952) and GOODY and DAVIES (1957).

The motion will be described in terms of the variables introduced in (32.85), $\omega = \vartheta + i\tau$. The alteration in the boundary condition consists in replacing (32.89) by

$$\frac{\partial \vartheta}{\partial \psi} = l \frac{g}{c^3} e^{-3\tau} \sin 3\vartheta \quad \text{for } \psi = 0, \tag{34.50}$$

where l is some fixed constant chosen so that $l \sin 3\vartheta$ is a good approximation to $\sin\vartheta$. If one wishes to consider (34.50) as the first term in a series approximation to (32.89), one may expand $\sin\vartheta$ in a series in $\sin 3\vartheta$ and express (32.89) as

$$\frac{\partial \vartheta}{\partial \psi} = \frac{g}{c^3} e^{3\tau}\left[\frac{1}{3}\sin 3\vartheta + \frac{4}{81}\sin^3 3\vartheta + \cdots\right]. \tag{34.51}$$

In this case (34.50) with $l=\frac{1}{3}$ represents the approximation obtained by keeping only the first term of (34.51). However, we shall not pursue the approximation procedure and refer to DAVIES (1951) for further information. It will be convenient to reformulate the boundary condition (34.50) as follows:

$$\operatorname{Im}\left\{\frac{d\omega}{df}+l\frac{g}{c^3}\,e^{i3\omega}\right\}=0 \quad \text{for } \psi=0, \tag{34.52}$$

or, after introducing the new variable $\chi(f)=e^{-i3\omega}=\omega^3/c^3$, as

$$\operatorname{Im}\left\{\frac{1}{\chi}\left(i\frac{d\chi}{df}+3l\frac{g}{c^3}\right)\right\}=0 \quad \text{for } \psi=0. \tag{34.53}$$

In order to proceed further, we must further specify the nature of the wave motion. Let us suppose the motion to be periodic with wavelength λ and take the fluid to be infinitely deep. We again introduce the ζ-plane of (32.91) and take coordinates as in Fig. 50. The expression in curly brackets in (34.53) is a regular analytic function of ζ inside the unit disc of the ζ-plane with vanishing imaginary part on the boundary, hence is a constant. Since, for $\zeta=0$ (i.e. as $\psi\to-\infty$), $\chi=1$ and $d\chi/df=0$, the constant must be $3lg/c^3$. Thus χ must satisfy the differential equation

$$i\frac{d\chi}{df}-3l\frac{g}{c^3}\chi=-3l\frac{g}{c^3}. \tag{34.54}$$

The solution is easily found to be

$$\chi=1+A\,e^{-i3lgf/c^3}. \tag{34.55}$$

Referring to Fig. 50, we see that, if $f=0$, $\chi=q_0^3/c^3$, where q_0 is the absolute velocity at a crest. Hence

$$A=\frac{q_0^3}{c^3}-1. \tag{34.56}$$

Since ϑ must also vanish at $\varphi=\pm\frac{1}{2}c\lambda$, i.e. the left-hand side of (34.55) must be real, we must also have $(3lg/c^3)\,\frac{1}{2}c\lambda=\pi$, or

$$c^2=3lg\,\lambda/2\pi. \tag{34.57}$$

Note that if $l=\frac{1}{3}$, the relation between c^2 and λ is the same as in the infinitesimal-wave theory. The solution (34.55) may now be put into the following form:

$$\chi=\frac{w^3}{c^3}=1-\left(1-\frac{q_0^3}{c^3}\right)e^{-i2\pi f/c\lambda}, \tag{34.58}$$

where $0\leqq q_0\leqq c$. If $q_0=c$, then $w=c$ and the flow is uniform. If $q_0=0$, then

$$w^3=c^3\left[1-e^{-i2\pi f/c\lambda}\right], \tag{34.59}$$

and near $f=0,\ \pm c\lambda,\ \pm 2c\lambda,\ \ldots$ there is a corner in the wave profile with the two tangents making the same angle 120° as in STOKES' theorem [near $f=0$, (33.5) gives $w^3=i\,\frac{3}{2}gf$, (34.58) gives $w^3=i\,3lgf$]. Hence this wave corresponds to MICHELL's highest periodic wave. The ratio of amplitude to length of this wave may be computed from the following expression for the trough:

$$\frac{1}{2}\lambda-i\,a=\frac{1}{c}\int\limits_0^{\frac{1}{2}c\lambda}\left[1-e^{-i2\pi\varphi/c\lambda}\right]^{-\frac{1}{3}}d\varphi.$$

By expanding in a series and integrating term by term, one finds

$$\frac{a}{\lambda} = 0.127. \tag{34.60}$$

We recall that the value for MICHELL'S wave was 0.142.

If the depth of fluid is finite, one must add the additional boundary condition, $\operatorname{Im}\{\chi\} = 0$ for $\psi = -Q$, as well as for $\varphi = 0$ and $\pm\frac{1}{2}\lambda c$ if the motion is to be periodic. The determination of χ now becomes too difficult to carry through briefly. However, an explicit solution is still possible and has been worked out by DAVIES (1952) and further investigated by GOODY and DAVIES (1957). Similarly, a "solitary wave" can be explicitly constructed which satisfies the boundary conditions $\operatorname{Im}\{\chi\} = 0$ for $\psi = -Q$ and for $\varphi = 0$, $0 > \psi \geqq -Q$ and $\chi \to 1$ as $\varphi \to \pm\infty$. This has been done by PACKHAM (1952). Either of these problems leads to the following differential-difference equation for $\chi(f)$:

$$\frac{1}{\chi(f+iQ)}\left[\chi'(f+iQ) - 3l\frac{g}{c^3}i\right] + \frac{1}{\chi(f-iQ)}\left[\chi'(f-iQ) + 3l\frac{g}{c^3}i\right] = 0; \tag{34.61}$$

it may be established in a manner similar to that used in deriving (22.30) or (32.80).

δ) Pure capillary waves. The first investigation of periodic progressive capillary waves satisfying the exact boundary condition is apparently due to N.A. SLËZKIN (1937). He formulated the boundary-value problem in the same manner as will be done below, reduced it to solution of a nonlinear integral equation analogous to NEKRASOV'S and proved existence and uniqueness of a solution. However, he apparently did not observe that an explicit solution was possible for infinite depth of fluid. This was discovered by CRAPPER (1957), following a different and, in fact, more elementary method.

We shall consider the motion as a steady one in which the fluid moves to the right with velocity c as $y \to -\infty$. The existence of a complex velocity potential $f(z) = \varphi + i\psi$ will be assumed and the free surface $y = \eta(x)$ will be taken to correspond to the streamline $\psi = 0$ as usual. It will also be convenient to make use of the variable $\omega = \vartheta + i\tau$ introduced in (32.85). If p_0 is atmospheric pressure, then from BERNOULLI'S integral

$$p + \tfrac{1}{2}\varrho q^2 = p_0 + \tfrac{1}{2}\varrho c^2 \tag{34.62}$$

(we recall that gravity is being neglected). The dynamical condition at the free surface [see (3.8) and (3.9)] is

$$p - p_0 = \frac{T}{R} = T\frac{\eta''}{[1+\eta'^2]^{\frac{3}{2}}}. \tag{34.63}$$

Before combining (34.62) and (34.63), we recall that the curvature of a streamline at any of its points is given by $d\vartheta/ds$ where s is arc length along the streamline. Hence, we may combine (34.62) and (34.63) to obtain the following boundary condition

$$\frac{1}{2}\varrho(c^2 - q^2) = T\frac{d\vartheta}{ds} = T\frac{\partial\vartheta}{\partial\varphi}\frac{d\varphi}{ds} = Tq\frac{\partial\vartheta}{\partial\varphi} \quad \text{for} \quad \psi = 0, \tag{34.64}$$

or

$$\frac{\varrho c}{2T}\left(\frac{c}{q} - \frac{q}{c}\right) = \frac{\partial\vartheta}{\partial\varphi} \quad \text{for} \quad \psi = 0. \tag{34.65}$$

Since $q = c e^{\tau}$ and since $\partial\vartheta/\partial\varphi = \partial\tau/\partial\psi$ from the Cauchy-Riemann equations, (34.65) may be written

$$\frac{\partial\tau}{\partial\psi} = \frac{\varrho c}{2T}\left(\mathrm{e}^{-\tau} - \mathrm{e}^{\tau}\right) = -\frac{\varrho c}{T}\sinh\tau \quad \text{for} \quad \psi = 0. \tag{34.66}$$

The problem is now to find a function $\omega(f)$ analytic for $\psi \leqq 0$, such that $\omega \to 0$ as $\psi \to -\infty$ and such that the imaginary part τ satisfies (34.66). However, since the boundary condition (34.66) involves only τ, unlike its analogue (32.89) for pure gravity waves, it is possible to solve first for the harmonic function $\tau(\varphi, \psi)$ and then to find ϑ later.

We assume that a solution can be found which satisfies

$$\frac{\partial\tau}{\partial\psi} = -h(\psi)\sinh\tau, \qquad h(0) = \frac{\varrho c}{T}, \tag{34.67}$$

and proceed to verify the assumption. Integrating (34.67), we obtain

$$\log\tanh\tfrac{1}{2}\tau = -H(\psi) + G(\varphi), \tag{34.68}$$

where $H'(\psi) = h(\psi)$ and $G(\varphi)$ is an arbitrary function, or

$$\tau = \log\frac{\mathrm{e}^{H} + \mathrm{e}^{G}}{\mathrm{e}^{H} - \mathrm{e}^{G}} = \log\frac{X(\psi) + Y(\varphi)}{X(\psi) - Y(\varphi)}. \tag{34.69}$$

Since τ is a harmonic function of φ and ψ, LAPLACE'S equation must be satisfied by (34.69). This yields an equation to be satisfied by X and Y in which the two variables can be separated. We shall not repeat the detailed analysis, which is typical of that occurring in separation-of-variables problems. The final result is that X and Y must satisfy

$$\left.\begin{aligned} X'^2 &= a_1 + a_2 X^2 + a_3 X^4, \\ Y'^2 &= -a_1 - a_2 Y^2 - a_3 Y^4, \end{aligned}\right\} \tag{34.70}$$

where a_1, a_2, a_3 are arbitrary constants. CRAPPER states that the full equations may be used to construct a solution for fluid of finite depth, but that it is sufficient to set $a_3 = 0$ for infinite depth (in view of SLËZKIN'S result, this is presumably also necessary). Since τ is real, we shall also take X and Y to be real. If one does set $a_3 = 0$ and assumes $a_1 < 0$, $a_2 > 0$, the following give real solutions of (34.70):

$$X(\psi) = \sqrt{\frac{-a_1}{a_2}}\cosh\left(\sqrt{a_2}\,\psi + E\right), \quad Y(\varphi) = \sqrt{\frac{-a_1}{a_2}}\cos\left(\sqrt{a_2}\,\varphi + F\right), \tag{34.71}$$

where E and F are real constants. A glance at (34.69) shows that τ is independent of the choice of a_1. It will be convenient to let $a_2 = m^2/c^2$, where $m > 0$. One may determine E from (34.67), for

$$\frac{\varrho c}{T} = H'(0) = \frac{d}{d\psi}\log X\big|_{\psi=0} = \frac{m}{c}\tanh E$$

or

$$\mathrm{e}^{2E} = \frac{m\,T/\varrho\,c^2 + 1}{m\,T/\varrho\,c^2 - 1} \equiv B^{-2}. \tag{34.72}$$

Since E is to be real, we must evidently have

$$\frac{m\,T}{\varrho\,c^2} \geqq 1. \tag{34.73}$$

Since F adds only a real constant to φ we may select it at our convenience; we take $F=0$. Substitution of (34.71) into (34.69) gives

$$\left.\begin{aligned}\tau &= \log\frac{\cosh(m\psi/c+E)+\cos(m\varphi/c)}{\cosh(m\psi/c+E)-\cos(m\varphi/c)}\\ &= \log\frac{\cos(i\,m\psi/c+i\,E)+\cos(m\varphi/c)}{\cos(i\,m\psi/c+i\,E)-\cos(m\varphi/c)}\\ &= \log\left[\cot\frac{1}{2}(m\,f/c+i\,E)\cot\frac{1}{2}(m\,\bar{f}/c-i\,E)\right].\end{aligned}\right\}\tag{34.74}$$

The analytic function ω, which has τ as imaginary part and which approaches zero as $\psi\to-\infty$, is given by

$$\omega = i\log\left[-\cot^2\tfrac{1}{2}(m\,f/c+i\,E)\right].\tag{34.75}$$

We then have

$$\frac{df}{dz} = c\,e^{-i\omega} = -c\cot^2\frac{1}{2}(m\,f/c+i\,E) = c\coth^2\frac{1}{2}(i\,m\,f/c-E).\tag{34.76}$$

From this one may solve for z in terms of f:

$$\left.\begin{aligned}c\,z &= f-\frac{2c}{m}\tan\frac{1}{2}\left(\frac{m\,f}{c}+i\,E\right)+\text{const}\\ &= f-i\frac{4c}{m}\frac{1}{1+e^{(i\,m\,f/c-E)}}+\text{const}\\ &= f-i\frac{4c}{m}\frac{1}{1+B\,e^{i\,m\,f/c}}+i\frac{4c}{m},\end{aligned}\right\}\tag{34.77}$$

where the constant has been chosen so as to make cz reduce to f when $B=0$. It is evident that

$$z\left(f+\frac{2\pi c}{m}\right) = z(f)+\frac{2\pi}{m},$$

so that the streamlines are periodic in the x-direction with wavelength $\lambda=2\pi/m$.

The surface streamline is obtained by setting $\psi=0$. After separation of real and imaginary parts in (34.77) the equation for the surface becomes:

$$\left.\begin{aligned}x &= \frac{\varphi}{c}-\frac{4}{m}\frac{B\sin m\varphi/c}{1+B^2+2B\cos m\varphi/c},\\ y &= \frac{4}{m}-\frac{4}{m}\frac{1+B\cos m\varphi/c}{1+B^2+2B\cos m\varphi/c},\end{aligned}\right\}\tag{34.78}$$

with φ serving as a parameter. There is a crest when $\varphi=0$ and a trough when $\varphi=\pi c/m$. The difference in the values of y yields the following expression for the ratio of total amplitude to wavelength:

$$\frac{a}{\lambda} = \frac{4}{\pi}\frac{B}{1-B^2}.\tag{34.79}$$

Eq. (34.72) then provides a relation between A/λ and $mT/\varrho c^2$, which we may write, for example, as

$$\left.\begin{aligned}c &= \sqrt{\frac{T\,m}{\varrho}}\left(1+\frac{1}{16}a^2m^2\right)^{-\frac{1}{4}}\\ &= \sqrt{\frac{T\,m}{\varrho}}\left(1-\frac{1}{64}a^2m^2+\cdots\right).\end{aligned}\right\}\tag{34.80}$$

If this formula is compared with (27.29), it should be kept in mind that a is here the total amplitude and that in (27.29) A is a length associated with the half amplitude. The formulas are consistent.

As a/λ increases, the surface profile becomes steeper and steeper near the troughs until the two sides finally touch. This occurs for $a/\lambda = 0.730$. A wave of these proportions may be considered as a "highest" capillary wave, an analogue of MICHELL's wave, although the nature of the limitation is different. Fig. 54, reproduced from CRAPPER's paper, shows the profile of this wave together with other streamlines. It is a consequence of the form of the dependence in (34.77) that the other streamlines in Fig. 54 may also serve as surface profiles for different values of a/λ, i.e. for different values of B. It is not surprising, of course, that the profiles are similar to the middle three of Fig. 35.

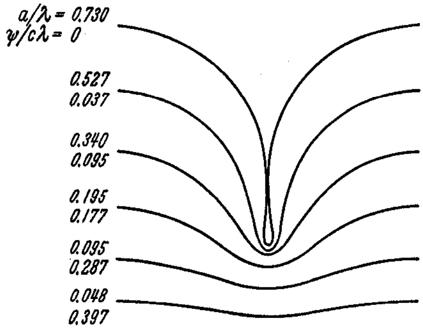


Fig. 54.

35. Existence theorems. In the various applications of the approximate theories of Chapt. D and E it is tacitly assumed that there is an exact solution which is being approximated. Without knowledge of conditions for existence and uniqueness of a solution to a particular problem, the status of an approximate solution is somewhat anomalous and one must rely upon comparison with experimental results for conviction concerning the correctness of the solution. However, such comparison is not a satisfactory criterion, for in the original formulation of a problem one will usually have already made a decision about the mathematical model of a fluid which will be used. Thus, if one has assumed a perfect fluid (as we usually have) and then made a further mathematical approximation in solving the problem at hand, the validity of this approximate solution must first be established before comparison of the predicted results with experimental measurements can be used as a criterion of the applicability of the fluid model. Without this additional knowledge, the comparison of approximate solutions with experimental results must be considered in some sense to be second best, even though valuable evidence may be provided by good agreement in a wide variety of situations.

Unfortunately, existence and uniqueness proofs in exact water-wave theory have generally been difficult to establish, and have usually been obtained for only rather restricted, although physically important, situations. Many of them are very recent and some rely upon methods of functional or topological analysis which cannot be briefly expounded. Although some proofs are so constructed that approximation methods are inherent in them, others are only able to assert the existence of a solution with no indication of how to obtain it approximately. Proofs are still lacking for many relatively simple but important problems, for example, MICHELL's highest wave and standing water waves.

No attempt will be made to give an exposition of the mathematical methods which have been used in establishing the various existing theorems. Instead only a discursive account will be given of the nature and limitations of the known theorems.

α) Irrotational waves—infinite depth. Proof of the existence of periodic waves of permanent type in infinitely deep water was first given by NEKRASOV (1921,

1922) in a journal of very restricted distribution. Shortly thereafter LEVI-CIVITA (1925) gave another proof along quite independent lines. Further proofs were given by NEUMANN (1929) and by LICHTENSTEIN (1931), these being more closely related to NEKRASOV'S. A new treatment of LEVI-CIVITA'S proof, due to LITTMAN and NIRENBERG, is contained in STOKER'S *Water waves* (1957, § 12.2). Also, NEKRASOV (1951) has recently published his researches in a more accessible form.

NEKRASOV'S method requires proving that there exists a solution $\vartheta(\gamma)$ to his nonlinear integral equation (32.104). His procedure, in brief, is to assume an expansion of $\vartheta(\gamma)$ in powers of the parameter $\mu'=\mu-3>0$,

$$\vartheta(\gamma)=\mu'\vartheta_1+\mu'^2\vartheta_2+\cdots, \tag{35.1}$$

then to derive equations relating each ϑ_k to ones of lower index, and finally to show that the series converges. $\mu=3$ is chosen as a starting point because it is the first eigenvalue of the "linearized" equation (32.104), i.e. the one obtained by replacing the quotient containing $\sin\vartheta$ by simply $\mu\vartheta(\beta)$. This corresponds to the infinitesimal-wave theory. Proof of convergence requires that μ' be sufficiently small, and positive, but no estimate of radius of convergence is obtained. On the other hand, the method does allow computation of explicit approximate formulas for quantities of interest.

LEVI-CIVITA also works with the variable ω, treating it as a function of the variable ζ introduced in (32.90). Hence his formulation of the problem is essentially the same as NEKRASOV'S, i.e. he is seeking a function $\omega(\zeta)$, regular in the disc $|\zeta|<1$, vanishing at $\zeta=0$ and satisfying (32.97) on $|\zeta|=1$ and some further condition assuring that $|w/c-1|<\beta<1$. His procedure for finding such a function is to expand both ω and $k\equiv 1-g\lambda/2\pi c^2$ in a power series in a parameter $\mu>0$:

$$\omega=\sum_{n=1}^{\infty}\omega_n(\zeta)\,\mu^n, \qquad k=\sum_{n=1}^{\infty}k_n\mu^n, \tag{35.2}$$

where the functions $\omega_n(\zeta)$ and the constants k_n are to be determined by the boundary conditions. The first terms, $\omega_1=-i\zeta$, $k_1=0$, correspond to infinitesimal waves, so that the parameter μ is essentially the amplitude/wavelength of this approximation. LEVI-CIVITA establishes the convergence of the series (35.2) for sufficiently small values of μ. No estimate of a radius of convergence is given, but HUNT (1953) has stated that an examination and refinement of LEVI-CIVITA'S inequalities show that convergence is established for amplitude-wavelength ratios up to $\frac{1}{98}$. The procedure lends itself to explicit computation of higher-order computations, and, in fact, he carries them out through $n=5$. LEVI-CIVITA further derives the interesting theorem that irrotational waves of permanent type must be symmetric about vertical lines through crest and trough. NEKRASOV assumed this at the outset.

NEUMANN and LICHTENSTEIN (the latter's approach is simpler) derive a coupled pair of nonlinear integral equations and put them into a form such that SCHMIDT'S theory of nonlinear integral equations is applicable. Iterative methods of solution can be used to obtain approximate formulas.

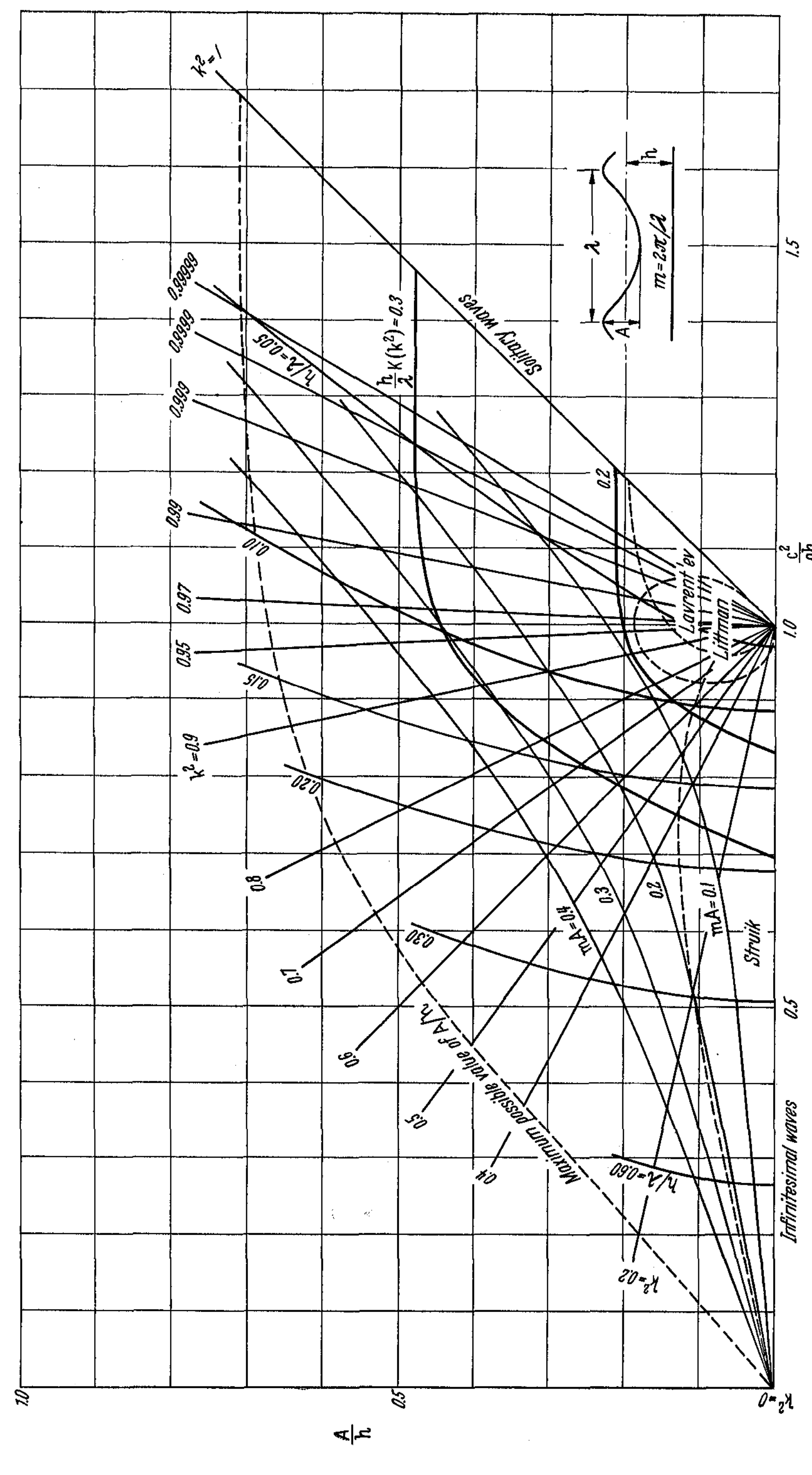
β) Irrotational waves—horizontal bottom. When the fluid is infinitely deep and the motion periodic, the only independent dimensionless parameter besides the amplitude-wavelength ratio is $c^2/g\lambda$. When the fluid is bounded below by a horizontal plane at mean depth h, then a new parameter, say c^2/gh, must enter into any solution. However, other independent sets of parameters may be used, and, in particular, different choices of a perturbation parameter have led earlier to different approximate solutions for finite depth. Thus, in Sects. 14β and 27 one finds the first and higher approximations for periodic waves of permanent type when A/λ is taken as a perturbation parameter, whereas in Sect. 31 one finds approximations to two further types of waves of permanent type, one of them periodic, corresponding to a different choice of parameter and a different method of approximation. In each of these cases there arises the question as to whether there exist waves of permanent type satisfying the exact boundary conditions for which these waves may be considered approximations. In each case the answer is affirmative.

Waves of small amplitude. The first proof of the existence of periodic progressive waves in fluid of finite depth is due to STRUIK (1926). His method of analysis is similar to LEVI-CIVITA'S for infinite depth. Existence of the desired wave is established for each value of $c^2/gh<1$ and for each sufficiently small value of A/λ, where the bound on A/λ depends upon c^2/gh. HUNT (1953) has recently corrected some errors in the proof which did not invalidate it but which resulted in incorrect approximate formulas.

NEKRASOV (1928, 1951) was also able to show that his integral equation for ϑ, as modified for finite depth [see (32.104) and (32.106)], had a solution, thus providing an independent proof. As was the case for infinite depth, NEKRASOV assumes that the waves are symmetric about verticals through trough and crest; STRUIK proves this. KRASNOSELSKII (1956) has recently applied topological methods of analysis to NEKRASOV'S equation and established not only existence of solutions for μ in the neighborhood of the eigenvalues of the linearized equation, but also their uniqueness and continuous dependence upon μ.

Solitary and cnoidal waves. LAVRENT'EV (1943, 1947) was the first one to establish the existence of cnoidal and solitary waves. Cnoidal waves are not mentioned by him by name, but, in fact, their existence for sufficiently large wavelength is established along with that of the solitary wave, the latter being obtained as a limiting case. The detailed exposition of the results (1947) is unfortunately both difficult of access and difficult to read, and relies upon earlier theorems of the author. Although the perturbation parameter appears at first glance to be taken as $\varepsilon^2=-1+gh^3/Q^2$, which for the solitary wave would be in contradiction with (32.52), the quantity h is not really mean depth but a related quantity which varies with the wavelength of the approximating periodic wave. FRIEDRICHS and HYERS (1954) by a completely different procedure have established the existence of the solitary wave. Their perturbation parameter is essentially $\varepsilon^2=1-gh^3/Q^2$ [actually it is $a^2=-\frac{1}{3}\log(gh^3/Q^2)$]. The point of departure is again the boundary condition (32.89) for the function ω. However, an integral equation is formulated, then altered by a change of variable $\hat{\varphi}=a\varphi$, $\hat{\psi}=\psi$. The different rates of stretching correspond to those of subsection 10β. (Something like this also occurs in LAVRENT'EV'S proof, but is disguised in his theorems on conformal mapping of narrow strip-like regions.) An iterative procedure is used to prove existence of a solution for sufficiently small values of ε^2.

LITTMAN (1957) has used a method somewhat similar to that of FRIEDRICHS and HYERS to establish the existence of cnoidal waves satisfying the exact boundary conditions. However, as a parameter he has used essentially h/λ, where h is the mean depth and λ the wavelength. It is demonstrated that solutions exist for values of c^2/gh which are both greater and less than 1. Fig. 55, modified slightly from one in LITTMAN'S paper, shows in a qualitative fashion the relation between the dimensionless parameters. The dotted lines enclose values of the parameters, again in a purely qualitative way, for which solutions have been demonstrated to exist. (Here k is the modulus of the elliptic functions



$k^2 = 1$
0.99999
0.9999
0.999
0.99
0.97
0.95
$k^2 = 0.9$
0.8
0.7
0.6
0.5
0.4
$k^2 = 0.2$
$k^2 = 0$
$h/\lambda = 0.05$
0.10
0.15
0.20
0.30
$h/\lambda = 0.60$
$\frac{h}{\lambda} K(k^2) = 0.3$
Solitary waves
Laurent'ev
Littman
Struik
Infinitesimal waves
Maximum possible value of A/h
$mA = 0.4$
0.3
0.2
$mA = 0.1$
0.2
λ
h
A
$m = 2\pi/\lambda$
$\frac{c^2}{gh}$
0.5
1.0
1.5
$\frac{A}{h}$
1.0
0.5
0

Fig. 55.

and K is the complete elliptic integral of the first kind. k serves as a parameter in certain approximate formulas.)

Fig. 55 was prepared by first computing the curves shown by means of both the cnoidal-wave theory and the theory of higher-order infinitesimal waves as developed in subsection 27α. [SKJELBREIA's tables (1959) facilitated the computation for the latter method.] Curves were then faired by eye in such a way as to pass smoothly from one set to the other. Hence, although they are claimed to be only qualitative, they have in fact a quantitative basis. An additional error has been introduced by taking the curves $k^2 = \text{const}$ as straight lines; they should, in fact, show some curvature as the radial distance from $c^2/gh = 1$, $A/h = 0$ increases. This additional complication of the computation did not seem necessary for the purpose at hand.

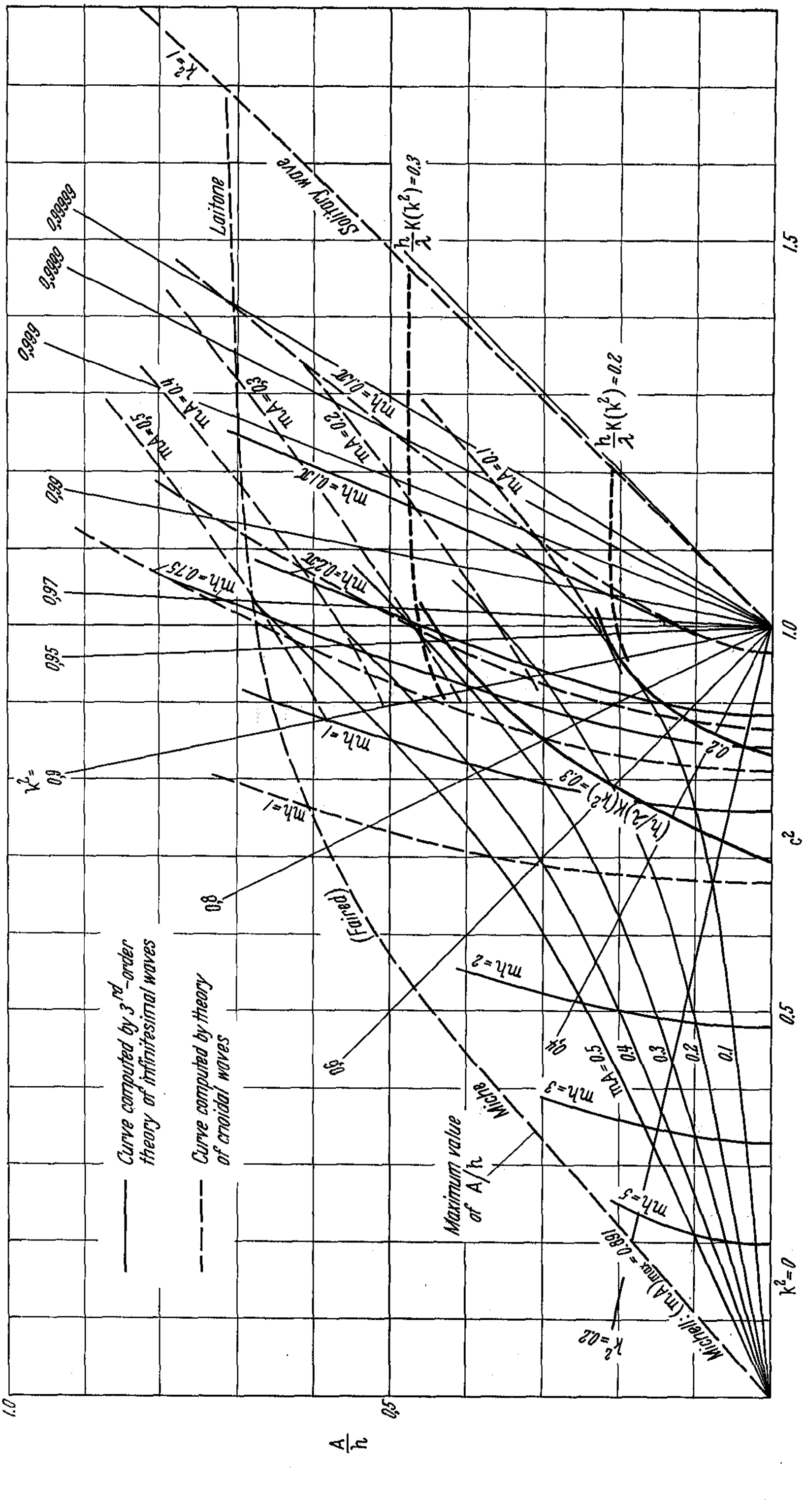
Although it is not strictly relevant to the material of the present section, it seems of interest to display the two sets of curves mentioned above, for they indicate in a rough way the ranges of validity of the two fundamental methods of approximation and show how they fit together. They are shown in Fig. 56. One expects the curves based on the infinitesimal-wave theory to be accurate near the horizontal axis, $A/h = 0$, those based on cnoidal-wave theory to be accurate near $c^2/gh = 1$, and the two to agree where these two regions overlap. The curves confirm this expected behavior. Computations based on the second-order cnoidal-wave theory of Eqs. (31.37) may be expected to produce better agreement over a wider range.

γ) Irrotational waves—other configurations. Flow over a wavy bottom. In connection with the study of inverse methods in subsection 34α an explicit example of a steady flow over a wave-shaped bottom was exhibited. However, there the surface profile was given and the bottom profile calculated. The direct problem, in which the bottom profile and other flow data are given, has also been considered by several persons. LAVRENT'EV (1943) announced theorems concerning this problem, but did not include them in his later (1947) exposition. GERBER (1955) has given a comprehensive treatment of the "supercritical" case and has announced further results for the "subcritical" case (1956). Let the bottom profile S be periodic and symmetric about vertical lines through the maxima and minima; let $\vartheta(s)$ be its intrinsic equation where s is arc length measured from a maximum and ϑ is the angle between the tangent and the x-direction. Let Q be the discharge rate for the fluid, and let q_0 be the velocity at a crest. In the first paper he considers flows in which the slope of the surface has the same sign as that of the bottom (we recall the two possible flows occurring in the linearized theory of subsection 20α). GERBER shows that there exists at least one solution of this type provided the following inequalities are satisfied in the interval between a maximum and the first minimum to the right:

$$\left.\begin{aligned} \frac{gQ}{q_0^3} + \max|\vartheta| &\leqq \pi - \varepsilon_1, \\ -\frac{1}{2}\pi + \varepsilon_2 \leqq \vartheta(s) &\leqq 0, \end{aligned}\right\} \tag{35.3}$$

where ε_1 and ε_2 are arbitrary small but positive quantities. If certain other inequalities, further limiting gQ/q_0^3, are satisfied, he is also able to prove uniqueness provided $\vartheta(s) \not\equiv 0$. In the second paper he announces that there exists at least one solution such that the profile has slope of opposite sign to that of the bottom if

$$\frac{gQ}{q_0^3} > (1+\varepsilon)\frac{\pi^3}{2} \tag{35.4}$$



Solitary wave
Laitone
$\frac{h}{\lambda}K(k^2)=0.3$
$\frac{h}{\lambda}K(k^2)=0.2$
$(h/\lambda)K(k^2)=0.3$
$k^2=1$
$k^2=$
0,99999
0,9999
0,999
0,99
0,97
0,95
0,9
0,8
0,6
$k^2=0$
$k^2=0.2$
$mA=0.1$
$mA=0.2$
$mA=0.3$
$mA=0.4$
$mA=0.5$
mh = 0.1π
mh = 0.2π
mh = 0.25π
mh = 0.75
mh = 1
mh = 2
mh = 3
mh = 5
(Faired)
Miche
Maximum value of A/h
Michell: $(mA)_{max}=0.881$
Curve computed by 3rd-order theory of infinitesimal waves
Curve computed by theory of cnoidal waves
1.0
0.5
$\frac{A}{h}$
0.5
1.0
1.5
$\frac{c^2}{gh}$

Fig. 56.

and $Q/L_0 q_0$ and $q_0 \Delta/Q$ are small enough; here L_0 is the arclength from a maximum to a minimum of S and Δ is the vertical distance. GERBER's methods are topological (Schauder-Leray theory) and do not yield effective methods of approximation.

MOISEEV (1957) has also considered this problem. By a modification of the method used to derive NEKRASOV's integral equation (32.104), he derives a pair of nonlinear integral equations to which the Lyapunov-Schmidt method is applicable. Let c be the average velocity defined by (7.5) for an allowable value of y (thus φ increases by $c\lambda$ over a wavelength), and let Q be the discharge rate. Then MOISEEV finds that there exists a sequence of velocities $c_1 > c_2 > \cdots > 0$ associated with the eigenvalues of a certain linear operator, such that, if $c \neq c_n$, there exists a unique flow provided the slope of the bottom is sufficiently small. Also, if $c > c_1$ or $c_{2n+1} < c < c_{2n}$ then the solution is such that the slopes of bottom and surface are of the same sign; if $c_{2n} < c < c_{2n+1}$, the slopes are of opposite sign.

see errata

Flow over a bottom with a declivity. Let the flow be from left to right and suppose the bottom profile to be asymptotic to horizontal lines as $x \to \pm\infty$, the one on the right being lower than that on the left. The discharge rate Q and velocity c at $x = -\infty$ should then be sufficient to determine the flow. The existence of a steady flow under these circumstances has been investigated by HAIMOVICI (1935) and GERBER (1955). The former derives a pair of nonlinear integral equations, similar to NEKRASOV's, relating ϑ and τ of (32.86). An iterative method is used to prove the existence of a solution. GERBER makes use again of the Schauder-Leray theory. The theorems established by each are very similar, but GERBER's is sharper. Let the bottom be given intrinsically by $\vartheta(s)$, measured from some fixed point. Then a solution exists if

$$\left.\begin{aligned} \frac{gQ}{c^3} < 1, \qquad \max|\vartheta(s)| + \frac{gQ}{c^3} < 1, \\ |\vartheta(s)| \leq A\,\mathrm{e}^{-\alpha|s|}, \qquad A, \alpha > 0. \end{aligned}\right\} \tag{35.5}$$

The last condition assures a rapid approach to the horizontal asymptotes. The case of subcritical flow does not appear to have been treated in the published literature.

Motion past a submerged vortex. TER-KRIKOROV (1958) has recently investigated steady flow past a submerged vortex of intensity Γ in a channel of depth h when the exact boundary conditions on the free surface are retained. If c is the velocity far upstream of the vortex, he proves existence and uniqueness of the flow provided that $c^2/gh > 1$ and Γ/ch is sufficiently small.

Interfacial waves. In subsection 14δ we considered the linearized theory of waves at an interface between two perfect fluids of different densities, bounded above and below by horizontal planes. The question naturally arises as to whether one can establish the existence of such waves when the exact boundary conditions at the interface are observed. KOCHIN (1927) extended the methods of LEVI-CIVITA and STRUIK to this problem and established the existence of (necessarily symmetric) interfacial waves of finite amplitude.

δ) *Rotational waves.* The explicit construction in subsection 34β of a periodic wave of permanent type which is rotational and the demonstrated existence of irrotational waves of this type which are of finite, if small, amplitude raises the question as to whether each of these waves is a special case of a more general type. This question has been discussed in a notable paper by DUBREIL-JACOTIN (1934) with results which include and generalize those of LEVI-CIVITA and STRUIK. We give a only a bare indication of the results.

Let us suppose that a coordinate system has been chosen so that we may treat the wave motion as a steady flow to the right. Although we do not assume the motion to be irrotational, there will still exist a stream function $\psi(x, y)$ by virtue of the continuity equation. The vorticity of the flow will be given by $-\Delta\psi$, and since by a classical theorem the vorticity is constant along a streamline, the following equation must be satisfied by ψ:

$$\Delta\psi = f(\psi), \tag{35.6}$$

where $f(\psi)$ is some unspecified function. The condition on the free surface $\psi=0$ may still be derived from the special Bernoulli theorem [see Eq. (2.10″)]

$$g\,\eta(x) + \tfrac{1}{2}\left[\psi_x^2 + \psi_y^2\right] = \text{const}. \tag{35.7}$$

For irrotational waves the function $f \equiv 0$; for GERSTNER's wave it is given by (34.47) after setting $b=\psi$, $\sigma = cm$. The question which DUBREIL-JACOTIN asked is whether a wave of finite amplitude exists for any distribution of vorticity $f(\psi)$. In order to encompass both of the known finite waves into her results, she limits f to functions of the following sort:

$$f(\psi) = -\mu\,\frac{Q}{h}\,m^2\,e^{2mh\psi/Q}\,F(e^{mh\psi/Q}), \qquad -Q \leqq \psi \leqq 0, \tag{35.8}$$

where Q is discharge rate, h the mean depth, and the function $F(\varrho)$ is bounded and satisfies a Hölder condition in ϱ; μ is a small parameter. If the depth is infinite, one must replace Q/h by c, the velocity at $\psi = -\infty$ (it is assumed that $\psi_y \to c$ as $\psi \to -\infty$).

DUBREIL-JACOTIN's theorem is as follows. For any $m = 2\pi/\lambda$, h and $f(\psi)$ satisfying (35.8) there exists a $\delta > 0$ such that for $\mu < \delta$ there exists a unique corresponding progressive wave of permanent type with vorticity distribution $f(\psi)$. The waves are also shown to be symmetric about vertical lines through crest or trough. She also demonstrates that among this class of waves for finite depth there is a unique analogue of the Gerstner wave, in the sense that the trajectories of individual particles are all closed. This wave has recently been investigated by KRAVTCHENKO and DAUBERT (1957). The development of means of calculating rotational waves has been the subject of a recent investigation by GOUYON (1958).

ε) *Waves in heterogeneous fluids—internal waves.* It has been shown in subsection 32β that irrotational waves of permanent type are not possible in a heterogeneous fluid, but that GERSTNER's rotational wave still provides a solution for infinite depth. DUBREIL-JACOTIN (1935) has shown that this is the only periodic wave of permanent type in infinitely deep fluid having this property. In a later paper (1937) she returned to this topic and made use of the methods developed by her for rotational waves to investigate the existence theory for the two problems described below. The first problem, a natural generalization of one investigated by KOCHIN and mentioned at the end of subsection 35γ, is the existence of periodic internal waves of permanent type in a heterogeneous fluid bounded both above and below by horizontal planes. In the second problem the upper surface is free.

The two problems may be formulated as follows. First we recall that in a steady flow of a heterogeneous fluid the density must be constant along streamlines. Hence, if ψ is the stream function, we may write $\varrho = \varrho(\psi)$. The equation analogous to (35.6) is now somewhat more complicated. It may be derived from (32.54) as follows. Apply the operators

$$\frac{\varrho'}{\varrho}\psi_y - \frac{\partial}{\partial y}, \qquad \frac{\varrho'}{\varrho}\psi_x - \frac{\partial}{\partial x}$$

to the two equations of (32.54), respectively, and subtract. This yields

$$\frac{\varrho'}{\varrho}\frac{\partial(E,\psi)}{\partial(x,y)} - \frac{\partial(\zeta,\psi)}{\partial(x,y)} = 0.$$

Since $\varrho = \varrho(\psi)$,

$$\frac{\varrho'}{\varrho}\frac{\partial(E,\psi)}{\partial(x,y)} = \frac{\partial(\varrho' E/\varrho,\psi)}{\partial(x,y)}$$

and hence

$$\frac{\partial(\varrho' E/\varrho - \zeta,\psi)}{\partial(x,y)} = 0$$

or, integrating and substituting $\zeta = -\Delta\psi$,

$$\Delta\psi + \frac{\varrho'}{\varrho}\left[g\,y + \frac{1}{2}(\psi_x^2 + \psi_y^2)\right] = f_1(\psi) \tag{35.9}$$

where $f_1(\psi)$ is an arbitrary function. This is the equation which ψ must satisfy. If $\psi = 0$ is the top streamline and $\psi = -Q$ the bottom streamline, then the boundary conditions are,

$$\psi = 0 \quad \text{for} \quad y = 0, \qquad \psi = -Q \quad \text{for} \quad y = -h \tag{35.10}$$

for the first problem, and

$$\left.\begin{aligned} \psi_x^2 + \psi_y^2 + 2g\,y &= \text{const} \quad \text{for} \quad \psi = 0, \\ \psi &= -Q \quad \text{for} \quad y = -h \end{aligned}\right\} \tag{35.11}$$

for the second problem [cf. (32.60)]. The function $\varrho(\psi)$ cannot be considered as an arbitrary given function in the same sense that $f_1(\psi)$ is arbitrary; it must be related to the density distribution when the fluid is at rest. DUBREIL-JACOTIN assumes that $\varrho(\psi)$ is the same as the density at the mean level of the streamline ψ when the fluid is at rest.

In order to obtain results analogous to those of subsection 35 δ, certain restrictions are placed upon the function $f_1(\psi)$ and the density distribution. Both problems are then reducible to integro-differential equations. In general there is no nontrivial solution. However, under certain conditions there are an infinite number of values of the parameter $\lambda g/2\pi c^2$ in the neighborhood of which there exist nontrivial symmetric waves of finite (but small) amplitude.

ζ) Waves with surface tension. It has already been mentioned in subsection 34 δ that SLËZKIN (1935b, 1937) had derived an integral equation for the motion of pure capillary waves and had proved both existence and uniqueness of solution under certain circumstances. The explicit solution for this problem derived by CRAPPER supersedes in a sense these earlier results.

SEKERZH-ZENKOVICH (1956) has formulated the exact boundary-value problem for combined gravity and capillary waves in terms of the function ω of (32.86) and announced that a proof of existence for sufficiently small amplitude-to-wavelength ratio can be carried out by LEVI-CIVITA's method for pure gravity waves.

G. Bibliography.

The following bibliography is not by any means a complete one for the subject. For the most part it consists of papers to which reference is made in the text. In a few cases a paper has been included because its subject has been neglected in the text, although this can hardly be considered a remedy. In general, the bibliography is weakest in its coverage of papers describing experimental work; many more such exist than are listed here.

The forms of reference are more or less standard and will not be explained. Titles of papers in Russian have been translated into English; these are indicated by an (R) following the title. Otherwise titles are given in the original language. Published translations into another language are occasionally listed after the original. However, no attempt has been made to search for them. Russian names have been transliterated by a system natural to English, i.e. the cyrillic letters which might be transliterated in a phonetic alphabet as "č, š, ž" are here "ch, sh, zh", respectively. However, when Russians write in a language which uses a roman alphabet, they customarily transliterate their names so that the pronunciation is approximately correct according to the rules of the language in which they are writing. Such variants have been included in brackets behind the transliteration used here.

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Subject Index

List of Errors and Corrections

Corrections to and comments on "Surface Waves" by J. V. Wehausen and E. V. Laitone, *Encyclopedia of Physics*, Volume 9, pp. 446–778, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960. This list has been compiled by the first author.

Identification of locations: First by page number, then by formula number: (x.y), or by line identification: n lines from top or bottom, or (x.y)+n to indicate n lines after formula (x.y), or something similar. If a formula number (x.y) consists of several equations, a subscript will be used to identify the one intended: e.g., $(x.y)_3$ for the third equation.

I am grateful to the many colleagues who have informed me of errors. Without their help the list would be much less complete.

p. 447, line 2: "und" should be "and".

p. 449, (2.10'): delete + following $\frac{1}{2}$.

p. 452, (3.4): a $-$ should precede the term $\mu(w_y + v_z)$.

p. 453, (3.8'): delete the exponent $\frac{1}{2}$ in the two denominators where it occurs.

p. 454, footnote: in the last term on the right $F_{,j}$ should be $F_{,i}$.

p. 457, (7.5), comment: (x, y) must be submerged.

p. 460, (8.8): In the last integral d_s should be ds .

p. 461, (9.4)+1: "orce fon" should be "force on".

p. 468, (10.34): the correctness of the fifth equation has been questioned.

p. 469, 4 lines from the top: (10.32) should be (10.33).

p. 469, (10.36)$-$2: $\alpha\eta^{(01)}$ should be followed by a +.

p. 475, line 3: for $\phi(x, y)$ read $\phi(x, z)$.

p. 475, (13.8)+8: for $(x - \alpha)^2$ read $(x - a)^2$.

p. 475, (13.9)+2: delete the final "n" from "ben".

p. 479, (13.21): in the third term on the first line the denominator should be r_1^{n+2} .

p. 481, bottom equation: in the exponential term c should be $\bar{c}$.

p. 482, $(13.31)_2$: the two occurrences of $-\frac{(-1)^{n-1}}{(n-1)!}$ should be replaced by $\frac{1}{(n-1)!}$.

p. 482, $(13.31)_3$: the two occurrences of $\frac{(-1)^{n-1}}{(n-1)!}$ should be replaced by $\frac{1}{(n-1)!}$.

p. 482, the unnumbered formula below (13.31): the + following $\log r$ should be $-$.

p. 483, (13.34): in the second line $-\frac{e^{-kh}}{k}$ should be $+\frac{e^{-kh}}{k}$; in the last line $\sin\sigma t$ should be $\cos\sigma t$.

p. 484, $(13.35)_5$: grad $\phi = O([(x-a)^2 + (z-c)^2]^{-1/4})$ as $x \to \infty$.

p. 484, (13.37): in each of the two integrals the terms on the second line should be inserted in the numerator after the two cosh terms so that all four terms multiply $(k\cos^2\theta+\nu)$. I would also like to call attention to the fact that a "corrected" version of this equation in my article on wave resistance in Advances in Applied Mechanics, **13**, equation (3.51b) is also in error because of some misplaced multipliers.

p. 485, (13.38): in the last line there is a missing $\}$ before $\sec^3\theta$.

p. 489, (13.43): in the third line the denominator should be $u-\bar{c}$; in the fifth line the exponent should be $-i\nu(z-\bar{c})$.

p. 490, (13.46-48): in order that $f(z)\to 0$ as $x\to\infty$, one must subtract a term $\frac{Q\nu}{2(\nu h-1)}(z-a+ih)$ from (13.46), add a term $\frac{\Gamma\nu(b+h)}{2(\nu h-1)}$ to (13.47), and add a term $\frac{M\nu}{2(\nu h-1)}\cos\alpha$ to (13.48).

p. 491, 11 lines from top: second term on right-hand side should be mr_1^{-1} .

p. 492, (13.49): on the last line, in the exponential term α should be a .

p. 495, (13.54): the numerator in the second term should be $\Gamma(t)-iQ(t)$.

p. 496, (14.2)+3: add to the equation ", $A=a\sigma/g$" and to the equation below add ", $A=(a\sigma/g)\cosh m_0h$".

p. 497, lines 9 and 10 from bottom: the final t on line 9 should be interchanged with the final period on line 10.

p. 497, line 6 from bottom: $\chi(x,y)$ should be $\chi(x,z)$.

p. 498, (14.6): in the second line the right-hand side should be $(\sigma^2/g)^2$.

p. 498, (14.11)−2: last word should be "depth".

p. 504, (14.35)+10: (14.39) should be (14.33).

p. 511, (15.11)+14: insert "near the observer" after η_R .

p. 512, second displayed formula: the numerator in the third term should be $-k_1'x^2/t^2-\sigma(k_1)+\sigma'(k_1)k_1'(x/t^2)t$.

p. 515, line 4: $2\rho g$ should be ρg .

p. 515, top graph on the right and bottom one on left: in second displayed formula $\sigma =$ should be $\sigma' =$. In bottom graph on right in second formula numerator of last term should be $2Tk^2/\rho g$.

p. 519, (15.23): the lower limit for the integral should be $-h$.

p. 520, (15.28): in the first summation σ should be σ_j .

p. 520, (15.29)−1: $+b_j$ should be $+ib_j$.

p. 521, second displayed formula after (15.33): at the end of the formula $dy\,dx$ should be $dy]dx$.

p. 523, (16.4)-1: add an "s" to "function"

p. 527, six lines from bottom: $[A_1^2+B_1^2]$ should be $[A_1^2-B_1^2]$.

p. 527, (17.2): in the coefficient of the second term m_0^2 should be $m_0^{(2)}$.

p. 528, (17.3)+3 and 4: in each occurrence the word "wave" should be followed by "potential".

p. 528, (17.3)+7: I note that I have inserted a comment at the bottom of the page: If A_1, B_1, A_2 are actually wave amplitudes then with $R = |B_1/A_1|$, $T = |A_2/A_1|$, $R^2 + T^2[\cosh m_1 h_1/\cosh m_2 h_2]^2 = 1$.

p. 529, 15 lines from bottom: $= 0$ is missing from the equation that begins $\Phi_{tt} + \ldots$.

p. 529, bottom line: l should be $-l$.

p. 530, line 3: in the second equation f_1 should be νf_l .

p. 531: in the middle of the page the sentence beginning "Let $E_1 = \ldots$" D should be D_1 .

p. 532, (17.4): in the second equation $+\beta_T$ should be $-\beta_T$.

p. 537, line following β) *Waves on beaches*: The first inequality statement should be replaced by $\tan\gamma \geq \frac{-y}{x} \geq 0$; in the next line α should be γ .

p. 540, line 4: eliminate the word "both".

p. 541, (17.52)+3: "or" should be "of".

p. 543, (18.4): in the third equation ν should be m .

p. 544, (18.9): $\cos\theta$ should be $\cos n\theta$.

p. 545, (18.11)−2: Y should be y .

p. 547, (18.24): The upper limit of the integral should be $+y$ and the term beginning $-2\pi i$ should begin $+2\pi i$.

p. 547, $(18.25)_4$: the last term should begin k_1^2 .

p. 550, (18.33): in the summation the argument of the exponential on the first line should terminate with $\gamma]$, as in the similar exponential on the second line.

p. 551, line 2: $\cos(z-\zeta)$ should be $\cos k(z-\zeta)$.

p. 556, (19.11): I have the following note in the margin: Here $\mathbf{n}$ points out of S_1 .

p. 558, (19.24): eliminate superfluous $+$.

p. 563, Fig. 18: the numbers in the abscissa scale should be multiplied by π .

p. 566, (19.66): read $\Phi_f = \mathrm{Re}\{-i\sigma[\varphi^1 a_0 + \varphi^2 b_0 + \varphi^3 c_0 + \varphi^4 \alpha_0 + \varphi^5 \beta_0 + \varphi^6 \gamma_0]\mathrm{e}^{-i\sigma t}\}$.

p. 570, (20.17)−1: for (20.10) read (20.16).

p. 575, (20.42): in the second integral the upper limit should be z .

p. 575, (20.45): for second term read $f_2 = \mathfrak{K}\{f_1\}$.

p. 576, (20.51)+11: (1939b) should be (1936b).

p. 578, (20.60)−11: (1958) should be (1959).

p. 581, $(20.69)_1$: in numerator of first coefficient eliminate c .

p. 585, the second of two lines in the middle of the page: delete the final "s" in "summations".

p. 592, (20.101)+1: "There" should be "These" .

p. 593, (21.1): replace $p(x, y, zt)$ by $p(x, z, t)$.

p. 593, $(21.3)_2$: $-i\sigma$ should be $+i\sigma$; lower limit of third integral should be 0; the sign of the last term should be changed from + to − .

p. 593, (21.4): same as (21.3).

p. 594, (21.6): in both lines the initial signs should be changed; the second integral in line one goes from 0 to ∞ .

p. 594, (21.6)+1: "simpel" should be "simple".

p. 594, (21.7): the right-hand side should be preceded by a − .

p. 594, (21.7)+2: 1953 should be 1958.

p. 594, (21.7)+5: lower limit should be 0 .

p. 594, (21.7)+6: I have written a note at the bottom of the page: Note that these are not really 'sources', for the source at $y = b$ is accompanied by a sink at $y = -b$. As $b \to 0$ these two cancel and the integral above is left.

p. 595, (21.10): the sign of each term should be reversed.

p. 596, (21.16): the right-hand side should be preceded by a − sign.

p. 596, (21.17): both terms on the right should be preceded by − signs.

p. 596, (21.18): the right-hand side should be preceded by a − sign.

p. 597, (21.19), (21.21), (21.22): in all these equations the signs on the right-hand sides should be changed.

p. 597, line 2 from bottom: c should be $c > 0$.

p. 598, (21.26): on the third line, the exponent $-\nu \sec^3 \theta$ should be replaced by $\nu y \sec^3 \theta$.

p. 599, (21.30)+1: after "potential" insert "as calculated by Lunde (1951b),".

p. 600, (21.34): on the first line, p in the denominator should be ρ ; in the second denominator replace $\sec^2$ by $\sec^2 \theta$.

p. 604, (22.1): the = in the second term should be deleted.

p. 604, (22.4): on the first line, in $G(...)$ and $G_t(...)$ the correct arguments should be be $(x, y, z; \xi, 0, \zeta; t, \tau)$.

p. 604, (22.4)+6: replace ξ, ζ by ξ, η, ζ in the first occurrence of G .

p. 605, (22.8): the upper limit for the two single integrals should be t; a subscript t should be added to the G in the third line and removed from the G in the fourth line.

p. 606, line 4: the multiplier of Φ_t should be $-\frac{1}{2}$.

p. 606, second displayed formula − 1: replace "f, F" by "$\eta(x, z, 0), \eta_t(x, z, 0)$".

p. 609: in the first group of three equations determining K, multiply the second equation by $2/\pi$ and the third equation by $1/\pi$; in the second group of three equations determining K, divide all terms by π; in the third equation (single line) determining K delete π from the coefficient; in (22.20) and all three equations of (22.21) replace π^2 by π .

p. 613, (22.42)+2 and 5: in line 2 delete "of"; in line 5 insert "in" before "the form".

p. 615, lines 1 and (22.50)−5: delete " 's" in "Green's".

p. 615, (22.49): an overbar is missing over $gk \tanh kh$.

p. 616, lines 7, 15, 26, 28: eliminate the " 's" in "Green's".

p. 616, (22.52)+2: delete the final "s" in "computations"; in the next line (1937) should be (1939).

p. 618, (22.60): insert $J_0(kR)$ before $\sigma(k)$.

p. 619, (22.65)+2: delete " 's" in "Green's".

p. 620, line 3: replace "an analytic" by "a closed".

p. 623, (23.13)+13: replace "McKnown" by "McNown".

p. 628, $(23.35)_1$: in the second term replace $\bar{x}_y$ by $\bar{x}$.

p. 631, line before **24. Gravity waves**: replace "with" by "within".

p. 633, (24.13)+1 and +3: in line 1 delete the 2 before ρg; in line 3 replace 0.33 by 0.47.

p. 636, (24.31)−3: (24.38) should be (24.28).

p. 637, second displayed equation: the $\pm$ should be reversed.

p. 637, (24.32)−5 (constant c): according to a marginal note c has been found by Pursco (reference missing); $c = \left[\sqrt{3}\,(2-\sqrt{3})\right]^{-1} = 2.1547$.

p. 640, $(25.1)_1$: w_y should be w_z

p. 641, (25.10)+1: delete "n" in "wheren".

p. 642, (25.23): for $T'\omega^3$ read $T'm^3$.

p. 644, (25.39)+3: read "he" for "the".

p. 645, line 3: for ω_1 read ω .

p. 646, line 18: for "Pulsing" read "Pulsating".

p. 647, (26.3)−2: for (26.1) read (26.2).

p. 654, (27.2): the last term on the left should be $c\eta_x$.

p. 655, $(27.7)_2$: in the second line +grad should be −grad; in the third line $+\eta^{(1)}$grad should be $-\eta^{(1)}$grad, and $-T'$ should should be $+T'$.

p. 655, (27.8): in the first equation C_0 should be c_0; in the next line (27.9) should be (27.6).

p. 657, line 6: "seen" should be "seem".

p. 657, last displayed equation: in the last term it should be $\sin nmx$.

p. 658, (27.25): I have a ? beside the = in front of A' .

p. 659, (27.30): $\cos m$ should be $\cos mx$.

p. 660, (27.34): A should be A' , where $A' = A\left\{1 + A^2m^2\dfrac{2\sinh^4 mh + 14\sinh^2 mh + 3}{16\sinh^4 mh}\right\}$.

p. 661, (27.43): in the double integral the limits of the second one should be $-\infty$ and η .

p. 662, (27.45): on line two, in both the exponential and the denominator $(m_1 - m_2)$ should be $|m_1 - m_2|$.

p. 664, (27.55): in the second equation c_0 should be σ_0 .

p. 664, (27.55)+4: replace "they" by "these".

p. 664, $(27.56)_2$: in the last line $\cos 2my$ should be $\cos 2mx$.

p. 665, $(27.60)_2$: there should be a bracket] after $\cos 3mx$.

p. 666, line 8: in the margin I have written Concus (1962).

p. 666, $(27.63)_{1,2}$: In line 2 the second appearance of $\coth^2 mh$ should be preceded by a $+$ and not a $-$; in the fourth line the initial $+$ should be $-$.

p. 666, (27.67): I have added an extra term $+A^2\sigma_0 m\exp(-4\nu m^2 t)\sin 2my$ together with a reference to Longuet-Higgins [1960, p. 296] for a correction to Harrison.

p. 668, line 12: after "converges" insert "asymptotically".

p. 683, (30.22)+8: χ_n should be χ_u .

p. 695, (30.47)+8: delete first t in "relations".

p. 715, line 5: insert "irrotational" before "motion".

p. 715, (32.1)−2: φ should be Φ .

p. 717, (32.17)+1: after "transport" insert "velocity".

p. 720, (32.44)−1: for (32.88) read (32.38).

p. 721, (32.48): before the integral insert $1/\lambda$. I have the following note in the margin: See Longuet-Higgins, Proc. Roy. Soc. Lond., Ser. A **342** (1975), p. 163.

p. 721, (32.50): at the end of the right-hand side insert $> \frac{1}{2}$.

p. 725, middle of the page: beside the reference to Lamb I have written: Lamb does not assume irrotational motion.

p. 728, (32.89)+1: for "It" read "If".

p. 730, (32.104): the factor preceding the integral should be $\frac{1}{6\pi}$; the limits should be $-\pi$ to π.

p. 734, (33.24): $\sqrt[?]{2}$ should be $\sqrt[3]{2}$, i.e. $2^{1/3}$.

p. 744, (34.49)+5: At the end of the sentence I have written "See also Boussinesq (1877)".

p. 755, end of paragraph beginning with Moiseev: in the maargin I have written: Seems to contradict p. 570 and Gerber.

p. 767, Kochin, fifth entry: "Sobranie" is misspelled.

p. 769, before Lewy: insert Lewis, D. J.: The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. II. Proc. Roy. Soc. Lond., Ser. A **202**, 81–96 (1950).

p. 770, McKnown: should be McNown.

p. 775, after Tamiya insert: Taylor, G. I.: The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I. Proc. Roy. Soc. Lond., Ser. A **201**, 192–196 (1950).